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Reliability approximations via asymptotic distribution

Sondipon Adhikari
University of Bristol, Bristol, United Kingdom

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ABSTRACT: In the reliability analysis of safety critical complex engineering structures a very large number of the system parameters can be considered to be random variables. The difficulty in computing the failure probability using the classical First and Second-Order Reliability Methods (FORM and SORM) increases rapidly with the number of variables or 'dimension'. There are mainly two reasons behind this. The first is the increase in computational time with the increase in the number of random variables. In principle this problem can be handled with superior computational tools. The second, which is perhaps more fundamental, is that there are some conceptual difficulties associated typically with high dimensions. This means that even one manages to carry out the necessary computations, the application of existing FORM and SORM may still lead to incorrect results in high dimensions. This paper is aimed at addressing this issue. Based on the asymptotic distribution of quadratic form in Gaussian random variables, two formulations for the case when the number of random variables $n \to \infty$ is provided. The first is called strict asymptotic formulation and the second is called weak asymptotic formulation. Both approximations results in simple closed-form expressions for the probability of failure of an engineering structure. The proposed asymptotic approximations are compared with existing approximations and Monte Carlo simulations using numerical examples.

1 INTRODUCTION

Uncertainties in specifying material properties, geometric parameters, boundary conditions and applied loadings are unavoidable in describing real-life engineering structural systems. Traditionally this has been catered for in an ad-hoc way through the use of safety factors at the design stage. Such an approach is unlikely to be satisfactory in today's competitive design environment, for example, in minimum weight design of aircraft structures. The situation may also arise when system safety is being jeopardized due to the lack of detailed treatment of uncertainty at the design stage, for example, finite probability of occurring a resonance is unlikely to be captured by a safety-factor based approach due to the intricate nonlinear relationships between the system parameters and the natural frequencies. For these reasons a scientific and systematic approach is required to predict the probability of failure of a structure at the design stage. Accurate reliability assessment is also critical for optimal design of structures. Suppose the random variables describing the uncertainties in the structural properties and loading are considered to form a vector $\mathbf{x} \in \mathbb{R}^n$ where n is the number of random variables. The statistical properties of the system are fully described by the joint probability density function $p(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$. For a given set of variables $\mathbf{x}$ the structure will either fail under the applied (random) loading or will be safe. The condition of the structure for every $\mathbf{x}$ can be described by a safety margin $g(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ such the structure has failed if $g(\mathbf{x}) \leq 0$ and is safe if $g(\mathbf{x}) > 0$. Thus, the probability of failure is given by

$$P_f = \int_{g(\mathbf{x})\leq 0} p(\mathbf{x}) d\mathbf{x}. \tag{1}$$

The function $g(\mathbf{x})$ is also known as the limit-state function and the $(n-1)$-dimensional surface $g(\mathbf{x}) = 0$ is known as the failure surface. The central theme of a reliability analysis is to evaluate the multidimensional integral (1). The exact evaluation of this integral, either analytically or numerically, is not possible for most practical problems because n is usually large and $g(\mathbf{x})$ is a highly nonlinear function of $\mathbf{x}$ which may not be available explicitly. Over the past three decades there has been extensive research (see for example, the books by Madsen et al., 1986, Melchers, 1999) to develop approximate numerical methods for the efficient calculation of the reliability integral. The approximate reliability methods can be broadly grouped into (a) first-order reliability method (FORM), and (b) second-order reliability method (SORM). In FORM and SORM it is assumed that all the basic random variables are transformed and scaled so that they are uncorrelated Gaussian random variables, each with zero mean and unit standard deviation.

The difficulty in computing the failure probability using the classical First and Second-Order Reliability Methods (FORM and SORM) increases rapidly with the number of variables or 'dimension'. There are mainly two reasons behind this. The first is the increase in computational time with the increase in the number of random variables. In principle this problem can be handled with superior computational tools and powerful computing machines. The second, which is perhaps more fundamental, is that there are some conceptual difficulties associated typically with high dimensions. In the context of FORM, using the Tchebysheff bound, Veneziano (1979) has shown that the probability of failure depends on the dimension n although the reliability index does not explicitly depend on n. In the context of SORM, using the χ^2-distribution Fiessler et al. (1979) have shown that there can be significant difference of probability of failure in higher dimension for a fixed value of the reliability index. This means that even one manages to carry out the necessary computations, the application of existing FORM and SORM may still lead to incorrect results in high dimensions. This paper is aimed at investigating this fundamental issue.

A new approach based on the asymptotic distribution of quadratic form of random variables is proposed in this paper. Two closed form asymptotically-equivalent approximate expressions of the integral in (1) are derived for the case when the number of random variables $n \to \infty$. It is assumed that the basic random variables are Gaussian or can be transformed to Gaussian, for example using Rosenblatt transformation (Rosenblatt, 1952). The first approximation is called *strict asymptotic formulation* as it requires some asymptotic conditions to be satisfied strictly. The second approximation, called *weak asymptotic formulation*, relaxes some of the strict asymptotic requirements of the first approach. The proposed asymptotic approximations are compared with existing approximations and Monte-Carlo simulations using numerical examples.

2 BRIEF REVIEW OF CLASSICAL FORM AND SORM

After suitable transformations and keeping only second-order terms, Madsen et al. (1986) have approximated the failure surface by a parabolic surface as

$$\widetilde{g} \approx -y_n + \beta + \mathbf{y}^T \mathbf{A} \mathbf{y}, \tag{2}$$

where $\mathbf{y} \sim \mathbb{N}_{n-1}(\mathbf{0}_{n-1}, \mathbf{I}_{n-1})$ and $y_n \sim \mathbb{N}_1(0,1)$. The parabolic function in (2) is normally used in the classical SORM approximations. With this approximation the failure probability is given by

$$P_f \approx \text{Prob}\left[\widetilde{g} \le 0\right] \approx \text{Prob}\left[y_n \ge \beta + \mathbf{y}^T \mathbf{A} \mathbf{y}\right] = \text{Prob}\left[y_n \ge \beta + U\right] \tag{3}$$

where

$$U : \mathbb{R}^{n-1} \mapsto \mathbb{R} = \mathbf{y}^T \mathbf{A} \mathbf{y}, \tag{4}$$

is a central quadratic form in standard Gaussian random variables.

Using quadratic approximation of the failure surface together with asymptotic analysis Breitung (1984) proved that when $\beta \to \infty$

$$P_f \to \Phi(-\beta) \left\| \mathbf{I}_{n-1} + 2\beta \mathbf{A} \right\|^{-1/2}. \tag{5}$$

Here $\Phi(\bullet)$ is the standard Gaussian cumulative distribution function. The eigenvalues of $\mathbf{A}$, say a_j, can be related to the principal curvatures of the surface κ_j as $a_j = \kappa_j/2$. This result is important because it gives the asymptotic behavior P_f under general conditions. It should be noted that equation (5) is the exact asymptotic expression and it cannot be improved as long as the asymptotic behavior of P_f in β is considered. Later, Hohenbichler and Rackwitz (1988) proposed the following formula

$$P_f \approx \Phi(-\beta) \left\| \mathbf{I}_{n-1} + 2\frac{\varphi(\beta)}{\Phi(-\beta)} \mathbf{A} \right\|^{-1/2} \tag{6}$$

where $\varphi(\bullet)$ is the standard Gaussian probability density function. This expression is more accurate than (5) for lower values of β (although both are asymptotically equivalent) and it was also rederived by Köylüoğlu and Nielsen (1994) and Polidori et al. (1999) using different approaches. In FORM, the failure surface is approximated by a hyperplane at the design point. This implies that the Hessian matrix at the design point is assumed to be a null matrix, that is $\mathbf{A} = \mathbf{O}$ and consequently $U = 0$ in equation (3). Thus, from equation (3), one obtains the probability of failure

$$P_f \approx \Phi(-\beta). \tag{7}$$

This is the simplest approximation to the integral (1). Breitung's formula (5) and the formula by Hohenbichler and Rackwitz (6) can be viewed as corrections to the FORM formula (7) to take account of the curvature of the failure surface at the design point.

If n is very large, the computation of P_f using any available methods will be difficult. Nevertheless, it is useful to ask the following questions of fundamental interest:

- Suppose we have followed the 'usual route' and did all the calculations (*i.e.*, obtained x^*, β and $\mathbf{A}$). Can we still expect the same level of accuracy from the classical FORM/SORM formula in high dimensions as we do in low dimensions? If not, what are the exact reasons behind it?

- From the point of view of classical FORM/SORM, what do we mean by 'high dimension'? Is it a problem dependent quantity, or is it simply our perception based on available computational tools so that what we regard as a high dimension today may not be considered as a high dimension in the future when more powerful computational tools will be available?

We tried to answer these questions using the asymptotic distribution of the quadratic form (4) as $n \to \infty$. It will be shown that minor modifications to the classical FORM and SORM formula can improve their accuracy in high dimensions. Based on an error analysis, we also attempt to provide a value of n above which the number of random variables can be considered as high from the point of view of classical FORM/SORM.

3 ASYMPTOTIC DISTRIBUTION OF QUADRATIC FORMS

Discussions on asymptotic distribution of quadratic forms may be found in Mathai and Provost (1992, Section 4.6b). Here, one of the simplest forms of asymptotic distribution of U will be used. We start with the moment generating function of U

$$M_U(s) = \mathrm{E}\left[e^{sU}\right] = \mathrm{E}\left[e^{s\mathbf{y}^T\mathbf{A}\mathbf{y}}\right] = \|\mathbf{I}_{n-1} - 2s\mathbf{A}\|^{-1/2} = \prod_{k=1}^{n-1}(1-2sa_k)^{-1/2}. \quad (8)$$

Now construct a sequence of new random variables $q = U/\sqrt{n}$. The moment generating function of q:

$$M_q(s) = M_U(s/\sqrt{n}) = \prod_{k=1}^{n-1}\left(1 - 2sa_k/\sqrt{n}\right)^{-1/2}. \quad (9)$$

From this

$$\ln\left(M_q(s)\right) = \frac{1}{2}\sum_{k=1}^{n-1} 2sa_k/\sqrt{n} + s^2\left(2a_k/\sqrt{n}\right)^2/2 + s^3\left(2a_k/\sqrt{n}\right)^3/3 + \cdots \quad (10)$$

provided

$$|2sa_k| < 1, \quad \text{for} \quad k = 1, 2, \cdots, n-1. \quad (11)$$

Consider a case when a_k and n are such that the higher-order terms of s vanish as $n \to \infty$, *i.e.*, we assume n is large such that the following conditions hold

$$\sum_{k=1}^{n-1}(2a_k/\sqrt{n})^2/2 < \infty \text{ or } \frac{2}{n}\mathrm{Trace}\left(\mathbf{A}^2\right) < \infty \quad (12)$$

and $\forall\ r \geq 3$

$$\sum_{k=1}^{n-1}(2a_k/\sqrt{n})^r/r \to 0 \text{ or } \frac{2^r}{n^{r/2}\ r}\mathrm{Trace}\left(\mathbf{A}^r\right) \to 0. \quad (13)$$

Under these assumptions, the series in equation (10) can be truncated after the quadratic term

$$\ln\left(M_q(s)\right) \approx \frac{1}{2}\sum_{k=1}^{n-1} s\left(2a_k/\sqrt{n}\right) + \frac{s^2}{2}\left(2a_k/\sqrt{n}\right)^2 = \mathrm{Trace}\left(\mathbf{A}\right)s/\sqrt{n} + \left(2\ \mathrm{Trace}\left(\mathbf{A}^2\right)\right)s^2/2n. \quad (14)$$

Therefore, the moment generating function of $U = q\sqrt{n}$ can be approximated by

$$M_U(s) \approx e^{\mathrm{Trace}(\mathbf{A})s+\ 2\,\mathrm{Trace}\ \mathbf{A}^2\ \ s^2/2}. \quad (15)$$

From the uniqueness of the Laplace Transform pair it follows that when the conditions (11)–(13) are satisfied, U asymptotically approaches a Gaussian random variable with mean $\mathrm{Trace}\left(\mathbf{A}\right)$ and variance $2\mathrm{Trace}\left(\mathbf{A}^2\right)$, that is

$$U \to \mathbb{N}_1\left(\mathrm{Trace}\left(\mathbf{A}\right), 2\ \mathrm{Trace}\left(\mathbf{A}^2\right)\right) \text{ when } n \to \infty. \quad (16)$$

For practical problems, the minimum number of random variables required for the accuracy of this asymptotic distribution will be helpful. The error in neglecting higher order terms in series (10) is of the form

$$\sum_{k=1}^{n-1}(2sa_k/\sqrt{n})^r/r = \frac{1}{r}\left(\frac{2s}{\sqrt{n}}\right)^r \mathrm{Trace}\left(\mathbf{A}^r\right), \text{ for } r \geq 3. \quad (17)$$

Values of s define the domain over which the moment generating function is used. For large β, it turns out that appropriate choice of s is $s = -\beta$ (see the Appendix). Using this, here we aim to derive a simple expression for the minimum value of n which is *sufficient* for the application of the asymptotic distribution method. From the expression of error (17), assume there exist a small real number ϵ (allowable error) such that

$$\left|\frac{1}{r}\frac{(-2\beta)^r}{n^{r/2}}\mathrm{Trace}\left(\mathbf{A}^r\right)\right| < \epsilon \quad (18)$$

$$\text{or} \quad n^{r/2} > \frac{(2\beta)^r}{r\epsilon}\mathrm{Trace}\left(\mathbf{A}^r\right) \quad (19)$$

$$\text{or} \quad n > \frac{4\beta^2}{\sqrt[r]{r^2\epsilon^2}}\left(\sqrt[r]{\mathrm{Trace}\left(\mathbf{A}^r\right)}\right)^2. \quad (20)$$

Since $\mathbf{A}$ is a positive definite matrix, the critical value of n is obtained for $r = 3$:

$$n_{\min} = \frac{4\beta^2}{\sqrt[3]{9\epsilon^2}} \left(\sqrt[3]{\text{Trace}\left(\mathbf{A}^3\right)} \right)^2 . \tag{21}$$

From equation (21), the following points may be observed: (a) the minimum number of random variables required would be more if ϵ (error) is considered to be small (as expected) and $n_{\min} \propto \frac{1}{\epsilon^{2/3}}$, (b) if β is large, more random variables are needed to achieve a desired accuracy and $n_{\min} \propto \beta^2$, and (c) if $\mathbf{A}$ has some large eigenvalues (principal curvatures), they would control the term in the bracket and consequently $n_{\min}$. In the next two sections the asymptotic distribution (16) is used to obtain the probability of failure.

4 FAILURE PROBABILITY USING STRICT ASYMPTOTIC FORMULATION

In the strict asymptotic formulation we start with equation (3). The probability of failure can be rewritten as

$$P_f \approx \text{Prob}\left[y_n \geq \beta + U\right] = \text{Prob}\left[y_n - U \geq \beta\right]. \tag{22}$$

Since from (16) the asymptotic pdf of U is Gaussian, the variable $z = y_n - U$ is also a Gaussian random variable with mean $(-\text{Trace}\,(\mathbf{A}))$ and variance $(1 + 2\,\text{Trace}\left(\mathbf{A}^2\right))$. Thus, when $n \to \infty$, the probability of failure can be obtained from equation (22) as

$$P_{f_{\text{Strict}}} \to \Phi\left(-\beta_1\right), \quad \beta_1 = \frac{\beta + \text{Trace}\,(\mathbf{A})}{\sqrt{1 + 2\,\text{Trace}\left(\mathbf{A}^2\right)}}. \tag{23}$$

This is the exact asymptotic expression of P_f *after* making the parabolic failure surface assumption and it cannot be improved or changed asymptotically. If the failure surface is close to linear and the number of random variables is not very large then it is expected that $\text{Trace}\,(\mathbf{A}) = \text{Trace}\left(\mathbf{A}^2\right) \to 0$, and it is easy to see that equation (23) reduces to the classical FORM formula (7). Therefore, the expressions derived here can be viewed as the 'correction' which need to be applied to the classical FORM formula when a large number of random variables are considered. A simple geometric interpretation of this asymptotic expression can be given.

From equation (22) the failure domain is given by

$$y_n - U \geq \beta. \tag{24}$$

We have already shown that when $n \to \infty$

$$U \simeq \mathbb{N}_1\left(m, \sigma^2\right),$$
$$\text{with}\, m = \text{Trace}\,(\mathbf{A}) \text{ and}\, \sigma = \sqrt{2\text{Trace}\left(\mathbf{A}^2\right)}. \tag{25}$$

Using the standardizing transformation $Y = (U - m)/\sigma$, equation (24) can be rewritten as

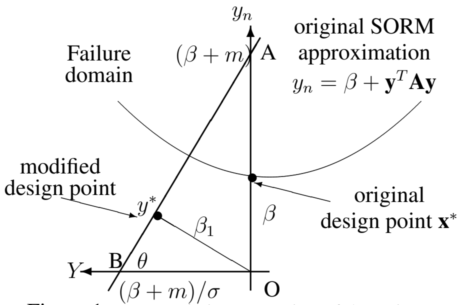

Figure 1: Geometric interpretation of the strict asymptotic formulation

$$\frac{y_n}{\beta + m} + \frac{Y}{-\frac{\beta+m}{\sigma}} \geq 1. \tag{26}$$

This implies that the original $(n-1)$-dimensional parabolic hypersurface asymptotically becomes a straight line in the two-dimensional (y_n, Y)-space as shown in figure 1. Considering the triangle AOB, $\tan\theta = \frac{\text{OA}}{\text{OB}} = \frac{(\beta+m)}{(\beta+m)/\sigma} = \sigma$. Therefore, $\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\sigma}{\sqrt{1+\sigma^2}}$. Now, considering the triangle OBy^* and noticing that O$y^* \perp$AB, $\sin\theta = \frac{\text{O}y^*}{\text{OB}} = \frac{\beta_1}{(\beta+m)/\sigma}$. From this, the modified reliability index

$$\begin{aligned} \beta_1 &= \frac{\beta + m}{\sigma}\sin\theta = \frac{\beta + m}{\sigma}\frac{\sigma}{\sqrt{1+\sigma^2}} \\ &= \frac{\beta + m}{\sqrt{1+\sigma^2}} = \frac{\beta + \text{Trace}\,(\mathbf{A})}{\sqrt{1 + 2\,\text{Trace}\left(\mathbf{A}^2\right)}}. \end{aligned} \tag{27}$$

Therefore, from figure 1 the failure probability $P_{f_{\text{Strict}}} = \Phi(-\beta_1)$, which has been derived in equation (23). If n is small, m and σ are also expected to be small. This would shift the point B towards $-\infty$ in the Y-axis and point A towards the β level in the positive y_n-axis. That is, when m and σ approch to 0, line AB will rotate clockwise and eventually it will be parallel to the Y-axis with a shift of $+\beta$. In this situation y^* will approach to the original design point in the y_n-axis and $\beta_1 \to \beta$ as expected. This geometric analysis explains why classical SORM approximations based on the original design point $\mathbf{x}^*$ do not work well when a large number of random variables are considered.

The value of n given by equation (21) can be viewed as the borderline between the low and high dimension. Beyond this value of n, significant 'trace effect' can be observed and consequently the modified reliability index β_1 instead of β should be used.

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5 FAILURE PROBABILITY USING WEAK ASYMPTOTIC FORMULATION

The expression of P_f given by (23) cannot be improved asymptotically. However, there are scope of 'improvements' if one does not strictly apply the asymptotic condition $n \to \infty$. The advantage of such non-asymptotic approximations is that the approximations may work well even when the asymptotic condition is not met. The disadvantage is that a non-asymptotic approximation will have unquantified errors and one cannot in general prove that such errors will vanish when the asymptotic condition is fulfilled. Nevertheless, it is worth perusing a non-asymptotic approximation since real-life structural systems have finite number of random variables.

Rewriting equation (3), the failure probability can be expressed as

$$\begin{aligned} P_f &\approx \mathrm{Prob}\left[y_n \geq \beta + U\right] \\ &= \int_{\mathbb{R}} \left\{ \int_{\beta+u}^{\infty} \varphi(y_n) dy_n \right\} p_U(u) du \\ &= \mathrm{E}\left[\Phi(-\beta - U)\right]. \end{aligned} \tag{28}$$

where $p_U(u)$ is the probability density function of U and $\mathrm{E}[\bullet]$ is the expectation operator. Extensive discussions on quadratic forms in Gaussian random variables can be found in the books by Johnson and Kotz (1970, Chapter 29) and Mathai and Provost (1992). In general a simple closed-form expression of $p_U(u)$ is not available. For this reason it is difficult to calculate the expectation $\mathrm{E}[\Phi(-\beta - U)]$ analytically. Several authors have used approximations of $\mathrm{E}[\Phi(-\beta - U)]$ to obtain closed-form expressions of P_f. A selected collection of such expressions can be found in Zhao and Ono (1999). Here asymptotic distribution of U in (16) is used to obtain P_f from equation (28).

From the definition of U in (4) note that $u \in \mathbb{R}^+$ since $\mathbf{A}$ is a positive definite matrix. We rewrite (28) as

$$\begin{aligned} P_f &\approx \int_{\mathbb{R}^+} \Phi(-\beta - u)\, p_U(u)\, du \\ &= \int_{\mathbb{R}^+} e^{\ln[\Phi(-\beta-u)] + \ln[p_U(u)]}\, du. \end{aligned} \tag{29}$$

The aim here is to expand the integrand in a first-order Taylor series about the most probable point or optimal point, say $u = u^*$. The optimal point is the point where the integrand in (29) reaches its maxima in $u \in \mathbb{R}^+$. The asymptotic approximation of $p_U(u)$ in (16) will only be used to find the maxima of the integrand and will *not* be used subsequently to calculate the expectation. The expectation operation will be carried out exactly by utilizing the expression of the moment generating function in equation (8). For this reason this approach is called weak asymptotic formulation.

For the maxima of the integrand in (29) we must have

$$\frac{\partial}{\partial u} \left\{ \ln\left[\Phi(-\beta - u)\right] + \ln\left[p_U(u)\right] \right\} = 0. \tag{30}$$

Recalling that

$$p_U(u) = (2\pi)^{-1/2} \sigma^{-1} e^{-(u-m)^2/(2\sigma^2)} \tag{31}$$

where m and σ are given in (25), equation (30) results

$$\frac{\varphi(\beta + u)}{\Phi(-(\beta + u))} = \frac{m - u}{\sigma^2}. \tag{32}$$

Because this relationship holds at the optimal point u^* we define a constant η as

$$\eta = \frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))} = \frac{m - u^*}{\sigma^2}. \tag{33}$$

Taking a first-order Taylor series expansion of $\ln[\Phi(-\beta - u)]$ about $u = u^*$ we have

$$\begin{aligned} \ln\left[\Phi(-\beta - u)\right] &\approx \ln\left[\Phi(-\beta - u^*)\right] \\ &\quad - \frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))}(u - u^*) \end{aligned} \tag{34}$$

or

$$\Phi(-\beta - u) \approx e^{\ln[\Phi(-(\beta+u^*))] - \frac{\varphi(\beta+u^*)}{\Phi(-(\beta+u^*))}(u-u^*)}. \tag{35}$$

Using equation (33) this reduces to

$$\Phi(-\beta - u) \approx \Phi(-\beta_2) e^{\eta u^*} e^{-\eta u} \tag{36}$$

where

$$\beta_2 = \beta + u^*. \tag{37}$$

Taking the expectation of (36) and utilizing the expression of the moment generating function in equation (8), the probability of failure can be expressed as

$$P_f \approx \Phi(-\beta_2) e^{\eta u^*} \left\| \mathbf{I}_{n-1} + 2\,\eta\, \mathbf{A} \right\|^{-1/2}. \tag{38}$$

The optimal point u^* should be obtained by solving the nonlinear equation (33). An exact closed-form solution of this equation does not exist. However, it can be easily solved numerically (for example the function 'fzero' in MATLAB® can be used) to obtain u^*. An approximate solution of (33) can be obtained by considering the asymptotic expansion of the ratio $\varphi(\beta + u^*)/\Phi(-(\beta + u^*))$,

$$\frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))} \approx (\beta + u^*) + (\beta + u^*)^{-1} - \cdots. \tag{39}$$

Keeping only the first term, the left-hand side of (33) becomes $(\beta + u^*)$ and consequently we obtain

$$\eta \approx (\beta + u^*) \approx \frac{m - u^*}{\sigma^2} \quad \text{or} \quad u^* \approx \frac{m - \beta\sigma^2}{1 + \sigma^2} \tag{40}$$

so that

$$\beta_2 = \beta + u^* \approx \frac{\beta + m}{1 + \sigma^2} = \frac{\beta + \mathrm{Trace}\,(\mathbf{A})}{1 + 2\,\mathrm{Trace}\left(\mathbf{A}^2\right)}. \tag{41}$$

In view of (37) and (40) it is also clear that

$$\eta \approx \beta_2. \tag{42}$$

Using this, from (40) u^* can be expressed in terms of

β_2 as

$$u^* \approx -\left(2\beta_2 \mathrm{Trace}\left(\mathbf{A}^2\right) - \mathrm{Trace}\left(\mathbf{A}\right)\right). \quad (43)$$

Now substituting η from (42) and u^* from (43) in equation (38), the failure probability using weak asymptotic formulation can be finally obtained as

$$P_{f_{\mathrm{Weak}}} \rightarrow \frac{\Phi\left(-\beta_2\right) e^{-\left(2\beta_2^2 \mathrm{Trace}\left(\mathbf{A}^2\right) - \beta_2 \mathrm{Trace}\left(\mathbf{A}\right)\right)}}{\sqrt{\|\mathbf{I}_{n-1} + 2\beta_2 \mathbf{A}\|}}, \ n \rightarrow \infty \quad (44)$$

where β_2 is defined in (41). If the number of random variables is not very large then it is expected that $\mathrm{Trace}\left(\mathbf{A}\right) = \mathrm{Trace}\left(\mathbf{A}^2\right) \rightarrow 0$. In that case it is easy to see that $\beta_2 \rightarrow \beta$ and equation (44) reduces to the classical Breitung's SORM formula (5). Therefore, (44) can be viewed as the 'correction' which needs to be applied to the classical SORM formula when a large number of random variables are considered. Unlike the strict formulation, a simple geometric explanation of this expression cannot be given. Also note that the modified reliability indices β_1 and β_2 for the two formulations are not identical.

6 NUMERICAL RESULTS AND DISCUSSIONS

We consider a problem for which the failure surface is *exactly* parabolic in the normalized space, as given by equation (2). The purpose of this numerical study is to understand how the proposed approximation work after making the parabolic failure surface assumption. Therefore, the effect of errors due to parabolic failure surface assumption itself cannot and will not be investigated here.

In numerical calculations we have fixed the number of random variables n and the trace of the coefficient matrix $\mathbf{A}$. It is assumed that the eigenvalues of $\mathbf{A}$ are uniform positive random numbers. Based on the values of n and $\mathrm{Trace}\left(\mathbf{A}\right)$ two cases are considered:

Case 1: small number of random variables: $n-1 = 35$, and $\mathrm{Trace}\left(\mathbf{A}\right) = 1$

Case 2: large number of random variables: $n-1 = 200$, and $\mathrm{Trace}\left(\mathbf{A}\right) = 1$

When $\mathrm{Trace}\left(\mathbf{A}\right) = 0$ the failure surface is effectively linear. Therefore, the more the value of $\mathrm{Trace}\left(\mathbf{A}\right)$ the more non-linear the failure surface becomes. Probability of failure obtained using the two asymptotic expressions is compared with Breitung's asymptotic result, the formula (6) derived by Hohenbichler and Rackwitz (1988) and Monte Carlo simulation. Monte Carlo simulation is carried out by generating 10000 samples of the quadratic form (4) and numerically calculating the expectation operation (28) for each value of β.

Figure 2 shows probability of failure (normalized by dividing with $\Phi(-\beta)$) for values of β ranging from 0 to 6 for case 1. For this problem, the minimum number of random variables required for the applicability of the asymptotic distribution can be obtained from (21). Considering $\epsilon = 0.01$, it can be shown from equation (21) that $n_{min} = 176$. Although this condition is not satisfied here, the results obtained from the weak asymptotic formulation are accurate. The results obtained from the strict asymptotic formulation are not accurate, especially when β is high. This is however expected as the asymptotic condition has not been met for this case.

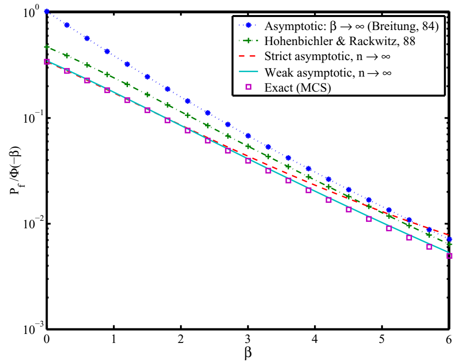


Figure 2: Normalized failure probability, Case 1: $n-1 = 35$, $\mathrm{Trace}\left(\mathbf{A}\right) = 1$.

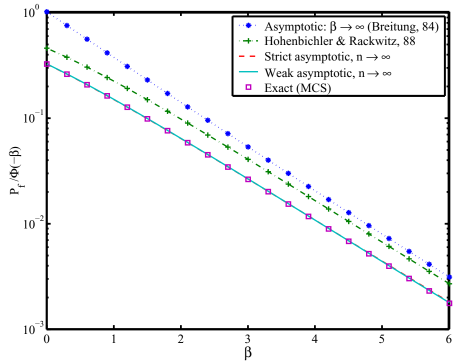
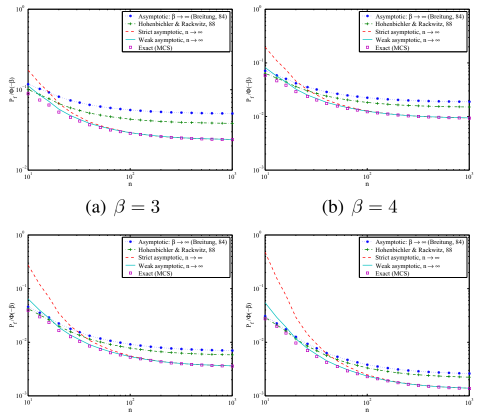


Figure 3: Normalized failure probability, Case 2: $n-1 = 200$, $\mathrm{Trace}\left(\mathbf{A}\right) = 1$.

Results obtained from the asymptotic analysis improve when the number of random variables becomes large. Figure 3 shows the probability of failure for case 2. As expected, with more random variables, results obtained from both asymptotic formulations match well with the Monte Carlo simulation result. For this case the maximum value of the curvature (a_j) is 0.0097 - which implies that the failure surface is almost linear. Even in such case it is interesting to note



(a) $\beta = 3$ (b) $\beta = 4$

(c) $\beta = 5$ (d) $\beta = 6$

Figure 4: Normalized failure probability for different values of β when $\text{Trace}\,(\mathbf{A}) = 1$.

the difference between the results obtained from existing approximations and the proposed methods.

From these selective numerical examples, the following points may be noted:

- For a fixed value of β and $\mathbf{A}$, the weak asymptotic formulation is more accurate for smaller values of n (say n_1) compared to the strict asymptotic formulation. The convergence of the proposed formulations for increasing values of n, when β is ranging from 3 to 6 is shown in figure 4. The strict asymptotic formulation becomes accurate when n is more than that given by equation (21) (say n_2). Although it was not possible to obtain an expression of n_1, it can be conceived conceptually (see figure 4). Overall, the applicability of the approximate analytical methods for structural reliability calculations as a function of number of random variables can be summarized in figure 5. When $n < n_2$ the existing FORM/SORM are applicable, but they may not be very accurate if $n > n_1$. When $n > n_1$, the weak asymptotic formulation can provide accurate result and when $n > n_2$ both of the proposed formulations yield similar results. We again recall that these conclusions are based on the validity of the parabolic failure surface approximation (2).
- For a fixed value of $\mathbf{A}$, from equation (21) it can be seen that the results from both approaches will be more accurate if β is small. This fact can also be observed in the numerical results shown in figures 2–4. However, proposed asymptotic approximation are based on the parabolic failure surface approximation (2) which is expected to be accurate when β is high. These two conflicting demands can only be met when n is significantly large.
- When n is large, computational cost to accurately obtain β and $\mathbf{A}$ can be prohibitive. In recent years there have been significant developments in numerical simulation methods specifically tailored to deal with reliability problems in high dimensions (see Au and Beck, 2003, Koutsourelakis et al., 2004). Proposed formulae nevertheless provides an alternative which can give physical insight and can be used in the early stages of reliability based optimal design. Moreover, the modified design point and the asymptotic density function can be used for importance sampling in high dimension. Further research however needed in this area.

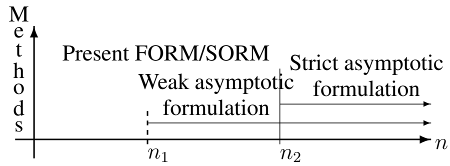


Figure 5: Approximate analytical methods for structural reliability calculations as a function of number of random variables n

7 CONCLUSIONS

The demands of modern engineering design have lead structural engineers to model a structure using random variables in order to handle uncertainties. Two approximations to calculate the probability of failure of an engineering structure when the number of random variables used for mathematical modeling $n \to \infty$ are provided. It is assumed that the basic random variables are Gaussian and the failure surface is approximated by a parabolic hypersurface in the neighborhood of the design point. The new approximations are based on the asymptotic distribution of a central quadratic form in Gaussian random variables. The main outcome of the asymptotic analysis is that the conventional reliability index β needs to be modified when $n \to \infty$. A simple geometric explanation is given for this fact. Two formulations, namely, *strict asymptotic formulation* and *weak asymptotic formulation* are presented. Both approximations results in simple closed-form expressions:

$$P_{f_{\text{Strict}}} \to \Phi\left(-\beta_1\right),$$

$$P_{f_{\text{Weak}}} \to \frac{\Phi\left(-\beta_2\right) e^{-\left(2\beta_2^2 \text{Trace}\left(\mathbf{A}^2\right) - \beta_2 \text{Trace}\left(\mathbf{A}\right)\right)}}{\sqrt{\|\mathbf{I}_{n-1} + 2\beta_2 \mathbf{A}\|}},$$

where β_1 and β_2 are given by equations (27) and(41). For small number of variables, the trace effects may not be significant and in such cases it is easy to see that these two formula reduce to the classical FORM and SORM formula respectively. A closed-form expression for the minimum number of random variables required to apply these asymptotic formulae is derived. Beyond this value of n, the reliability problem can be considered as high-dimensional since the trace effects become significant. The proposed approximations are compared with some existing approximations and Monte-Carlo simulations using numerical examples. The results obtained from both asymptotic formulations match well with the Monte Carlo simulation results in high dimensions. Numerical studies show that the weak formulation is in general applicable for low number of variables compared to the the strict formulation. In many real-life problems the number of random variables is expected to be large. In such situations the asymptotic results derived here will be useful.

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APPENDIX: THE CHOICE OF s IN THE MOMENT GENERATING FUNCTION OF U

To select the value of s we begin with the approximation of P_f in equation (28) as

$$P_f \approx \mathrm{E}\left[\Phi(-\beta - U)\right]. \tag{A.1}$$

Because $u \in \mathbb{R}^+$ as $\mathbf{A}$ is positive definite, the maxima of $\ln\left[\Phi(-\beta-u)\right]$ in $\mathbb{R}^+$ occurs at $u=0$. Therefore, the maximum contribution to the expectation of $\ln\left[\Phi(-\beta-U)\right]$ comes from the neighborhood of $u=0$. Expanding $\ln\left[\Phi(-\beta-u)\right]$ in a first-order Taylor series about $u=0$ we obtain

$$\Phi(-\beta-u) \approx e^{\ln[\Phi(-\beta)]-\frac{\varphi(\beta)}{\Phi(-\beta)}u} = \Phi(-\beta)e^{-\frac{\varphi(\beta)}{\Phi(-\beta)}u}. \tag{A.2}$$

The reason for keeping only one term in the Taylor series is to exploit the expression of the moment generating function in (8). Substituting $\Phi(-\beta-u)$ in equation (A.1) we have

$$P_f \approx \Phi(-\beta)\mathrm{E}\left[e^{-\frac{\varphi(\beta)}{\Phi(-\beta)}u}\right] = \Phi(-\beta)\, M_U\left(s = -\frac{\varphi(\beta)}{\Phi(-\beta)}\right). \tag{A.3}$$

The preceding equation indicates that in order to calculate P_f, the appropriate choice of s to be used in the moment generating function of U is given by

$$s = -\frac{\varphi(\beta)}{\Phi(-\beta)}. \tag{A.4}$$

If β is large, then using the asymptotic series (39) we have $s = -\frac{\varphi(\beta)}{\Phi(-\beta)} \approx -\beta$. This analysis explains the rationale behind choosing $s=-\beta$ is equation (17).