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HYBRID PERTURBATION-POLYNOMIAL CHAOS APPROXIMATE SOLUTION TO THE ALGEBRAIC RANDOM EIGENVALUE PROBLEM

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Abstract. *The analysis of structures is affected by uncertainty in the structure's material properties, geometric parameters, boundary conditions and applied loads. In the stochastic finite element method, the random quantities are represented by stochastic models. Amongst the various problems affected by uncertainty, the random eigenvalue problem is specially important when analyzing the dynamic behavior of a structure. The methods that stand out in research dealing with the random eigenvalue problem are the perturbation method and methods based on Monte Carlo Simulation. In the past few years, methods based on Polynomial Chaos (PC) have been developed for this problem, where each eigenvalue and eigenvector are represented by a PC expansion. In this paper two new methods hybridizing perturbation approach and Polynomial Chaos approach are proposed and compared. The methods use Rayleigh quotient, the power method and the inverse power method. Polynomial Chaos expansions of eigenvalues and eigenvectors are obtained with the proposed methods. The new methods are applied to the problems of a Bernouilli beam and a thin plate with stochastic properties.*

1 Introduction

The algebraic eigenvalue problem arises in a variety of fields, for example, in buckling of columns and shells [1], vibration of elastic bodies, etc. Accurate methods to calculate the eigensolution of a deterministic matrix have been available for long (see, for example, [2]), but such is not the case when the matrix considered is random. When random matrices are considered, the joint pdf of eigenvalues is only available for some special random matrix distributions, as the Gaussian Orthogonal Ensemble [3] and Wishart matrices [4]. Randomness can be introduced in the system by random parameters (e.g. Young's modulus, mass density) and consequently propagated to the system matrices such as the mass and stiffness matrices. Using the stochastic finite element method [5], these matrices in turn can be represented by a linear combination of deterministic matrices, where the coefficients are random variables [6, 7].

Several methods have been developed to solve the algebraic random eigenvalue problem. Methods dealing with large amounts of uncertainty are based on Monte Carlo Simulation (MCS), where different initialization strategies for the power method have been developed. These strategies are based on ordering the samples depending on the distance between them. This ordering can be based on algorithms from the traveling salesman problem and space reduction [8] or component mode synthesis [9], or can be done in a tree-type data structure [10]. The start-vector used is the result from the iteration process of the previous sample. The initialization strategies and size reduction methods reduce the computational time of MCS, but for smaller uncertainties, more efficient methods are available.

Methods that can be applied to small uncertainties are based on the perturbation method [11]. First applications date from the late sixties [12, 13], and a series of modified methods have been developed. A comparison of several of these methods is given by Chen *et al.* [14]. Other perturbation-based methods use iterations or linear combination of deterministic and perturbed eigenvectors to deal with larger uncertainties or to allow reanalysis of structures [15–18]. Other methods available are based on crossing theory [19], kronecker product [20], the dimensional decomposition method [21, 22], use of Laplace's method to obtain moments of eigenvalues [23], collocation methods [24], use of interpolations and meta-models [25, 26], the use of an auxiliary function where the derivative of the eigenvector equals the eigenvalue multiplied by the eigenvector [27] and possibilistic approaches [28].

More recently, Polynomial Chaos (PC) [5] has been applied to the random algebraic eigenvalue problem. A PC expansion of eigenvalues and eigenvectors was obtained by Ghosh *et al.* [29] using MCS. Verhoosel *et al.*[30] developed an iterative procedure based on the inverse power method and Rayleigh quotient to obtain PC expansions of the eigensolutions. Ghanem and Ghosh [31] substituted eigenvalues and eigenvectors by their PC expansion in the eigenvalue problem. Coefficients were obtained from the nonlinear problem with the help of a norm equation for the eigenvectors. A modification of the previous method using enrichment functions was derived by Ghosh and Ghanem [32].

It can be observed that even if research has been carried out both on the perturbation and PC methods for the random eigenvalue problem, no method hybridizing both approaches is yet available. Efficient methods hybridizing PC and other methods have been proposed for the elliptic problem [33, 34], where a reduction of the size of the linear system to be solved was achieved. The aim of the present paper is to gain efficiency on the PC algorithms for random eigenvalue problems through the use of results from the perturbation method. The outline of the paper is as follows. The basic theories of the perturbation method and Polynomial Chaos are discussed respectively in subsection 2.1 and subsection 2.2. PC expansion of eigenvalues

is obtained in subsection 3.1 using the Rayleigh quotient where eigenvectors are obtained from the perturbation method. Two new methods, namely reduced spectral power method (RSPM) and reduced spectral inverse power method (RSIPM), are proposed to update the eigenvectors in section 4. Both methods allow us to obtain an updated PC expansion of the eigenvectors and eigenvalues using Rayleigh quotient. A summary of the proposed methods is given in section 5. A comparison of the two methods is performed for the problems of a beam and a thin plate with stochastic properties in section 6.

2 Stochastic Finite Element method for the random eigenvalue problem

The deterministic eigenvalue problem is given by the equation

$$\mathbf{A}\mathbf{u}^{(j)} = \lambda^{(j)}\mathbf{u}^{(j)} \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the system matrix, $\mathbf{u}^{(j)}$ is the j-th eigenvector and $\lambda^{(j)}$ is the corresponding eigenvalue. The system matrix is obtained from the generalized eigenvalue problem $\mathbf{K}\mathbf{y}^{(j)} = \lambda^{(j)}\mathbf{M}\mathbf{y}^{(j)}$ so that $\mathbf{A} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$ and $\mathbf{u}^{(j)} = \mathbf{M}^{-1/2}\mathbf{y}^{(j)}$ [35]. In a dynamic problem, matrix $\mathbf{K}$ is the stiffness matrix and $\mathbf{M}$ is the mass matrix [36]. Randomness in the matrix $\mathbf{A}$ can be introduced by a parameter (e.g. Young's modulus) represented by a random field. The random field can be approximated with a finite set of random variables using a discretization procedure (see, e.g. [6, 37]). Here, the random field is discretized using the Karhunen-Loéve (KL) expansion [38, 39] and truncated after M terms. In this paper it is assumed that the stiffness matrix is random and the mass matrix is deterministic. As a result, the system matrix $\mathbf{A}$ can be approximated by the following expansion

$$\mathbf{A} = \mathbf{A}_0 + \sum_{i=1}^{M} \xi_i \mathbf{A}_i \tag{2}$$

Here $\mathbf{A}_0$ is the mean of the system matrix and $\mathbf{A}_i$ are the matrices appearing from the propagation of the Karhunen-Loéve expansion into the system matrix. In the next subsections, the perturbation method and the Polynomial Chaos method are used to approximate the eigenvalues and eigenvectors of the stochastic system matrix.

2.1 Perturbation method for the random eigenvalue problem

Among the various methods developed to solve the random eigenvalue problem, the perturbation method is widely used due to its efficiency. The different perturbation methods available to analyze the random eigenvalue problem are based on keeping different number of terms in the Taylor series expansions. These perturbation methods are mainly applied to eigenvectors, while the eigenvalues can then be obtained using the Rayleigh quotient. This approximation of eigenvalues is more accurate than the one obtained by directly applying the perturbation method via the Taylor series expansions [14]. If $\lambda_0^{(j)}$ and $\mathbf{u}_{j0}$ are the j-th deterministic eigenvalue and the corresponding eigenvector, a first-order expression for the perturbed eigenvector can be given by [13]

$$\mathbf{u}^{(j)} = \mathbf{u}_{j0} + \sum_{i=1}^{M} \xi_i \frac{\partial \mathbf{u}^{(j)}}{\partial \xi_i} \tag{3}$$

The deterministic eigenvectors satisfy the following properties

$$\mathbf{u}_{j0}^T\mathbf{u}_{j0} = 1 \qquad \text{and} \qquad \mathbf{u}_{j0}^T\frac{\partial \mathbf{u}^{(j)}}{\partial \xi_i} = 0 \tag{4}$$

Different methods have been developed to calculate the derivatives of the eigenvectors. One of these methods expands the derivative of eigenvectors as a linear combination of deterministic eigenvectors [12, 40], so that

$$\mathbf{u}_{ji} = \frac{\partial \mathbf{u}^{(j)}}{\partial \xi_i} = \sum_{m=1, m\neq j}^{N} \alpha_{jim} \mathbf{u}_{m0} \qquad \text{where} \qquad \alpha_{jim} = \frac{1}{\lambda_0^{(j)} - \lambda_0^{(m)}} \mathbf{u}_{m0}^T (\mathbf{A}_i) \mathbf{u}_{j0} \tag{5}$$

This equation is used when all deterministic eigenvectors are calculated. If only a limited number of eigenvectors were calculated, other methods described by Nelson [41] could be applied. The case of close or repeated eigenvalues is not dealt with here. The perturbation method for such cases is derived, for example, in [42–44].

2.2 Polynomial Chaos approach for the random eigenvalue problem

In our system, uncertainty is represented by a finite set of random variables $\{\xi_1, \ldots, \xi_M\}$ defined on the probability space $(\Xi, \mathcal{B}_\Xi, P_\xi)$. Any random quantity of interest of the system considered is then defined on this probability space, in particular, eigenvalues and eigenvectors. The eigensolution is assumed to have finite second-order moments, and can be represented in the space of square integrable functions $\mathcal{L}^2(\Xi, dP_\xi)$ [6, 45], and a basis of functions Γ_k in $\mathcal{L}^2(\Xi, dP_\xi)$ can be defined. The representations of $\lambda^{(j)}$ and $\mathbf{u}^{(j)}$ on the basis functions Γ_k truncated after P terms can be given by

$$\lambda^{(j)}(\xi_1, \ldots \xi_M) = \sum_{k=1}^{P} \lambda_{jk} \Gamma_k(\xi_1, \ldots \xi_M) \tag{6}$$

$$\mathbf{u}^{(j)}(\xi_1, \ldots \xi_M) = \sum_{k=1}^{P} \mathbf{u}_k^{(j)} \Gamma_k(\xi_1, \ldots \xi_M) \tag{7}$$

where λ_{jk} and $\mathbf{u}_k^{(j)}$ are unknowns and the basis functions considered here are Polynomial Chaos [5]. The basis functions Γ_k are obtained from the terms of the tensor product of several univariate Hermite polynomials up to a fixed total-order specification, this approach is referred to as "total-order expansion" [46]. The univariate Hermite polynomials can be calculated from the equation $H_n(\xi) = (-1)^n \exp(\xi^2/2) \partial^n \exp(-\xi^2/2)/\partial^n \xi$, where each polynomial depends on a Gaussian random variable ξ_i.

Polynomial Chaos has already been used in the context of the algebraic random eigenvalue problem by Ghanem and Ghosh [31]. In [31], eigenvalues and eigenvectors were substituted by their expansions, i.e., $\lambda^{(j)}$ and $\mathbf{u}^{(j)}$ are replaced in Equation (1) by their expansions from Equations (6) and (7). The resulting equation is projected on the basis in $\mathcal{L}^2(\Xi, \mathcal{B}_\Xi, P_\xi)$ through the Galerkin method, that is, the equation is multiplied by each basis function Γ_p and subsequently the mean is taken. The resulting system of equations of size $P \times n$ relating the unknowns λ_{jk} and $\mathbf{u}_k^{(j)}$ can be given by

$$\left(\mathbf{c}_0 \otimes \mathbf{A}_0 + \sum_{i=1}^{M} \mathbf{c}_{1i} \otimes \mathbf{A}_i \right) \begin{bmatrix} \mathbf{u}_1^{(j)} \\ \vdots \\ \mathbf{u}_P^{(j)} \end{bmatrix} = \sum_{l=1}^{P} \lambda_{jl} \mathbf{e}_{0l} \otimes \mathrm{I} \begin{bmatrix} \mathbf{u}_1^{(j)} \\ \vdots \\ \mathbf{u}_P^{(j)} \end{bmatrix} \tag{8}$$

Here the elements in the k-th row and p-th column of $P \times P$ matrices $\mathbf{c}_0$, $\mathbf{c}_{1i}$ and $\mathbf{e}_{0l}$ are respectively $\mathbf{c}_{0kp} = \mathrm{E}[\Gamma_k \Gamma_p]$, $\mathbf{c}_{1ikp} = \mathrm{E}[\xi_i \Gamma_k \Gamma_p]$, with $i = 1, \ldots M$ and $\mathbf{e}_{0lkp} = \mathrm{E}[\Gamma_k \Gamma_p \Gamma_l]$ with

$l = 1, \dots P$. If the norm of the eigenvectors is also prescribed, the nonlinear system of equations can be solved with iterative techniques. A good initial approximation is often needed to obtain fast convergence.

An iterative procedure to obtain PC expansions of eigenvalues and eigenvectors has been proposed by Verhoosel *et al.* [30] based on the inverse power method. The algorithm of the iteration can be concisely described by the following steps:

1. $\lambda^{(j)}_{(q+1)} = \left(\mathbf{u}^{(j)}_{(q)}\right)^T (\mathbf{A}_0 + \sum_{i=1}^M \mathbf{A}_i)\mathbf{u}^{(j)}_{(q)}$.
2. $\mathbf{u}^{(j)}_{(q+1)} = \left(\lambda^{(j)}_{(q+1)} - \lambda^{(j)}_0\right)\left[\mathbf{A}_0 + \sum_{i=1}^M \mathbf{A}_i - \lambda^{(j)}_0 \mathrm{I}\right]^{-1} \mathbf{u}^{(j)}_{(q)}$.
3. $\mathbf{u}^{(j)}_{(q+1)} \rightarrow \frac{\mathbf{u}^{(j)}_{(q+1)}}{||\mathbf{u}^{(j)}_{(q+1)}||_{L^2_S}}$ where $||\mathbf{u}^{(j)}_{(q+1)}||_{L^2_S} = \sqrt{\sum_{k=1}^P \mathrm{E}\left[\Gamma_k^2\right]\left(\mathbf{u}^{(j)}_{k(q+1)}\right)^T \mathbf{u}^{(j)}_{k(q+1)}}$.
4. Define errors $\epsilon_{1_{q+1}} = ||[\mathbf{A}_0 + \sum_{i=1}^M \mathbf{A}_i - \lambda^{(j)}_{(q+1)}\mathrm{I}]||_{L^2_S}$ and $\epsilon_{2_{(q+1)}} = \frac{|V_{\lambda^{(j)}_{(q+1)}} - V_{\lambda^{(j)}_{(q)}}|}{|V_{\lambda^{(j)}_{(q)}}|}$ with $V_{\lambda^{(j)}_{(q)}} = \frac{\sqrt{\sum_{k=1}^P \lambda^2_{jk(q)}\mathrm{E}\left[\Gamma_k^2\right]}}{\lambda_{j1(q)}}$, the coefficient of variation of the eigenvalue.

For all the steps of the iterative procedure where it is needed, the coefficients of the PC expansions are obtained using the Galerkin method. The subscripts (q) and $(q+1)$ denote the number of the iteration. That is, $\lambda^{(j)}_{(q)}$ is the PC expansion of the j-th eigenvalue at iteration q, while $\lambda_{jk(q)}$ denotes the k-th coefficient of the expansion, so that $\lambda^{(j)}_{(q)} = \sum_{k=1}^P \lambda_{jk(q)}\Gamma_k$. Similarly, $\mathbf{u}^{(j)}_{(q)}$ is the PC expansion of the j-th eigenvector at iteration q, so that $\mathbf{u}^{(j)}_{(q)} = \sum_{k=1}^P \mathbf{u}^{(j)}_{k(q)}\Gamma_k$.

The two methods discussed here can be used to obtain the eigenvalues and eigenvectors coefficients λ_{jk} and $\mathbf{u}^{(j)}_k$ of the Polynomial Chaos expansions given by Equations (6) and (7). From these PC expansions, the pdf of eigenvalues and eigenvectors can be obtained if a Monte Carlo simulation is performed. This is done by sampling the set of independent Gaussian random variables $\xi_1, \dots, \xi_M$. The corresponding values of the basis functions Γ_k are calculated and subsequently introduced in the Polynomial Chaos expansions of eigenvalues and eigenvectors given by Equations (6) and (7). A numerical approximation to the pdf of eigenvalues from the samples of eigenvalues can then be obtained. Moments of eigenvalues and eigenvectors can also be derived from Equations (6) and (7). The first and second moments of eigenvalues are given by

$$\mathrm{E}\left[\lambda^{(j)}\right] = \sum_{k=1}^P \lambda_{jk}\mathrm{E}\left[\Gamma_k\right] = \lambda_{j1} \tag{9}$$

$$\mathrm{E}\left[\left(\lambda^{(j)}\right)^2\right] = \sum_{k,l=1}^P \lambda_{jk}\lambda_{jl}\mathrm{E}\left[\Gamma_k\Gamma_l\right] = \sum_{k=1}^P \lambda^2_{jk}\mathrm{E}\left[\Gamma_k^2\right] \tag{10}$$

Similarly, first and second moments of elements of eigenvectors could be calculated. Expressions are not derived here as our interest lays in the moments of eigenvalues.

3 Rayleigh quotient method for the Polynomial Chaos expansion of eigenvalues

The Rayleigh quotient can be used to obtain an approximation to the eigenvalue $\lambda^{(j)}$ if an approximation to eigenvector $\mathbf{u}^{(j)}$ is available

$$\lambda^{(j)} = \frac{\left(\mathbf{u}^{(j)}\right)^T \mathbf{A}\mathbf{u}^{(j)}}{\left(\mathbf{u}^{(j)}\right)^T \mathbf{u}^{(j)}} \tag{11}$$

This method to obtain eigenvalues is similar to step 1 of the algorithm by Verhoosel *et al.* [30], but here, the eigenvectors do not need to be normalized. The eigenvalues of the random algebraic eigenvalue problem can be expanded with a PC expansion such that the j-th eigenvalue is given by Equation (6). Substituting this expansion in the expression of the Rayleigh quotient in Equation (11) can lead to

$$\left(\sum_{k=0}^{P} \lambda_{jk}\Gamma_k\right)\left(\mathbf{u}^{(j)}\right)^T \mathbf{u}^{(j)} = \left(\mathbf{u}^{(j)}\right)^T \left(\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i \mathbf{A}_i\right)\mathbf{u}^{(j)} \tag{12}$$

In the following subsections, the eigenvectors $\mathbf{u}^{(j)}$ will be substituted by different approximations, allowing to obtain λ_{jk} coefficients from a linear system of equations.

3.1 Perturbation of the eigenvectors

Using the first-order perturbation of the j-th eigenvector, the corresponding eigenvalue can be approximated using Rayleigh quotient by substituting $\mathbf{u}^{(j)}$ from Equations (3) and (5) into Equation (12). The Galerkin method is applied to the resulting equation (i.e. the equation is multiplied by the p-th Polynomial Chaos basis function Γ_p and mean of the equation is taken) and the resulting equation is simplified using properties given by Equation (4). Then, coefficients λ_{jk} of the PC expansion of $\lambda^{(j)}$ can be obtained from the equations

$$\begin{aligned}\sum_{k=1}^{P} \lambda_{jk}\left(\mathbf{c}_{0kp} + \sum_{i,g=1}^{M} \mathbf{u}_{ji}^T\mathbf{u}_{jg}\mathbf{c}_{2igkp}\right) = \lambda_0^{(j)}\mathbf{d}_{0p} + \sum_{i=1}^{M} \mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{j0}\mathbf{d}_{1ip} + \sum_{i,g=1}^{M}\big(2\mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{jg} + \\ \mathbf{u}_{jg}^T\mathbf{A}_0\mathbf{u}_{ji}\big)\,\mathbf{d}_{2igp} + \sum_{i,g,h=1}^{M} \mathbf{u}_{jg}^T\mathbf{A}_i\mathbf{u}_{jh}\mathbf{d}_{3ighp}\end{aligned} \tag{13}$$

where $p = 1, \ldots, P$. Matrix $\mathbf{c}_0$ has already been defined as a diagonal matrix with element in the k-th row and p-th column given by $\mathbf{c}_{0kp} = \mathrm{E}\left[\Gamma_k\Gamma_p\right]$. Similarly, the $P \times P$ matrix $\mathbf{c}_{2ig}$ has elements $\mathbf{c}_{2igkp} = \mathrm{E}\left[\Gamma_k\Gamma_p\xi_i\xi_g\right]$ and column vectors of length P $\mathbf{d}_0$, $\mathbf{d}_{1i}$, $\mathbf{d}_{2ig}$ and $\mathbf{d}_{3igh}$ have their p-th term given by $\mathbf{d}_{0p} = \mathrm{E}\left[\Gamma_p\right]$, $\mathbf{d}_{1ip} = \mathrm{E}\left[\Gamma_p\xi_i\right]$, $\mathbf{d}_{2igp} = \mathrm{E}\left[\Gamma_p\xi_i\xi_g\right]$ and $\mathbf{d}_{3ighp} = \mathrm{E}\left[\Gamma_p\xi_i\xi_g\xi_h\right]$ respectively. This notation for matrices $\mathbf{c}$ and $\mathbf{d}$ is such that for each matrix the mean of the product of, respectively, one and two polynomials Γ_k is calculated. The numerical subindex (e.g. 1, 2) indicate the number of random variables multiplying those polynomials inside the mean operator. Equation (13) can be represented by the linear system of equations

$$\begin{aligned}\left(\mathbf{c}_0 + \sum_{i,g=1}^{M} \mathbf{u}_{ji}^T\mathbf{u}_{jg}c_{2ig}\right)\begin{bmatrix}\lambda_{j1}\\ \vdots \\ \lambda_{jP}\end{bmatrix} = \lambda_0^{(j)}\mathbf{d}_0 + \sum_{i=1}^{M} \mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{j0}\mathbf{d}_{1i} + \sum_{i,g=1}^{M}\big(2\mathbf{u}_{j0}^T\mathbf{A}_i\mathbf{u}_{jg} + \\ \mathbf{u}_{jg}^T\mathbf{A}_0\mathbf{u}_{ji}\big)\,\mathbf{d}_{2ig} + \sum_{i,g,h=1}^{M} \mathbf{u}_{jg}^T\mathbf{A}_i\mathbf{u}_{jh}\mathbf{d}_{3igh}\end{aligned} \tag{14}$$

and the coefficients λ_{jk} of the eigenvalue expansion can be obtained by solving the $P \times P$ linear system given by Equation (14).

3.2 Reduced Polynomial Chaos eigenvectors

The PC expansion of an eigenvector is given by Equation (7). A possible approach to reduce the size of the system is to assume the vectors $\mathbf{u}_k^{(j)}$ can be expressed as a linear combination of the vectors $\mathbf{u}_{j0}$ and $\mathbf{u}_{ji}$, such that

$$\mathbf{u}_k^{(j)} = a_{0k}^{(j)} \mathbf{u}_{j0} + \sum_{i=1}^{M} a_{ik}^{(j)} \mathbf{u}_{ji} \tag{15}$$

The idea of expanding the random eigenvectors into a basis formed by the deterministic eigenvector and its first derivative has already been used by Nair and Keane [18], and an accurate approximation to eigenvectors was obtained. This Equation (15) is introduced in Equation (7), so that the eigenvector can be expanded as

$$\mathbf{u}^{(j)} = \left(\sum_{k=1}^{P} a_{0k}^{(j)} \Gamma_k \right) \mathbf{u}_{j0} + \sum_{i=1}^{M} \left(\sum_{k=1}^{P} a_{ik}^{(j)} \Gamma_k \right) \mathbf{u}_{ji} \tag{16}$$

This expansion will be used in the next section, where coefficients $a_{0k}^{(j)}$, $a_{ik}^{(j)}$ will be obtained through the reduced spectral power method and the reduced spectral inverse power method. After obtaining the set of coefficients $a_{0k}^{(j)}$, $a_{ik}^{(j)}$, the PC expansion of the j-th eigenvalue is obtained using the Rayleigh quotient. That is, the new eigenvectors from Equation (16) are now introduced in Equation (12) and as formerly, the equation is multiplied by Γ_p and mean is taken to obtain

$$\begin{aligned}
\sum_{k=1}^{P} \lambda_{jk} \sum_{l,m=1}^{P} \mathbf{g}_{lmpk} \left(a_{0l}^{(j)} a_{0m}^{(j)} + \sum_{i,g=1}^{M} a_{ik}^{(j)} a_{gl}^{(j)} \mathbf{u}_{ji}^T \mathbf{u}_{jg} \right) = \\
\sum_{l,m=1}^{P} a_{0l}^{(j)} a_{0m}^{(j)} \left(\lambda_{0j} \mathbf{e}_{0lmp} + \sum_{i=1}^{M} \mathbf{u}_{j0}^T \mathbf{A}_i \mathbf{u}_{j0} \mathbf{e}_{1ilmp} \right) + \\
\sum_{l,m=1}^{P} a_{0l}^{(j)} \sum_{i,g=1}^{M} a_{im}^{(j)} 2 \mathbf{u}_{j0}^T \mathbf{A}_g \mathbf{u}_{ji} \mathbf{e}_{1glmp} + \\
\sum_{l,m=1}^{P} \sum_{i,g}^{M} a_{ik}^{(j)} a_{gl}^{(j)} \left(\mathbf{u}_{ji}^T \mathbf{A}_0 \mathbf{u}_{jg} \mathbf{e}_{0lmp} + \sum_{h=1}^{M} \mathbf{u}_{ji}^T \mathbf{A}_h \mathbf{u}_{jg} \mathbf{e}_{1hlmp} \right)
\end{aligned} \tag{17}$$

Here $p = 1, \ldots, P$, $\mathbf{e}_{0lmp} = \mathrm{E}\left[\Gamma_l \Gamma_m \Gamma_p\right]$, $\mathbf{e}_{1glmp} = \mathrm{E}\left[\Gamma_m \Gamma_l \Gamma_p \xi_g\right]$, $\mathbf{g}_{klmp} = \mathrm{E}\left[\Gamma_k \Gamma_l \Gamma_m \Gamma_p\right]$. As formerly, the set of coefficients λ_{jk} can be found solving a linear equation similar to the one obtained in Equation (14). The set of coefficients $a_{0k}^{(j)}$, $a_{ik}^{(j)}$ have yet not been defined. Their role is to improve the approximation of the eigenvectors. They are calculated in the next section, following two approaches.

4 Updating of the eigenvectors

When considering a deterministic eigenvalue problem, several iterative methods are available to approximate each eigenvector with a desired accuracy [2]. The power method is based on the

fact that a symmetric matrix $\mathbf{M}$ with eigenvalues χ_i and eigenvectors $\mathbf{x}_i$ multiplied s times by itself can be expressed in terms of the s-th power of its eigenvalues through $\mathbf{M}^s = \sum_{i=1}^{n} \chi_i^s \mathbf{x}_i \mathbf{x}_i^T$. If the eigenvalues have distinct values, the expression is dominated by $\sum_{i=1}^{r} \chi_i^s \mathbf{x}_i \mathbf{x}_i^T$ where χ_i are the r larger or dominant eigenvalues. Therefore, the power method is an iterative method that allows to obtain the eigenvector corresponding to the largest eigenvalue though $\mathbf{M}^s\mathbf{v}$, where $\mathbf{v}$ is the start-vector of the iteration algorithm, and s is the step of the iteration. A shift of origin (i.e. $(\mathbf{M} - p\mathrm{I})$, where $p \in \mathbb{R}$) allow convergence of $(\mathbf{M} - p\mathrm{I})^s\mathbf{v}$ to different eigenvectors. When the inverse of $\mathbf{M}$ is considered, it is observed that $\mathbf{M}^{-s}$ is dominated by $\sum_{i=1}^{r} (1/\chi_i^s)\, \mathbf{x}_i \mathbf{x}_i^T$ where χ_i are the r smallest eigenvalues. If the power method is applied to the matrix $(\mathbf{M} - p\mathrm{I})^{-1}$, the inverse power method is obtained, and the product $(\mathbf{M} - p\mathrm{I})^{-s}\mathbf{v}$ converges to the eigenvector corresponding to the eigenvalue closest to p [2]. In a deterministic system, these two methods allow to update a given approximation to an eigenvector. In the following subsections, the power and inverse power methods are extended to the stochastic case for updating the PC expansions of the eigenvectors.

4.1 Reduced spectral power method

The power method has been used in the context of spectral stochastic finite element method (SSFEM) by Lee and Singh [20] to obtain the mean and covariance of the eigenvectors. Only one iteration was performed, so that the random eigenvectors were obtained by multiplying the deterministic eigenvectors by the system stochastic matrix from Equation (2). Here, the equation used to derive the reduced spectral power method (RSPM) is based on the deterministic power method equation

$$\lambda^{(j)} \mathbf{u}_{(q+1)}^{(j)} = \mathbf{A}\mathbf{u}_{(q)}^{(j)} \tag{18}$$

where the subscripts (q) and $(q+1)$ indicate the number of the iteration. Substituting the KL expansion of eigenvalues from Equation (6) into Equation (18) leads to

$$\left(\sum_{k=1}^{P} \lambda_{jk} \Gamma_k \right) \mathbf{u}_{(q+1)}^{(j)} = \mathbf{A}\mathbf{u}_{(q)}^{(j)} \qquad \text{where} \quad \mathbf{u}_{(0)}^{(j)} = \mathbf{u}_{j0} + \sum_{i=1}^{M} \xi_i \mathbf{u}_{ji} \tag{19}$$

The q-th iteration $\mathbf{u}_{(q+1)}^{(j)}$ to approximate the j-th eigenvector can be expressed as the eigenvector from Equation (16). In the iterative case, eigenvector and constants $\mathbf{u}^{(j)}$, $a_{0k}^{(j)}$ and $a_{ik}^{(j)}$ are obtained from an iteration and, therefore, are expressed respectively as $\mathbf{u}_{(q)}^{(j)}$, $a_{0k,(q)}^{(j)}$ and $a_{ik,(q)}^{(j)}$. An expression for $\mathbf{u}_{(1)}^{(j)}$ is calculated, which corresponds to one iteration. The iterative algorithm could be derived easily, but an accurate solution is expected from one iteration as the perturbed eigenvectors are close to the exact solution. Furthermore, accuracy might be compromised if a large number of iterations are performed. This is so because the deterministic power method converges to the eigenvector corresponding to the largest eigenvalue when a considerable number of iterations are performed. A rectangular matrix $\mathbf{U}$ is formed with the columns of the deterministic eigenvector and its derivatives with respect to each random variable

$$\mathbf{U}_j = \begin{bmatrix} \mathbf{u}_{j0} & \mathbf{u}_{j1} & \dots & \mathbf{u}_{jM} \end{bmatrix} \in \mathbb{R}^{n \times (M+1)} \tag{20}$$

Equation (18) is multiplied by each basis function Γ_p and mean of the equation is taken. Then, the reduced spectral power method is represented by the linear system

$$\left(\sum_{l=1}^{P} (\lambda_l^{(j)} \mathbf{U}_j^T \mathbf{U}_j \otimes \mathbf{e}_{0l}) \right) \mathbf{a} = \mathbf{f} \tag{21}$$

with $\mathbf{a} = [a^{(j)}_{01(1)} \ \dots \ a^{(j)}_{0M(1)} \ a^{(j)}_{11(1)} \ \dots \ a^{(j)}_{ip(1)} \ \dots \ a^{(j)}_{MP(1)}]^T$ the vector of unknown coefficients, the $P \times P$ matrices $\mathbf{e}_{0l}$ are such that their k-th row and p-th element is $\mathbf{e}_{0lkp}$ and

$$\mathbf{f} = \begin{bmatrix} \lambda^{(j)}_0 \mathbf{d}_0 + \sum_{i=1}^{M} \mathbf{u}^T_{j0} \mathbf{A}_i \mathbf{u}_{j0} \mathbf{d}_{1i} + \sum_{i,k=1}^{M} \mathbf{u}^T_{j0} \mathbf{A}_i \mathbf{u}_{jk} \mathbf{d}_{2ik} \\ \sum_{i=1}^{M} \left(\mathbf{u}^T_{j1} \mathbf{A}_0 \mathbf{u}_{ji} + \mathbf{u}^T_{j1} \mathbf{A}_i \mathbf{u}_{0j} \right) \mathbf{d}_{1i} + \sum_{i,k=1}^{M} \mathbf{u}^T_{j1} \mathbf{A}_i \mathbf{u}_{jk} \mathbf{d}_{2ik} \\ \vdots \\ \sum_{i=1}^{M} \left(\mathbf{u}^T_{jM} \mathbf{A}_0 \mathbf{u}_{ji} + \mathbf{u}^T_{jM} \mathbf{A}_i \mathbf{u}_{0j} \right) \mathbf{d}_{1i} + \sum_{i,k=1}^{M} \mathbf{u}^T_{jM} \mathbf{A}_i \mathbf{u}_{jk} \mathbf{d}_{2ik} \end{bmatrix} \tag{22}$$

The deterministic power method can update the eigenvectors if the initial approximation is close to the solution, but as the number of iterations increases, the method converges to the eigenvector corresponding to the largest eigenvalue. The reduced spectral power method is likely to suffer from the same drawback. The reduced spectral inverse power method is derived in the next subsection to try to overcome this difficulty.

4.2 Reduced spectral inverse power method

The reduced spectral inverse power method (RSIPM) developed here is based on the deterministic inverse power method equation

$$(\mathbf{A} - \lambda^{(j)}_0 \mathbf{I}) \mathbf{u}^{(j)}_{(q+1)} = \left(\lambda^{(j)} - \lambda^{(j)}_0 \mathbf{I} \right) \mathbf{u}^{(j)}_{(q)} \tag{23}$$

where the subscripts (q) and $(q+1)$ indicate the number of the iteration. Substitution of the KL expansion of eigenvalues from Equation (6) into Equation (23) leads to

$$(\mathbf{A} - \lambda^{(j)}_0 \mathbf{I}) \mathbf{u}^{(j)}_{(q+1)} = \left(\sum_{k=1}^{P} \lambda_{jk} \Gamma_k - \lambda^{(j)}_0 \mathbf{I} \right) \mathbf{u}^{(j)}_{(q)} \qquad \text{where} \qquad \mathbf{u}^{(j)}_{(0)} = \mathbf{u}_{j0} + \sum_{i=1}^{M} \xi_i \mathbf{u}_{ji} \tag{24}$$

As in the previous subsection, $\mathbf{u}^{(j)}_{(q+1)}$ can be expressed as the eigenvector from Equation (16) where subscripts $(q+1)$ are added to $\mathbf{u}^{(j)}$, $a^{(j)}_{0k}$ and $a^{(j)}_{ik}$. For the first iteration, the approximation to the j-th eigenvalue $\sum_{k=1}^{P} \lambda_{jk} \Gamma_k$ can be obtained from Equation (14), that is, after applying the Rayleigh quotient using perturbed eigenvectors. Substituting the eigenvalue Polynomial Chaos expansion resulting from Equation (14) into Equation (24) and using the perturbation eigenvectors as the first approximation leads to

$$\begin{aligned} &\left(\mathbf{A}_0 + \sum_{i=0}^{M} \xi_i \mathbf{A}_i - \lambda^{(j)}_0 \mathbf{I} \right) \left(\left(\sum_{k=1}^{P} a_{0k} \Gamma_k \right) \mathbf{u}_{j0} + \sum_{i=1}^{M} \left(\sum_{k=1}^{P} a_{ik} \Gamma_k \right) \mathbf{u}_{ji} \right) = \\ &\left(\sum_{k=1}^{P} \lambda^{(j)}_k - \lambda^{(j)}_0 \mathbf{I} \right) \left(\mathbf{u}_{j0} + \sum_{i=1}^{M} \xi_i \mathbf{u}_{ji} \right) \end{aligned} \tag{25}$$

The unknown coefficients a_{0k} and a_{ik} can be found by multiplying Equation (25) by $\mathbf{u}^T_{j0} \Gamma_p$ and $(\mathbf{u}_{ji})^T \Gamma_p$ for $p = 1, \dots, P$ and then taking the mean of the resulting equation. Then, a linear system of equation of size $(M+1) \times P$ can be obtained with the matrix

$$\mathbf{S} = \left(\sum_{i=1}^{M} (\mathbf{U}^T_j \mathbf{A}_i \mathbf{U}_j) \otimes \mathbf{c}_{1i} + (\mathbf{U}^T_j (\mathbf{A}_0 - \lambda^{(j)}_0 \mathbf{I}) \mathbf{U}_j) \otimes \mathbf{c}_0 \right) \tag{26}$$

such that

$$\mathbf{S}\begin{bmatrix} a_{01} \\ \vdots \\ a_{0M} \\ a_{11} \\ \vdots \\ a_{ip} \\ \vdots \\ a_{MP} \end{bmatrix} = \begin{bmatrix} \lambda_{j1}\mathrm{E}\left[\Gamma_1^2\right] - \lambda_0^{(j)}\mathrm{E}\left[\Gamma_1\right] \\ \vdots \\ \lambda_{jP}\mathrm{E}\left[\Gamma_P^2\right] - \lambda_0^{(j)}\mathrm{E}\left[\Gamma_P\right] \\ \sum_{g=1}^{M}(\partial\mathbf{u}_j/\partial\xi_1)^T(\partial\mathbf{u}_j/\partial\xi_g)(\sum_{k=1}^{P}\lambda_{jk}\mathbf{c}_{1gk1} - \lambda_0^{(j)}\mathbf{d}_{1g1}) \\ \vdots \\ \sum_{g=1}^{M}(\partial\mathbf{u}_j/\partial\xi_i)^T(\partial\mathbf{u}_j/\partial\xi_g)(\sum_{k=1}^{P}\lambda_{jk}\mathbf{c}_{1gkp} - \lambda_0^{(j)}\mathbf{d}_{1gp}) \\ \vdots \\ \sum_{g=1}^{M}(\partial\mathbf{u}_j/\partial\xi_M)^T(\partial\mathbf{u}_j/\partial\xi_g)(\sum_{k=1}^{P}\lambda_{jk}\mathbf{c}_{1gkP} - \lambda_0^{(j)}\mathbf{d}_{1gP}) \end{bmatrix} \tag{27}$$

In the next section, the proposed methods are summarized.

5 Summary of the proposed method

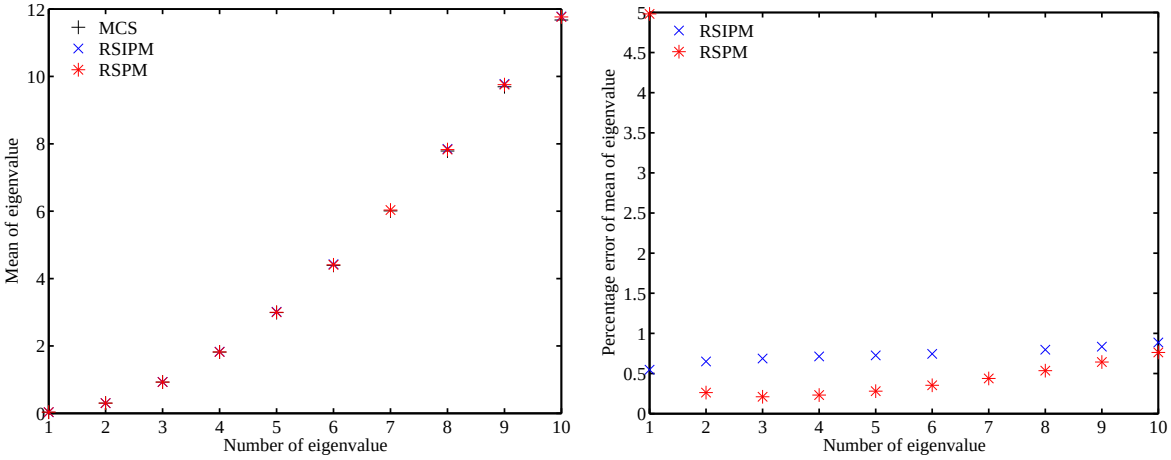
Two hybrid perturbation-Polynomial Chaos approximations are proposed for the solution of the algebraic random eigenvalue problem arising in structural dynamics. The random eigenvalues are firstly approximated with a Polynomial Chaos (PC) expansion using the perturbed eigenvectors in the Rayleigh quotient, $(\sum_{k=0}^{P}\lambda_{jk}\Gamma_k)\mathbf{u}^T\mathbf{u} = \mathbf{u}^T\mathbf{A}\mathbf{u}$. These results can be further improved if the eigenvectors are updated using the reduced spectral power method or the reduced spectral inverse power method (RSPM and RSIPM) respectively. These methods allow us to obtain PC expansions of the eigenvectors. These expansions are then used in the Rayleigh quotient to obtain an improved PC expansion of each eigenvalue. The proposed methods can be implemented by the following steps:

1. Calculate the system KL expansion matrices $\mathbf{A}_i\ \forall i = 1, \ldots, M$ and $\mathbf{A}_0$.
2. Obtain the deterministic eigenvalues and eigenvectors $\lambda_0^{(j)}$, $\mathbf{u}_{0j}$.
3. Use Equation (5) to obtain the perturbed eigenvectors $\mathbf{u}_j = \mathbf{u}_{0j} + \sum_{i=1}^{M}\xi_i\partial\mathbf{u}_j/\partial\xi_i$.
4. Calculate the PC expansion of eigenvalues using the Rayleigh quotient from Equation (14), where eigenvectors are given by the perturbed eigenvectors and the stiffness matrix by its truncated Karhunen-Loève expansion.
5. Calculate a new approximation to eigenvectors using one of the two proposed methods
   - The coefficients of the Polynomial Chaos expansions involved in each eigenvector approximation are obtained with the reduced spectral power method (RSPM) from Equation (21).
   - The coefficients of the Polynomial Chaos expansions involved in each eigenvector approximation are obtained with the reduced spectral inverse power method (RSIPM) from Equation (27).
6. Calculate the PC expansion of eigenvalues using Rayleigh quotient as in step 4.
7. Calculate the first and second moments of the eigenvalues using Equation (9) and (10).

6 Numerical examples

6.1 Euler-Bernouilli beam with stochastic properties

A clamped-free beam of length $L = 1$ m is considered. The system is discretized with the finite element method, using $n = 20$ elements. Details of the method can be found, for example, in the book by Dawe [47]. Uncertainty is introduced in the system by a Gaussian random field representing the bending rigidity, that is, $w(\theta, x) = EI_z$ with mean $\mathrm{E}[w(\theta, x)] = 1$. The discretization of $w(\theta, x)$ is done by the KL expansion of the exponential autocorrelation function $C(x_1, x_2) = e^{-5|x_1 - x_2|/L}$, that is $w(x, y, \theta) = \sum_{i=1}^{\infty} \lambda_i f_i \xi_i$ where the eigenvalues and eigenfunctions of the correlation function λ, $f(x_1)$ are obtained after solving the equation $\int_{-L/2}^{L/2} C(x_1, x_2) f(x_2) dx_2 = \lambda f(x_1)$ and ξ_i is a set of Gaussian independent random variable. The KL expansion is truncated at $M = 2$, so that the corresponding KL expansion of the stiffness matrix is $\mathbf{A} = \mathbf{A}_0 + \sum_{i=1}^{2} \mathbf{A}_i \xi_i$. The standard deviation of the random field is equal to 20% of the mean value and is included in the $\mathbf{A}_i$ matrices. More details on the KL expansion of the autocorrelation function used can be found in [5]. The maximum order of the Hermite polynomial used is 4, so that $P = 15$ polynomials are used as basis functions. Results obtained with the proposed methods are compared against results from MCS with 5000 samples. Figures 1 and 2 show the mean, standard deviation obtained with the proposed methods and the error of the corresponding quantities with respect to MCS results of the first 10 eigenvalues. It is



(a) Mean of the eigenvalues.

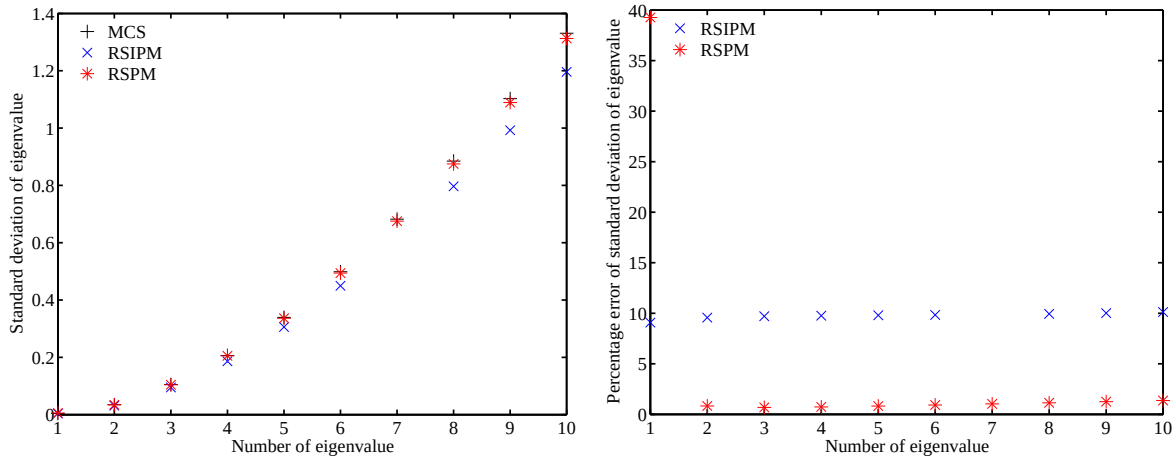
(b) Percentage error in the mean.

Figure 1: Mean and corresponding percentage error of the first ten eigenvalues of the beam obtained with Monte Carlo Simulation (MCS) using 5000 samples, the proposed reduced spectral power method (RSPM) and reduced spectral inverse power method (RSIPM). The standard deviation of the discretized random field is 20% of the mean value.

observed that the spectral power method results are more accurate than the ones obtained with the spectral inverse power method.

6.2 Thin plate with stochastic properties

The case of a thin plate clamped on one side is considered. The equation of motion is discretized using the FE method, as in the book by Dawe [47]. The plate has a length $L = 1.0$ m divided into 10 elements, and a width $W = 0.6$ m divided into 6 elements. All elements are rectangular. Uncertainty is introduced by the flexural rigidity of the plate $D = Eh^3/(12(1 - \nu^2))$, modelled as a Gaussian random field $w(x, y, \theta)$ with autocorrelation function $C(x_1, x_2, y_1 y_2) =$



(a) Standard deviation of the eigenvalues.

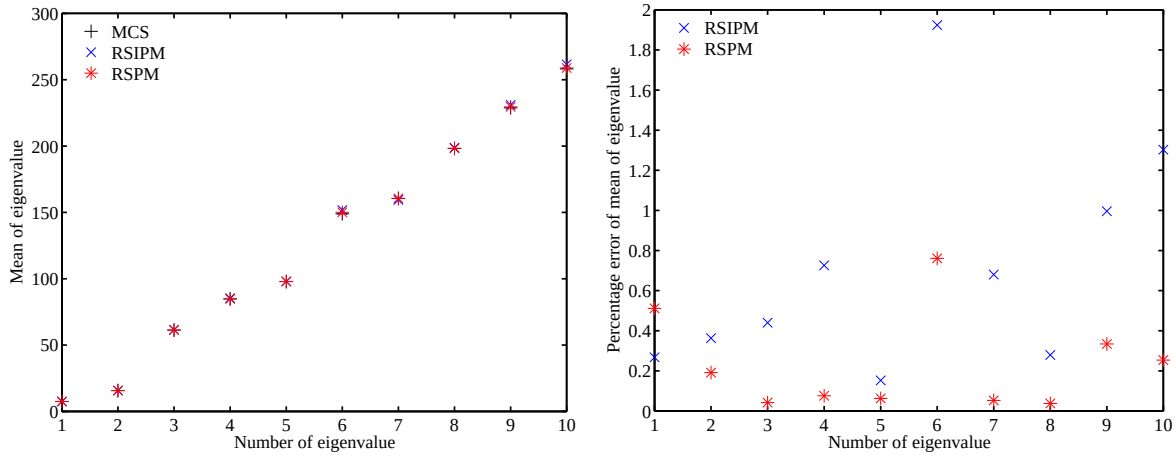
(b) Percentage error in the standard deviation.

Figure 2: Standard deviation and corresponding percentage error of the first ten eigenvalues of the beam obtained with Monte Carlo Simulation (MCS) using 5000 samples, the proposed reduced spectral power method (RSPM) and reduced spectral inverse power method (RSIPM). The standard deviation of the discretized random field is 20% of the mean value.

$e^{-|x_1-x_2|/b_x-|y_1-y_2|/b_y}$. The mean of the random field is obtained from values $E = 200$ GPa, $h = 3$ mm and $\nu = 0.3$. The correlation length in each direction is assumed to be $1/5$-th of the length of that direction, that is, $b_x = 0.2$, $b_y = 0.12$. The Karhunen-Loève expansion of the random field is given by $w(x, y, \theta) = \sum_{i=1}^{\infty} \lambda_i f_i \xi_i$ with ξ_i a set of Gaussian independent random variables. The eigenvalues and eigenfunctions of the correlation function $\lambda,\ f(x_1, y_1)$ are obtained after solving the equation $\int_{-L/2}^{L/2} \int_{-W/2}^{W/2} C(x_1, x_2, y_1 y_2) f(x_2, y_2) dy_2 dx_2 = \lambda f(x_1, y_1)$. The number of terms kept in the KL expansion in each direction (x and y) is two, so that The corresponding KL expansion of the stiffness matrix is $\mathbf{A} = \mathbf{A}_0 + \sum_{i=1}^{4} \mathbf{A}_i \xi_i$, where the standard deviation of the random field (here, $\sigma = 0.2\mathrm{E}[D]$) is included in the $\mathbf{A}_i$ matrices. As formerly, the maximum order of the Hermite polynomial used is 4, and now $P = 70$ polynomials are used as basis functions. Results obtained with the proposed method are compared against results from MCS using 5000 samples. Figures 3 and 4 show the mean, standard deviation and error with respect to MCS results of the first 10 eigenvalues. It is observed that the spectral power method results are more accurate than the ones obtained with the spectral inverse power method.

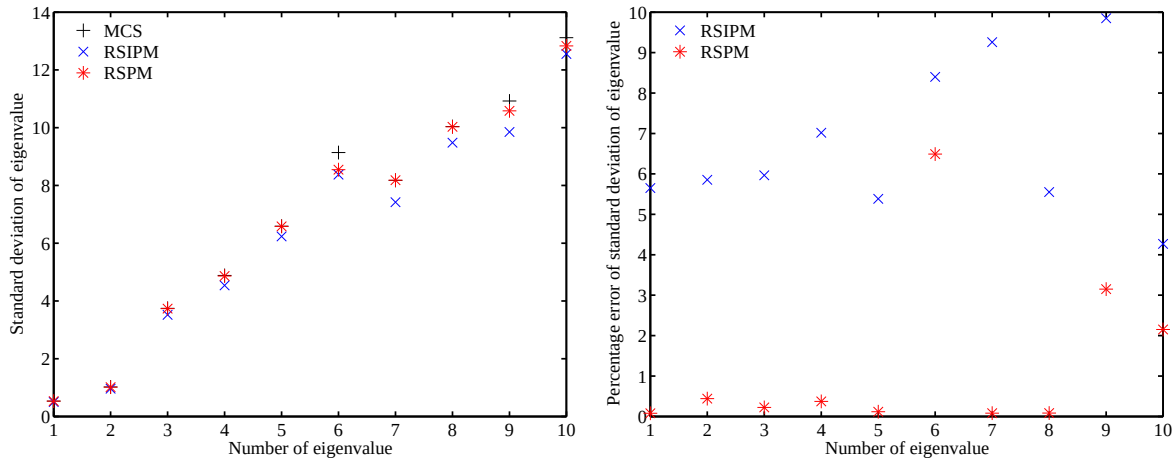
7 Conclusions

Two new methods, namely the reduced spectral power method and the reduced spectral inverse power method, have been proposed to improve the accuracy of the approximate solution of the random eigenvalue problem for symmetric matrices. Each method is based on approaches used in the context of the deterministic eigenvalue problem to obtain eigenvectors, namely, the power method and the inverse power method. Eigenvalues are obtained using the Rayleigh quotient. The deterministic methods are adapted to the stochastic case by projecting the power method, the inverse power method and the Rayleigh quotient in the basis functions Γ_p of the Hilbert space $\mathcal{L}^2(\Xi, dP_\xi)$. These basis functions are considered to be the multivariate Hermite polynomials used in the Polynomial Chaos method. Furthermore, a size reduction of the equations is achieved by assuming that, in the Polynomial Chaos expansion of eigenvectors, the coefficient vectors belong to the subspace spanned by the deterministic vector and its derivatives



(a) Mean of the eigenvalues. (b) Percentage error of the mean.

Figure 3: Mean and corresponding percentage error of the first ten eigenvalues of the plate obtained with Monte Carlo Simulation (MCS) using 5000 samples, the proposed reduced spectral power method (RSPM) and reduced spectral inverse power method (RSIPM). The standard deviation of the discretized random field is 20% of the mean value.



(a) Standard deviation of the eigenvalues. (b) Percentage error of the standard deviation.

Figure 4: Standard deviation and corresponding percentage error of the first ten eigenvalues of the plate obtained with Monte Carlo Simulation (MCS) using 5000 samples, the proposed reduced spectral power method (RSPM) and reduced spectral inverse power method (RSIPM). The standard deviation of the discretized random field is 20% of the mean value.

with respect to the random variables. Numerical results on beam and plate problems indicate that the reduced spectral power method leads to a better approximation. Further studies will address the case of uncertainty in the mass matrix.

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