RELIABILITY ANALYSIS IN HIGH DIMENSIONS

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Abstract
In the reliability analysis of a complex engineering structure a very large number of the system parameters can be considered to be random variables. The difficulty in computing the failure probability using the First and Second-Order Reliability Methods (FORM and SORM) increases rapidly with the number of variables or 'dimension'. There are mainly two reasons behind this. The first is the increase in computational time with the increase in the number of random variables. In principle this problem can be handled with superior computational tools [see Schuller *et al.* (2003)]. The second, which is perhaps more fundamental, is that there are some conceptual difficulties associated typically with high dimensions. This means that even one manages to carry out the necessary computations, the application of existing FORM and SORM may still lead to incorrect results in high dimensions. This issue has received little attention in the literature and this paper is aimed at addressing it. An *asymptotic approximation* for the case when the number of random variables $n \to \infty$ is provided. The new asymptotic SORM is based on parabolic approximation of the failure surface in the transformed standard Gaussian space. A simple closed-form asymptotic expression is derived. The proposed asymptotic approximation for $n \to \infty$ case is compared with existing approximations and Monte-Carlo simulations using numerical examples.

Introduction

In reliability based structural analysis and design it is required to calculate the probability of failure (or survival) of a structure, either to assess the risk associated with an existing structural facility or to determine if a structural design has met the prescribed reliability criteria. Suppose the random variables describing the uncertainties in the structural properties and loading are considered to form a vector $\mathbf{x} \in \mathbb{R}^n$. The statistical properties of the system are fully described by the joint probability density function $p(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$. For a given set of variables $\mathbf{x}$ the structure will either fail under the applied (random) loading or will be safe. The condition of the structure for every $\mathbf{x}$ can be described by a safety margin $g(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ such the structure has failed if $g(\mathbf{x}) \leq 0$ and is safe if $g(\mathbf{x}) > 0$. Thus, the probability of failure is given by

$$P_f = \int_{g(\mathbf{x}) \leq 0} p(\mathbf{x}) d\mathbf{x} \tag{1}$$

The function $g(\mathbf{x})$ is also known as the failure surface or the limit-state function. The central theme of a reliability analysis is to evaluate the multidimensional integral (1). The exact evaluation of this integral, either analytically or numerically, is not possible for most practical problems because (a) n is large, (b) $p(\mathbf{x})$ is non-gaussian, and (c) $g(\mathbf{x})$ is a highly nonlinear function of $\mathbf{x}$. Even direct Monte Carlo simulation is quite expensive because P_f is usually small.

Over the past three decades there has been extensive research (see for example, the books by Thoft-Christensen and Baker, 1982, Madsen *et al.*, 1986, Ditlevsen and Madsen, 1996, Melchers, 1999) to develop approximate numerical methods for the efficient calculation of the reliability integral. The approximate reliability methods can be broadly grouped into (a) first-order reliability method (FORM), and (b) second-order reliability method (SORM). In FORM and SORM it is assumed that all the basic random variables are transformed and scaled so that they are uncorrelated Gaussian random variables, each with zero mean and unit standard deviation. In the transformed space Hasofer and Lind (1974) defined the reliability index

$$\beta = (\mathbf{x}^{*T} \mathbf{x}^*)^{1/2} \quad \text{where} \quad \mathbf{x}^* : \min\{(\mathbf{x}^T \mathbf{x})^{1/2}\} \quad \text{subject to} \quad g(\mathbf{x}) = 0 \tag{2}$$

Here $\mathbf{x}^*$, the 'design point' or the 'checking point'. Once the reliability index and the design point is known, the probability of failure can be obtained using FORM or SORM. In FORM the failure surface is approximated by a hyperplane which is tangent to the failure surface at the design point. In SORM, the actual failure surface is approximated by a quadratic hypersurface in the neighborhood of the design point.

SORM Approximations

After suitable transformations and keeping only second-order terms, Madsen *et al.* (1986) have approximated the failure surface by a parabolic surface as

$$\widetilde{g} \approx -y_n + \beta + \mathbf{y}^T \mathbf{A} \mathbf{y}, \quad \text{where} \quad \mathbf{y} \sim \mathbb{N}_{n-1}(\mathbf{0}_{n-1}, \mathbf{I}_{n-1}) \text{ and } y_n \sim \mathbb{N}_1(0,1) \tag{3}$$

The matrix **A** can be diagonalized by a further orthogonal transformation using the eigenvectors of **A**. The parabolic surface in (3) has been used by a number of authors, for example, Hohenbichler and Rackwitz (1988), Köylüoğlu and Nielsen (1994), Cai and Elishakoff (1994), Zhao and Ono (1999a,b), Polidori *et al.* (1999) and Hong (1999). Der-Kiureghian *et al.* (1987) and Der-Kiureghian and Stefano (1991) have also used this parabolic surface as the basis for point fitted SORM. With this approximation the failure probability is given by

$$P_f \approx \text{Prob}\left[\frac{\widetilde{g}}{|\boldsymbol{\nabla} g|} \le 0\right] \approx \text{Prob}\left[y_n \ge \beta + \mathbf{y}^T \mathbf{A} \mathbf{y}\right] \tag{4}$$

Denoting

$$U : \mathbb{R}^{n-1} \mapsto \mathbb{R} = \mathbf{y}^T \mathbf{A} \mathbf{y}, \tag{5}$$

as a central quadratic form in standard Gaussian random variables, the failure probability can be rewritten as

$$P_f \approx \text{Prob}\left[y_n \ge \beta + U\right] = \int_{\mathbb{R}} \left\{ \int_{\beta+u}^{\infty} \varphi(y_n) dy_n \right\} p_U(u) du = \text{E}\left[\Phi(-\beta - U)\right] \tag{6}$$

Using quadratic approximation of the failure surface together with asymptotic analysis Breitung (1984) proved that

$$P_f \to \frac{\Phi(-\beta)}{\prod_{j=1}^{n-1} \sqrt{1 + 2\beta a_j}} \quad \text{when} \quad \beta \to \infty \tag{7}$$

Here a_j, the eigenvalues of **A**, can be related to the principal curvatures of the surface κ_j as $a_j = \kappa_j/2$. This result is important because it gives the asymptotic behavior P_f under general conditions. Later, Hohenbichler and Rackwitz (1988) proposed an improved formula, where P_f is given by

$$P_f = \text{E}\left[\Phi(-\beta - U)\right] \approx \Phi(-\beta) \left\| \mathbf{I}_{n-1} + 2\frac{\varphi(\beta)}{\Phi(-\beta)} \mathbf{A} \right\|^{-1/2} \tag{8}$$

This equation was also rederived by Köylüoğlu and Nielsen (1994), Polidori *et al.* (1999) and Adhikari (2004) using different approaches. Using probability density function of the quadratic form U, Adhikari (2004) has derived some new closed-form approximations of P_f. In this paper an asymptotic approximation is presented for the case when the number of random variables become very large.

Failure Probability Using Asymptotic Distribution

We consider the case when the number of random variables is very large, that is, when asymptotically $n \to \infty$. From equations (3) and (4) the probability of failure can be rewritten as

$$P_f \approx \text{Prob}\left[\frac{\widetilde{g}}{|\boldsymbol{\nabla} g|} \le 0\right] = \text{Prob}\left[z \ge \beta\right] \tag{9}$$

where

$$z = y_n - \mathbf{y}^T \mathbf{A} \mathbf{y} \in \mathbb{R} \tag{10}$$

From the central limit theorem it is expected that when $n \to \infty$ the random variable z will asymptotically approach a Gaussian random variable. Discussions on asymptotic distribution of quadratic forms may be found in Mathai and Provost (1992, Section 4.6b). Here one of the simplest forms of asymptotic distribution of z will be used to obtain P_f.

We start with the moment generating function of z

$$M_z(s) = \text{E}\left[e^{sz}\right] = \text{E}\left[e^{sy_n - s\mathbf{y}^T \mathbf{A} \mathbf{y}}\right] \tag{11}$$

Because y_n and $\mathbf{y}^T\mathbf{A}\mathbf{y}$ are independent random variables this equation reduces to

$$M_z(s) = \mathrm{E}\left[e^{sy_n}\right]\mathrm{E}\left[e^{-s\mathbf{y}^T\mathbf{A}\mathbf{y}}\right] \tag{12}$$

Considering the eigenvalues of **A** we have

$$M_z(s) = e^{s^2/2}\prod_{k=1}^{n-1}\left(1+2sa_k\right)^{-1/2} \tag{13}$$

Now construct a sequence new random variables $q = z/\sqrt{n}$. The moment generating function of q:

$$M_q(s) = M_z(s/\sqrt{n}) = e^{s^2/2n}\prod_{k=1}^{n-1}\left(1+2sa_k/\sqrt{n}\right)^{-1/2} \tag{14}$$

From this

$$\begin{aligned}\ln\left(M_q(s)\right) &= s^2/2n - \frac{1}{2}\sum_{k=1}^{n-1}\ln\left(1+2sa_k/\sqrt{n}\right)\\ &= s^2/2n - \frac{1}{2}\sum_{k=1}^{n-1}2sa_k/\sqrt{n} - s^2\left(2a_k/\sqrt{n}\right)^2/2 + s^3\left(2a_k/\sqrt{n}\right)^3/3 - \cdots\end{aligned} \tag{15}$$

provided

$$|2sa_k| < 1, \quad \text{for} \quad k = 1, 2, \cdots, n-1 \tag{16}$$

Consider a case when a_k and n are such that the higher-order terms of s vanish as $n \to \infty$, *i.e.*, we assume n is large such that the following conditions hold

$$\sum_{k=1}^{n-1}(2a_k/\sqrt{n})^2/2 < \infty \quad \text{or} \quad \frac{2}{n}\mathrm{Trace}\left(\mathbf{A}^2\right) < \infty \tag{17}$$

$$\text{and} \quad \sum_{k=1}^{n-1}(2a_k/\sqrt{n})^r/r \to 0 \quad \text{or} \quad \frac{2^r}{n^{r/2}\,r}\mathrm{Trace}\left(\mathbf{A}^r\right) \to 0, \quad \forall \quad r \geq 3 \tag{18}$$

Under these assumptions, the series in equation (15) can be truncated after the quadratic term

$$\begin{aligned}\ln\left(M_q(s)\right) &\approx s^2/2n - \frac{1}{2}\sum_{k=1}^{n-1}s\left(2a_k/\sqrt{n}\right) - s^2\left(2a_k/\sqrt{n}\right)^2/2\\ &= -\mathrm{Trace}\left(\mathbf{A}\right)s/\sqrt{n} + \left(1 + 2\;\mathrm{Trace}\left(\mathbf{A}^2\right)\right)s^2/2n\end{aligned} \tag{19}$$

Therefore, the moment generating function of $z = q\sqrt{n}$ can be approximated by

$$M_z(s) \approx e^{-\mathrm{Trace}\left(\mathbf{A}\right)s + \left(1+2\;\mathrm{Trace}\left(\mathbf{A}^2\right)\right)s^2/2} \tag{20}$$

From the uniqueness of the Laplace Transform pair it follows that when the conditions (16)–(18) are satisfied, z asymptotically approaches a Gaussian random variable with mean $(-\mathrm{Trace}\left(\mathbf{A}\right))$ and variance $(1+2\mathrm{Trace}\left(\mathbf{A}^2\right))$, that is

$$z \simeq \mathbb{N}_1\left(-\mathrm{Trace}\left(\mathbf{A}\right), \sqrt{1 + 2\;\mathrm{Trace}\left(\mathbf{A}^2\right)}\right) \quad \text{when} \quad n \to \infty \tag{21}$$

Using this pdf of z, the asymptotic probability of failure can be obtained from equation (9) as

$$P_f \to \Phi\left(-\frac{\beta + \mathrm{Trace}\left(\mathbf{A}\right)}{\sqrt{1 + 2\;\mathrm{Trace}\left(\mathbf{A}^2\right)}}\right) \quad \text{when} \quad n \to \infty \tag{22}$$

Like Breitung's result in equation (7), this is another asymptotic result. Here the asymptotic behavior when $n \to \infty$, instead of $\beta \to \infty$, is considered. If the failure surface is linear or close to linear then $\mathrm{Trace}\left(\mathbf{A}\right) = \mathrm{Trace}\left(\mathbf{A}^2\right) \to 0$, and it is easy to see that equation (22) reduces to the classical FORM formula $P_f \approx \Phi(-\beta)$. In many real-life problems the number of random variables is expected to be large. In such situations this asymptotic result may turn out to be useful.

Minimum Number of Random Variables Required for the Accuracy of the Asymptotic Approach

The error in neglecting higher order terms in series (15) is of the form

$$\sum_{k=1}^{n-1} (2sa_k/\sqrt{n})^r/r = \frac{1}{r}\left(\frac{2s}{\sqrt{n}}\right)^r \text{Trace}\left(\mathbf{A}^r\right), \quad \text{for} \quad r \geq 3 \tag{23}$$

Values of s define the domain over which the moment generating function is used. For large β, it turns out that a good choice of s is $s = \beta$. Using this, here we aim to derive a simple expression for the minimum value of n which is *sufficient* for the application of the asymptotic distribution method. From condition (18), assume there exist a small real number ϵ (the error) such that

$$\frac{1}{r}\frac{(2\beta)^r}{n^{r/2}}\text{Trace}\left(\mathbf{A}^r\right) < \epsilon \quad \text{or } n^{r/2} > \frac{(2\beta)^r}{r\epsilon}\text{Trace}\left(\mathbf{A}^r\right) \quad \text{or } n > \frac{4\beta^2}{\sqrt[r]{r^2\epsilon^2}}\left(\sqrt[r]{\text{Trace}\left(\mathbf{A}^r\right)}\right)^2 \tag{24}$$

Since $\mathbf{A}$ is a positive definite matrix, the critical value of n is obtained for $r = 3$:

$$n_{\min} = \frac{4\beta^2}{\sqrt[3]{9\epsilon^2}}\left(\sqrt[3]{\text{Trace}\left(\mathbf{A}^3\right)}\right)^2 \tag{25}$$

From equation (25), the following points may be observed: (a) minimum number of random variables required would be more if ϵ (error) is considered to be small (as expected) and $n_{\min} \propto \frac{1}{\epsilon^{2/3}}$, and (b) if β is large, more random variables are needed to achieve a desired accuracy and $n_{\min} \propto \beta^2$.

Geometric Interpretation of the Asymptotic Expression

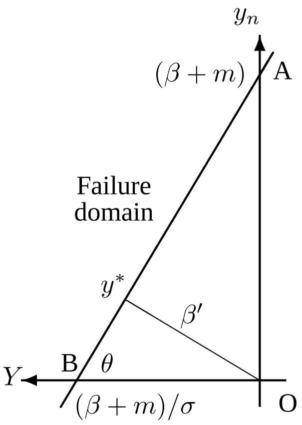


Figure 1: Geometric interpretation

The asymptotic expression of the failure probability given in equation (22) can be obtained using geometrical approach. From equations (4) and (5), the failure domain can be expressed as

$$y_n - U \geq \beta \tag{26}$$

Using asymptotic approach it was shown that when $n \to \infty$

$$U \simeq \mathbb{N}_1(m, \sigma), \quad \text{with } m = \text{Trace}\left(\mathbf{A}\right) \text{ and } \sigma^2 = 2\text{Trace}\left(\mathbf{A}^2\right) \tag{27}$$

Using the standardizing transformation $Y = (U - m/)\sigma$, equation (26) can be rewritten as

$$\frac{y_n}{\beta+m} + \frac{Y}{-\frac{\beta+m}{\sigma}} \geq 1 \tag{28}$$

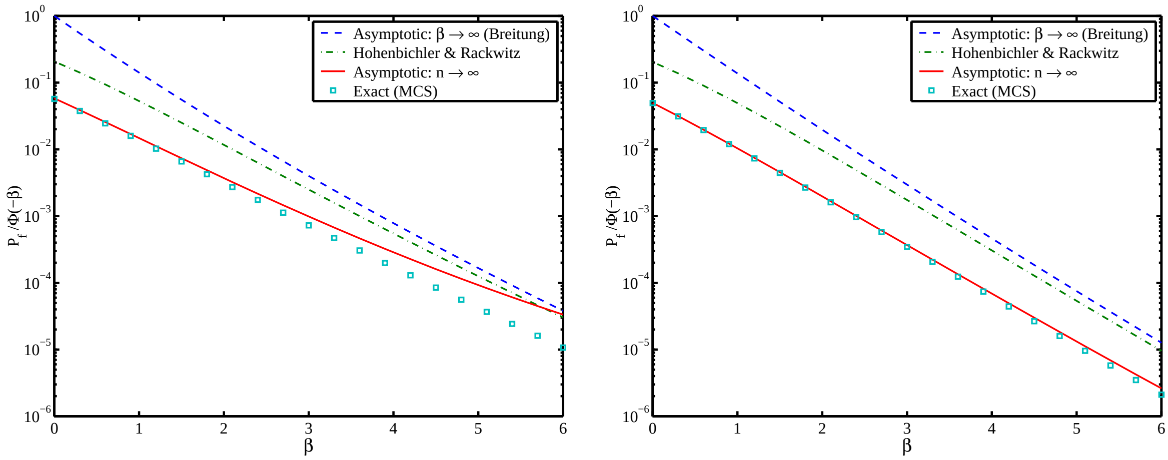
This equation (a straight line) is shown in Figure 1. Considering the triangle AOB, $\tan\theta = \frac{\text{OA}}{\text{OB}} = \frac{(\beta+m)}{(\beta+m)/\sigma} = \sigma$. Therefore, $\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\sigma}{\sqrt{1+\sigma^2}}$. Now, considering the triangle OBy^* and noticing that O$y^*\perp$AB, $\sin\theta = \frac{\text{O}y^*}{\text{OB}} = \frac{\beta'}{(\beta+m)/\sigma}$. From this, the modified reliability index

$$\beta' = \frac{\beta+m}{\sigma}\sin\theta = \frac{\beta+m}{\sigma}\frac{\sigma}{\sqrt{1+\sigma^2}} = \frac{\beta+m}{\sqrt{1+\sigma^2}} = \frac{\beta+\text{Trace}\left(\mathbf{A}\right)}{\sqrt{1+2\,\text{Trace}\left(\mathbf{A}^2\right)}} \tag{29}$$

Therefore, asymptotic failure probability $P_f = \Phi(-\beta')$, which has been derived in equation (22). If n is small m and σ will be very small. This would shift point B towards $-\infty$ in Y-axis and point A towards β level in in y_n-axis. That is, when $m, \sigma \to 0$, line AB will rotate clockwise and eventually will be parallel to Y-axis with a shift of $+\beta$. In this situation $\beta' \to \beta$ as expected.

Numerical Example

We consider a problem for which the failure surface is *exactly* parabolic in the normalized space, as given by equation (3). The purpose of this hypothetical example is to understand how the proposed approximation work after making a parabolic failure surface assumption. Therefore, the effect of errors due to parabolic failure surface assumption itself cannot and will not be investigated here. Probability of failure obtained using the asymptotic distribution is compared with both Breitung's asymptotic result and the formula (8) derived by Hohenbichler and Rackwitz (1988). Figure 2(a) shows asymptotic probability of failure (normalized by dividing with $\Phi(-\beta)$) for values of β ranging from 0 to 6. It is assumed that $n-1=100$ and $\text{Trace}\,(\mathbf{A})=2.0$. The eigenvalues of **A** are spaced in a quadratic manner. For this problem, the minimum number of random variables required for the applicability of the asymptotic distribution formula can be obtained from (25). Considering $\epsilon=0.01$, it can be shown from equation (25) that $n_{min}=220$. Although this condition is not satisfied here, the results obtained from this approach are quite acceptable, especially for lower values of β. Results obtained from the asymptotic



(a) $n-1=100$, $\text{Trace}\,(\mathbf{A})=2.0$

(b) $n-1=300$, $\text{Trace}\,(\mathbf{A})=2.0$

Figure 2: Failure probability using asymptotic distribution

analysis improve when the number of random variables becomes large. Figure 2(b) shows asymptotic probability of failure for $n-1=300$ and $\text{Trace}\,(\mathbf{A})=2.0$. As expected, with more random variables, asymptotic probability of failure match very well with the Monte Carlo simulation result.

Conclusions

An asymptotic approximation for the probability of failure when the number of random variables $n\to\infty$ is provided. It is assumed that the basic random variables are Gaussian and the the failure surface is approximated by a parabolic hypersurface in the neighborhood of the design point. The new asymptotic SORM is based on the asymptotic distribution of a central quadratic form in Gaussian random variables. The main outcome of the asymptotic analysis is that the conventional reliability index β needs to be modified when $n\to\infty$. A geometric interpretation of the modified reliability index is given. A closed-form expression for minimum number of random variables needed to apply the asymptotic formula is derived. Once a value of error (ϵ) is selected, this expression can be used to check whether the proposed asymptotic approximation can be used for the problem. The proposed asymptotic approximation is compared with some existing approximations and Monte-Carlo simulations using numerical examples. If the number of random variables is small then the proposed approach do not provide accurate result. However, when the number of random variables is large, then the asymptotic approximation produces excellent accuracy. In many real-life problems the number of random variables is expected to be large. In such situations the asymptotic result derived here will be useful.

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Nomenclature

z	random variable used for asymptotic distribution
β	Hasofer and Lind reliability index
$\mathbf{I}_k$	unit matrix of order k
$\mathbf{x}$	basic random variables
$\mathbf{x}^*$	design point
$\mathbf{y}$	$n-1$ dimensional uncorrelated Gaussian random vector
ϵ	small real number
κ_j	jth principal curvature of the failure surface at the design point
$\Phi(\bullet)$	standard Gaussian cumulative distribution function
$\varphi(\bullet)$	standard Gaussian probability density function
a_j	jth eigenvalue of $\mathbf{A}$
$g(\mathbf{x})$	failure surface in the space of basic variables
n	number of basic random variables
$p(\mathbf{x})$	probability density function of $\mathbf{x}$
P_f	probability of failure
$p_U(u)$	probability density function of u
q	a scaled random variable used for asymptotic distribution
U	quadratic form $\mathbf{y}^T\mathbf{A}\mathbf{y}$ in standard Gaussian random variables
u	a real scalar variable for the values of the random variable U
y_n	standard Gaussian random variable for nth coordinate
$(\bullet)^T$	matrix transpose
$\mathbb{N}_n(\mathbf{m},\mathbf{\Sigma})$	n dimensional Gaussian random vector with mean $\mathbf{m}\in\mathbb{R}^n$ and covariance matrix $\mathbf{\Sigma}\in\mathbb{R}^{n\times n}$
$\mathbb{R}$	space of real numbers
$\mathrm{E}[\bullet]$	expectation operator
$\in$	belongs to
$\mapsto$	maps into
$\|\| \bullet \|\|$	determinant of a matrix
$\propto$	proportional to
$\rightarrow$	approaches to
$\sim$	distributed as