Stochastic Structural Analysis Using Matrix Variate Distributions

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1 Introduction

Description of real-life engineering structural systems is inevitably associated with some amount of uncertainty in specifying material properties, geometric parameters, boundary conditions and applied loads. In order to come up with high-fidelity numerical models, it is required to consider these uncertainties explicitly. Within the past three decades deterministic finite element methods [1] have been generalized to consider uncertainties in a systematic and logical manner by means of Stochastic Finite Element Method [2, 3, 4, 5] (SFEM). Using SFEM one usually obtains the mass and stiffness matrices which are random in nature. Descriptions of these random matrices depend on the distributions of the randomness of the system parameters (e.g., Gaussian or Non-Gaussian) and also on sophistication of the analysis method used (e.g., random field discretisation methods). The starting point of this paper is that we know the joint probability density functions of the elements of the system matrices.

Once the random system matrices are 'known' the next step is to solve the governing equations of motion. In discrete linear structural mechanics, the following problems ore of fundamental interest:

- *Random algebraic equations*: Here the aim is to solve the following equation

$$\mathbf{K}\mathbf{y} = \mathbf{f} \tag{1}$$

  where $\mathbf{y}$ is the random displacement vector which we want to obtain and $\mathbf{p}$ is the forcing vector. For static problems the matrix $\mathbf{K}$ is the static stiffness matrix. For dynamic problems $\mathbf{K}$ is the dynamic stiffness matrix which is in general complex and frequency dependent.

- *Random eigenvalue problems*: Here the aim is to obtain the statistics of the eigenvalues (λ_j) and eigenvectors ($\boldsymbol{\phi}_j$) arising from the following eigenvalue problem

$$\mathbf{K}\boldsymbol{\phi}_j = \lambda_j \mathbf{M}\boldsymbol{\phi}_j \tag{2}$$

In this paper, random matrix theory [6, 7, 8] is used to solve the random algebraic equations (1).

2 Problem Statement and Background

Suppose $\mathbf{X} \in \mathbb{R}^{p\times p}$ is a random matrix with given probability density function (pdf) $p_{\mathbf{X}}(\mathbf{X})$. The matrix $\mathbf{X}$ stands for a static stiffness matrix or a dynamic stiffness matrix at a given frequency for an undamped system. We want to obtain the pdf of the response quantity $\mathbf{Y}$ given by

$$\mathbf{Y} = \mathbf{A}\mathbf{X}^{-1}\mathbf{B} \in \mathbb{R}^{n\times m} \tag{3}$$

here $\mathbf{A} \in \mathbb{R}^{n\times p}$ and $\mathbf{B} \in \mathbb{R}^{p\times m}$ are constant matrices. Problem (3) is considered for the sake of generality. The matrix $\mathbf{B}$ can be selected such that multiple loading conditions can be casted in different columns. The matrix $\mathbf{A}$ can be chosen, for example, to select only few elements of particular interest from the response matrix $\mathbf{Y}$ or to linearly combine different elements of $\mathbf{Y}$. The standard problem (1) is a special case when $\mathbf{A}$ is an identity matrix $\mathbf{B}$ is a $p \times 1$ dimensional matrix.

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The only assumption in problem (3) is that the random matrix $\mathbf{X}$ is real and invertible. The dimensions m, n and p are general and also no assumption regarding the symmetry of $\mathbf{X}$ will be used in this study. The conventional method to solve this type of problems is to use perturbation methods [3] or Neumann series expansion [9, 10]. Such methods may not be accurate when randomness is large. Moreover, the determination of pdf of the response vector when the elements of $\mathbf{X}$ are non-Gaussian or when higher order terms are used in Neumann series can pose serious challenge. Here we have used an approach based on matrix variate distributions, a subject in mathematical statistics which has seen significant development in recent years (see the book by Gupta and Nagar [8] and the references therein). The following two types of random matrices will be considered in this study:

Definition 1. *Normal Random Matrix*: The random matrix $\mathbf{X} \in \mathbb{R}^{p\times p}$ is said to have a matrix variate normal distribution with mean matrix $\mathbf{M} \in \mathbb{R}^{p\times p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}^{p\times p} > 0$ and $\mathbf{\Psi} \in \mathbb{R}^{p\times p} > 0$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-p^2/2}|\mathbf{\Sigma}|^{-p/2}|\mathbf{\Psi}|^{-p/2}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \tag{4}$$

This distribution is usually denoted as $\mathbf{X} \sim N_{p,p}(\mathbf{M}, \mathbf{\Sigma}\otimes\mathbf{\Psi})$.

In the above, $\otimes$ is Kronecker product, $|\bullet|$ is determinant of a matrix, $(\bullet)^T$ is matrix transposition, $\mathrm{etr}\{\bullet\}$ is $\exp\{\mathrm{Trace}(\bullet)\}$ and $\mathrm{Trace}(\bullet)$ is the sum of the diagonal elements of a matrix.

Although, normal matrices have been used extensively in literature, it is well known that for many physical parameters, such as mass or stiffness distributions which are strictly positive quantities, they are not suitable. For this reason we consider a non-Gaussian random matrix, known as the Wishart matrix. If $\mathbf{X} \sim N_{p,N}(\mathbf{O}, \mathbf{\Sigma}\otimes\mathbf{I}_N)$, then $\mathbf{S} = \mathbf{X}\mathbf{X}^T$ is a Wishart matrix denoted as $\mathbf{S} \sim W_p(N, \mathbf{\Sigma})$.

Definition 2. *Wishart matrix*: A $p\times p$ random symmetric positive definite matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters p, N and $\mathbf{\Sigma} \in \mathbb{R}^{p\times p}$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{2^{\frac{1}{2}pN}\,\Gamma_p\left(\frac{1}{2}N\right)|\mathbf{\Sigma}|^{\frac{1}{2}N}\right\}^{-1}|\mathbf{S}|^{\frac{1}{2}(N-p-1)}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S}\right\} \tag{5}$$

The advantage of using a Wishart matrix is that it is symmetric and positive definite. This makes it a more realistic model for structural matrices compared to a normal matrix.

3 Main Results

The usefulness of random matrix theory comes from the fact that many results for scalar random variables can be immediately generalized to the matrix case with suitable adaptations. Extending the well know result for the transformation of scalar random variables [11] to the matrix case we have

$$p_{\mathbf{Y}}(\mathbf{Y}) = \frac{1}{|J(\mathbf{X}_1|}p_{\mathbf{X}}(\mathbf{X}=\mathbf{X}_1) \tag{6}$$

where $\mathbf{X}_1$ is the unique solution of $\mathbf{X}$ from equation (3) which can be obtained as

$$\mathbf{X}_1 = \mathbf{B}\mathbf{Y}^{-1}\mathbf{A} \tag{7}$$

Note that that $\mathbf{Y}$ is in general a rectangular matrix so that one has to use pseudo-inverse in Eq. (7). The jacobian of the transformation in (3) can be obtained [12, 13] as

$$|J| = (\mathbf{B}^T\otimes\mathbf{A})|\mathbf{X}|^{-2p} \tag{8}$$

Substituting $\mathbf{X} = \mathbf{X}_1$ in Eq. (8) and considering $\mathbf{X}$ is a normal random matrix, after some algebra we have

$$p_{\mathbf{Y}}(\mathbf{Y}) = (2\pi)^{-p^2/2}|\widehat{\mathbf{\Sigma}}|^{-n/2}|\widehat{\mathbf{\Psi}}|^{-m/2}|\mathbf{B}\mathbf{Y}^{-1}\mathbf{A}|^{2p}\,\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{B}\mathbf{Y}^{-1}\mathbf{A}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{B}\mathbf{Y}^{-1}\mathbf{A}-\mathbf{M})^T\right\} \tag{9}$$

where

$$\widehat{\mathbf{\Sigma}} = \mathbf{A}\mathbf{\Sigma}\mathbf{A}^T \in \mathbb{R}^{n\times n} \quad \text{and} \quad \widehat{\mathbf{\Psi}} = \mathbf{B}^T\mathbf{\Sigma}\mathbf{B} \in \mathbb{R}^{m\times m} \tag{10}$$

Equation (9) is the exact closed-form pdf of the response quantity of interest. A similar expression for the case of Wishart matrix can also be derived using Eqs. (6)–(8). Since Eq. (9) is a function of several variables, it is difficult to visualize. For this reason in the numerical works the matrix matrix **A** is selected such that only two elements of **Y** are considered at a time.

4 Conclusions

An exact expression of the probability density function of the static response of a linear stochastic system has been derived in closed-form. Future works will address other non-normal random matrix models (such as random β-matrices), derivation of the cumulative density functions and response moments.

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