Random Matrix Eigenvalue Problems in Probabilistic Structural Mechanics

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1. Introduction

Characterization of the natural frequencies and the mode-shapes play a fundamental role in the analysis and design of engineering dynamic systems. The determination of natural frequencies and mode shapes require the solution of an eigenvalue problem. Eigenvalue problem also arises in the context of stability analysis of structures. This problem could either be a differential eigenvalue problem or a matrix eigenvalue problem, depending on whether a continuous model or a discrete model is used to describe the given vibrating system. Description of real-life engineering structural systems is inevitably associated with some amount of uncertainty in specifying material properties, geometric parameters, boundary conditions and applied loads. When we take account of these uncertainties, it is necessary to consider *random eigenvalue problems*. Several studies have been conducted on this topic since the mid-sixties. The study of probabilistic characterization of the eigensolutions of random matrix and differential operators is now an important research topic in the field of stochastic structural mechanics. The paper by Boyce [1] and the book by Scheidt and Purkert [2] are useful source of information on early works in this area of research and also provide a systematic account of different approaches to random eigenvalue problems.

In this paper we obtain a closed-form expression of arbitrary order joint moments of the eigenvalues of discrete linear systems or discretized continuous systems. The random eigenvalue problem of undamped or proportionally damped systems can be expressed by

$$\mathbf{K}(\mathbf{x})\boldsymbol{\phi}_j = \lambda_j \mathbf{M}(\mathbf{x})\boldsymbol{\phi}_j \tag{1}$$

Here λ_j and $\boldsymbol{\phi}_j$ are the eigenvalues and the eigenvectors of the dynamic system. $\mathbf{M}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$ and $\mathbf{K}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$, the mass and stiffness matrices, are assumed to be smooth, continuous and at least twice differentiable functions of a random parameter vector $\mathbf{x} \in \mathbb{R}^m$. The vector $\mathbf{x}$ may consist of material properties, e.g., mass density, Poisson's ratio, Young's modulus; geometric properties, e.g., length, thickness, and boundary conditions. Statistical properties of the system are completely described by the joint probability density function (pdf) $p_{\mathbf{X}}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$. For mathematical convenience we express

$$p_{\mathbf{X}}(\mathbf{x}) = \exp\left\{-L(\mathbf{x})\right\} \tag{2}$$

where $-L(\mathbf{x})$ is often known as the log-likelihood function. For example, if $\mathbf{x}$ is a m-dimensional multivariate Gaussian random vector with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ then $L(\mathbf{x}) = \frac{m}{2}\ln(2\pi) + \frac{1}{2}\ln\|\boldsymbol{\Sigma}\| + \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$. It is assumed that in general the random parameters are non-Gaussian and correlated, i.e., $L(\mathbf{x})$ can have any general form provided it is a smooth, continuous and at least twice differentiable function. It is further assumed that $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite random matrices so that all the eigenvalues are real and positive.

2. Background

The current literature on random eigenvalue problems arising in engineering systems is dominated by the mean-centered perturbation methods. These methods work well when the uncertainties are small and the parameter distribution is Gaussian. Methods which are *not* based on mean-centered perturbation but still have the generality and computational efficiency to be applicable for engineering dynamic systems are rare. Grigoriu [3] has examined the roots of characteristic polynomials of real symmetric random matrices using distribution of zeros of random polynomials. Lee and Singh [4] have proposed a direct matrix product (Kronecker product) method to obtain the first two moments of the eigenvalues of discrete linear systems. Under special circumstances when the matrix $\mathbf{H} = \mathbf{M}^{-1}\mathbf{K} \in \mathbb{R}^{N\times N}$ is Gaussian unitary ensemble (GUE) or Gaussian orthogonal ensemble (GOE) an exact closed-form expression can be obtained for the joint pdf of the eigenvalues using random matrix theory (RMT). See the book by Mehta [5] and references therein for discussions on random matrix theory. RMT has been extended to other type of random matrices. If $\mathbf{H}$ has Wishart distribution then the exact joint pdf of the eigenvalues can be obtained from Muirhead [6] (Theorem 3.2.18). Edelman [7] has obtained the pdf of the minimum eigenvalue (first natural frequency squared) of a Wishart matrix. A more general case when the matrix $\mathbf{H}$ has β-distribution has been obtained by Muirhead [6] (Theorem 3.3.4) and Dumitriu and Edelman [8]. However, unfortunately the system matrices of real structures may not always follow such distributions and consequently some kind of approximate analysis is required. Recently Adhikari and Langley [9] and Adhikari and Friswell [10] have proposed some non-perturbative methods to obtain moments and pdf of the eigenvalues for the general case. In this paper we obtain joint statistics of the eigenvalues.

3. Arbitrary Joint Moments of Two Eigenvalues

The eigenvalues, $\lambda_j(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}$ are non-linear functions of the parameter vector $\mathbf{x}$. Each $\lambda_j(\mathbf{x})$ is a smooth and twice differentiable function since the mass and the stiffness matrices are also smooth and twice differentiable function of the random parameter vector. We begin with arbitrary order joint moment of two eigenvalues

$$\mu_{jl}^{(rs)} = \mathrm{E}\left[\lambda_j^r(\mathbf{x})\lambda_l^s(\mathbf{x})\right] = \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x})\lambda_j^s(\mathbf{x}) p_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$
$$= \int_{\mathbb{R}^m} \exp\left\{-\left(L(\mathbf{x}) - r\ln\lambda_j(\mathbf{x}) - s\ln\lambda_l(\mathbf{x})\right)\right\} d\mathbf{x} \tag{3}$$

We evaluate the above multidimensional integral using asymptotic theory of integrals, see Papadimitriou et al [11] for example. Consider a function $f(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}$ which is smooth and at least twice differentiable and we want to evaluate an integral of the following form:

$$\mathcal{J} = \int_{\mathbb{R}^m} \exp\left\{-f(\mathbf{x})\right\} d\mathbf{x} \tag{4}$$

Suppose there exists a point $\boldsymbol{\theta}$ where $f(\mathbf{x})$ reaches its global minimum in $\mathbb{R}^m$, i.e., at $\mathbf{x} = \boldsymbol{\theta}$

$$\mathbf{d}_f(\boldsymbol{\theta}) = \mathbf{0} \tag{5}$$

where $\mathbf{d}_{(\bullet)}(\mathbf{x})$ is the gradient vector of $(\bullet)$ at $\mathbf{x}$. It can be shown that

$$\mathcal{J} \approx (2\pi)^{m/2} \exp\left\{-f\left(\boldsymbol{\theta}\right)\right\} \left\|\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\|^{-1/2} \tag{6}$$

where $\| \bullet \|$ denotes the determinant of a matrix and $\mathbf{D}_{(\bullet)}(\mathbf{x})$ is the Hessian matrix of $(\bullet)$ at $\mathbf{x}$. Choosing

$$f(\mathbf{x}) = L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x}) - s \ln \lambda_l(\mathbf{x}) \tag{7}$$

the optimal point $\boldsymbol{\theta}$ for the integral (3) can be obtained by differentiating the above equation with respect to x_k as

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = \frac{\partial L(\mathbf{x})}{\partial x_k} - \frac{r}{\lambda_j(\mathbf{x})} \frac{\partial \lambda_j(\mathbf{x})}{\partial x_k} - \frac{s}{\lambda_l(\mathbf{x})} \frac{\partial \lambda_l(\mathbf{x})}{\partial x_k} \tag{8}$$

Combining for all k and equating to 0, $\boldsymbol{\theta}$ can be obtained from

$$\mathbf{d}_L(\boldsymbol{\theta}) = \frac{r}{\lambda_j(\boldsymbol{\theta})} \mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) + \frac{s}{\lambda_l(\boldsymbol{\theta})} \mathbf{d}_{\lambda_l}(\boldsymbol{\theta}) \tag{9}$$

To obtain the Hessian matrix we differentiate (8) by x_i. After some simplification it can be shown that

$$\mathbf{D}_f\left(\boldsymbol{\theta}\right) = \mathbf{D}_L(\boldsymbol{\theta}) + \frac{r}{\lambda_j^2\left(\boldsymbol{\theta}\right)} \mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) \mathbf{d}_{\lambda_j}(\boldsymbol{\theta})^T - \frac{r}{\lambda_j\left(\boldsymbol{\theta}\right)} \mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) + \frac{s}{\lambda_l^2\left(\boldsymbol{\theta}\right)} \mathbf{d}_{\lambda_l}(\boldsymbol{\theta}) \mathbf{d}_{\lambda_l}(\boldsymbol{\theta})^T - \frac{s}{\lambda_l\left(\boldsymbol{\theta}\right)} \mathbf{D}_{\lambda_l}(\boldsymbol{\theta}) \tag{10}$$

Using the asymptotic approximation (6), the joint moment of two eigenvalues can be obtained as

$$\mu_{jl}^{(rs)} \approx (2\pi)^{m/2} \lambda_j^r(\boldsymbol{\theta}) \lambda_l^s(\boldsymbol{\theta}) \exp\left\{-L\left(\boldsymbol{\theta}\right)\right\} \left\|\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\|^{-1/2} \tag{11}$$

The optimal point $\boldsymbol{\theta}$ needs to be calculated by solving non-linear set of equations Eq. (9). Since explicit analytical expression of $\mathbf{d}_{\lambda_j}$ in terms of the derivative of the mass and stiffness matrices is avaliable [12], expensive numerical differentiation of $\lambda_j(\mathbf{x})$ at each step is not needed. Moreover, for most $p_{\mathbf{X}}(\mathbf{x})$, a closed-form expression of $\mathbf{d}_L(\mathbf{x})$ is available. If $\mathbf{x}$ is Gaussian random vector, then a simple iterative method [9, 10] can be used to obtain $\boldsymbol{\theta}$.

4. General Joint Moments of Multiple Eigenvalues

Above formulation can be extended to obtain general order joint moments of arbitrary number of eigenvalues. We want to obtain

$$\mu_{j_1 j_2 \cdots j_n}^{(r_1 r_2 \cdots r_n)} = \int_{\mathbb{R}^m} \left\{\lambda_{j_1}^{r_1}(\mathbf{x}) \lambda_{j_2}^{r_2}(\mathbf{x}) \cdots \lambda_{j_n}^{r_n}(\mathbf{x})\right\} p_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{12}$$

Following the method outlined the previous section it can be shown that

$$\mu_{j_1 j_2 \cdots j_n}^{(r_1 r_2 \cdots r_n)} \approx (2\pi)^{m/2} \left\{\lambda_{j_1}^{r_1}\left(\boldsymbol{\theta}\right) \lambda_{j_2}^{r_2}\left(\boldsymbol{\theta}\right) \cdots \lambda_{j_n}^{r_n}\left(\boldsymbol{\theta}\right)\right\} \exp\left\{-L\left(\boldsymbol{\theta}\right)\right\} \left\|\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\|^{-1/2} \tag{13}$$

where $\boldsymbol{\theta}$ is obtained from

$$\mathbf{d}_L(\boldsymbol{\theta}) = \frac{r_1}{\lambda_{j_1}(\boldsymbol{\theta})} \mathbf{d}_{\lambda_{j_1}}(\boldsymbol{\theta}) + \frac{r_2}{\lambda_{j_1}(\boldsymbol{\theta})} \mathbf{d}_{\lambda_{j_2}}(\boldsymbol{\theta}) + \cdots \frac{r_n}{\lambda_{j_n}(\boldsymbol{\theta})} \mathbf{d}_{\lambda_{j_n}}(\boldsymbol{\theta})$$

and the Hessian matrix is given by

$$\mathbf{D}_f\left(\boldsymbol{\theta}\right) = \mathbf{D}_L(\boldsymbol{\theta}) + \sum_{\substack{j = j_1, j_2, \cdots \\ r = r_1, r_2, \cdots}}^{j_n, r_n} \frac{r}{\lambda_j^2\left(\boldsymbol{\theta}\right)} \mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) \mathbf{d}_{\lambda_j}(\boldsymbol{\theta})^T - \frac{r}{\lambda_j\left(\boldsymbol{\theta}\right)} \mathbf{D}_{\lambda_j}(\boldsymbol{\theta})$$

Eq. (13) is perhaps the most general formula to obtain the moments of the eigenvalues of linear stochastic dynamic systems. Once the joint moments are known, the joint probability density functions of the eigenvalues are obtained using the maximum entropy principle. Proposed method is applied to dynamic analysis of a framed structure with uncertain properties.

5. Conclusions

The statistics of the eigenvalues of linear dynamic systems with parameter uncertainties have been considered. It is assumed that the mass and stiffness matrices are smooth and at least twice differentiable functions of a set of random variables. The random variables are in general assumed to be non-Gaussian. The usual assumption of small randomness employed in most mean-centered based perturbation analysis is not employed in this study. Closed-form asymptotically correct expression for general order joint moments of arbitrary number of eigenvalues of linear stochastic systems has been derived for the first time. Using the results derived in this paper it is possible to construct the joint propagability density function of the eigenvalues.

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