AENG M3102: Dynamic analysis of wind and marine turbines

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Abstract

Wind and marine turbine structures are long slender columns with a rotor and blade assembly placed on the top. These slender structures vibrate due to dynamic environmental forces and its own dynamics. Analysis of the dynamic behavior of wind and marine turbines is fundamental to the stability, performance, operation and safety of these systems. In this note a simplified approach is outlined for linear dynamic analysis of these long, slender structures. The method is based on an Euler Bernoulli beam-column with elastic end supports. The elastic end-supports are considered to model the flexible nature of the interaction of these systems with soil. Within these assumptions, a general approach is taken to obtain the natural frequency of the system. Theoretical developments are explained by practical examples. Design issues of wind and marine turbine structures are discussed based on the numerical results. Suitable `Maple`® and `Matlab`® are also given to support this study.

Contents

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Nomenclature

α	mass ratio, $\alpha = \frac{M}{mL}$
β	nondimensional rotary inertia, $\beta = \frac{J}{mL^3}$
η_r	nondimensional rotational end stiffness, $\eta_r = \frac{k_r L}{EI}$
η_t	nondimensional translational end stiffness, $\eta_t = \frac{k_t L^3}{EI}$
γ_k	stiffness correction factor, $\gamma_k = \frac{k_e}{k_{CL}}$
γ_m	mass correction factor
μ	nondimensional radius of gyration, $\mu = \frac{r}{L}$
ν	nondimensional axial force, $\nu = \frac{PL^2}{EI}$
Ω	nondimensional frequency parameter, $\Omega^2 = \omega^2 \frac{mL^4}{EI}$
ω	angular frequency (rad/s)
ϖ	passing frequency of the blades
ξ	non-dimensional length parameter, $\xi = x/L$
EI	bending stiffness of the beam
$f(x,t)$	applied time depended load on the beam
f_0	natural frequency scaling parameter (s^{-1}), $f_0 = \sqrt{\frac{EI}{mL^4}}$
f_i	natural frequency (Hz)
J	rotary inertia of the rotor system
k_e	equivalent end stiffness of the beam
k_r	rotational end stiffness of the elastic support
k_r	translational (linear) end stiffness of the elastic support
k_{CL}	normalizing stiffness value, $k_{CL} = \frac{3EI}{L^3}$
L	length of the beam
M	mass of the rotor system
m	mass per unit length of the beam, $m = \rho A$
M_b	mass of the beam, $M_b = mL$
M_e	equivalent mass of the SDOF system
P	constant axial force in the beam
p_L	normalized forcing amplitude at the top end $p_L = \epsilon \frac{ML^4}{EI} \varpi^2$
t	time
$W(\xi)$	transverse deflection of the beam
$w(x,t)$	time depended transverse deflection of the beam
x	spatial coordinate along the length of the beam
$(\bullet')$	derivative with respect to the spatial coordinate
$(\bullet)^T$	matrix transposition
$(\dot{\bullet})$	derivative with respect to time
$\lvert\bullet\rvert$	determinant of a matrix

1 Introduction

Wind and marine turbine structures are long slender columns with a rotor and blade assembly placed on the top. These slender structures vibrate due to dynamic environmental forces and its own dynamics. Analysis of the dynamic behavior of wind and marine turbines is fundamental to the stability, performance, operation and safety of these systems. For the design and analysis of real-life systems, detailed finite-element models are often used. A multi-physics model of a modern wind or marine turbine with (non-linear) structural dynamics, fluid-structure interaction, soil-structure interaction and rotor dynamics can easily lead to computational model consisting several million degrees-of-freedom. While such a detailed analysis can give incredible resolution of the dynamic behavior of the system, the understanding of basic physical principles which govern the overall design may be somewhat difficult to deduce from such a complex analysis. For this reason, in this note we will focus on a simplified analysis with the aim of understanding fundamental physics which underpins the overall dynamic behavior of the system. Our approach involves the following key steps:

- Idealisation of the complex system and related assumptions
- Derivation of the equation of motion
- Derivation of the boundary conditions
- Analytical solution of the eigenvalue problem to obtain natural frequencies of the system

2 Equation of Motion and Boundary Conditions

We consider a typical wind turbine structure as shown in Figure 1. This system is idealized by an Euler Bernoulli beam. The bending stiffness of the beam is $EI(x)$ and it is 'standing' on soil. Here x is the spatial coordinate along the height of the structure. The interaction of the structure with the surrounding soil is modeled using two springs. The rotational spring with spring stiffness k_r and the translational spring with spring stiffness k_t constrains the system at the bottom ($x = 0$). The beam has a top mass with rotary inertia J and mass M. This top mass is used to idealise the rotor and blade system. The mass per unit length of the beam is $m(x)$, $r(x)$ is the radius of gyration and the beam is subjected to a constant compressive axial load P. For the static case the value of P will be equal to Mg. But for dynamic case the value of the true force P coming into the beam can change. Therefore, for generality we will use the notation P in this note. If necessary, a specific value of P can always be used.

The equation of motion of the beam is given by (see the book by Géradin and Rixen [10] for the derivation of this equation):

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(P(x)\frac{\partial w(x,t)}{\partial x}\right) - \frac{\partial}{\partial x}\left(mr^2(x)\frac{\partial \ddot{w}(x,t)}{\partial x}\right) + m\,\ddot{w}(x,t) = f(x,t). \quad (1)$$

Here $w(x,t)$ is the transverse deflection of the beam, t is time, $(\dot{\bullet})$ denotes derivative with respect to time and $f(x,t)$ is the applied time depended load on the beam. The height of the structure is considered to be L. Equation (1) is a fourth-order partial differential equation [19] and has

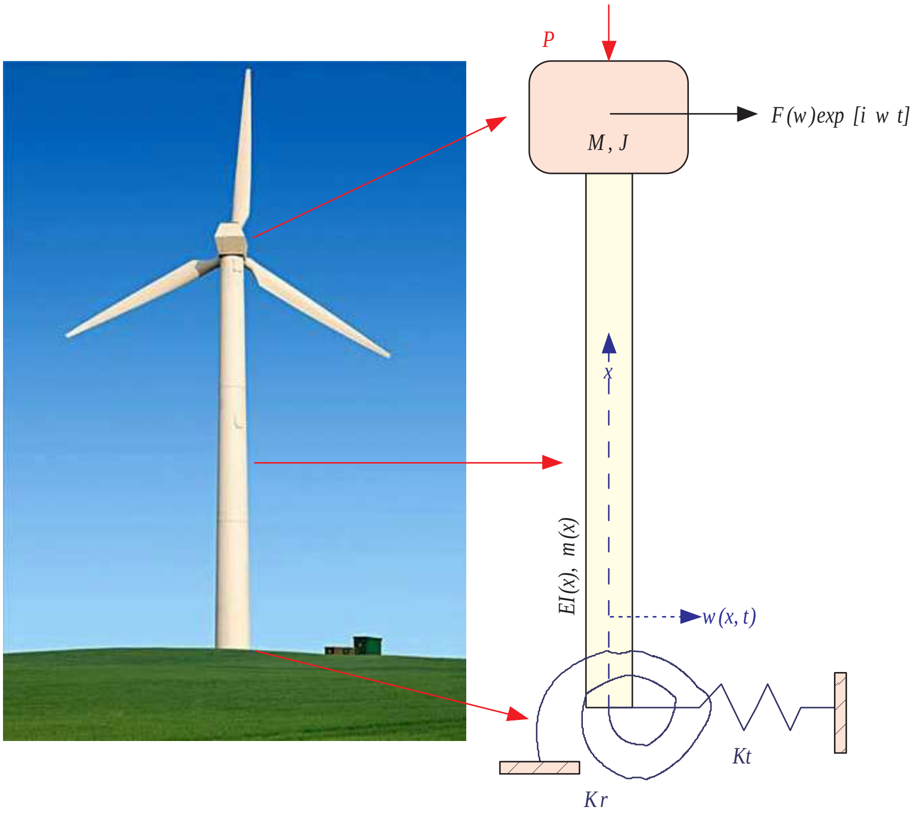


Figure 1: Idealisation of a wind turbine (the first picture is taken from http://www.segen.co.uk) using Euler Bernoulli beam with a top mass. Flexible springs are assumed to model the soil-structure interaction. The weight of the rotor-hub and blades are assumed to be P. The forcing due to the rotation of the blades is assumed to be harmonic in nature with frequency ϖ.

been used extensively in literature for various problems (see for example, references [1–3, 5–9, 11–18, 20, 23–25, 27–29]). Our central aim is to obtain the natural frequency of the system. The book by Blevins [4] lists several expressions of the natural frequencies of similar systems but this particular case has not been covered. For this reason we need to develop a new analysis method for this problem. Here we develop an approach based on the non-dimensionalisation of the equation of motion (1).

The forcing due to the rotation of the blades is assumed to be harmonic in nature with frequency ϖ. This implies that the system is subjected to a forcing

$$F(\varpi)\exp[i\varpi t] \tag{2}$$

at $x = L$. We can assume that there is a slight imbalance between the rotor and the column. Such an imbalance in unavoidable due to construction defects of such complex systems. Assuming the amount of imbalance is ϵ, we have $F(\varpi) = (M\varpi^2\epsilon)$ due to the centrifugal force. Therefore, the dynamic loading on the system at $x = L$ can be idealized as

$$f(x,t) = (M\varpi^2\epsilon)\exp[i\varpi t]\delta(x-L) \tag{3}$$

where $\delta(\bullet)$ is the Dirac delta function. This equation shows that the magnitude of the force acting on the system is proportional to the square of the blade passing frequency.

Equation (1) is a quite general equation. It is possible to consider any variation in the bending stiffness $EI(x)$ and mass density $m(x)$ of the structure with height (such as a tapered column). Consideration of such variation normally leads to the case where closed-form solutions are impossible to obtain due to the complex nature of the resulting equations. For simplicity, we therefore assume a special case where all properties are constant along the height of the structure. This assumption will definitely affect the numerical results, but unlikely to change the physical understandings, which are the primary aims of this lecture. In summary, the main assumptions of the proposed analysis are:

- The inertial and the elastic properties of the structure are constant along the height of the structure.
- The effect of soil stiffness is elastic and linear and can be captured by a translational and a rotational spring at the point of contact. In effect, the soil-structure interaction can be completely captured by the boundary condition at the bottom of the structure.
- The end-mass is rigidly attached to the structure.
- The axial force in the structure is constant and remains axial during vibration.
- Deflections due to shear force are negligible and a plain section in the structure remains plane during the bending vibration (standard assumptions in the Euler-Bernoulli beam theory).
- The flexible dynamics of the rotor and the blades above the top point is assumed to be uncoupled with the dynamics of the column.
- Forcing to the system is harmonic and arising to due to the imbalance in the rotor-beam system.
- None of the properties are changing with time. In other words, the system is time invariant.

Equation (1) can be applied to marine turbines also as the equation of motion for free vibration does not change even when the system is submerged in water. The main change for the marine turbine would be the addition of viscous damping forces arising due to the surrounding water. Since the effect of damping on the natural frequencies is very small, the mathematical formulations proposed here are equally applicable to marine turbines also.

Noting that the properties are not changing with x, equation (1) can be simplified as

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + P\frac{\partial^2 w(x,t)}{\partial x^2} - mr^2\frac{\partial^2 \ddot{w}(x,t)}{\partial x^2} + m\,\ddot{w}(x,t) = (M\varpi^2\epsilon)\exp[i\varpi t]\delta(x-L). \tag{4}$$

The four boundary conditions associated with this equation can be expressed as

- Bending moment at $x = 0$:

$$\left. EI\frac{\partial^2 w(x,t)}{\partial x^2} - k_r\frac{\partial w(x,t)}{\partial x} = 0 \right|_{x=0} \quad \text{or} \quad EIw''(0,t) - k_r w'(0,t) = 0. \tag{5}$$

- Shear force at $x = 0$:

$$\begin{aligned} EI\frac{\partial^3 w(x,t)}{\partial x^3} + P\frac{\partial w(x,t)}{\partial x} + k_t w(x,t) - mr^2\frac{\partial \ddot{w}(x,t)}{\partial x} = 0\Big|_{x=0} \\ \text{or} \quad EIw'''(0,t) + Pw'(0,t) + k_t w(0,t) - mr^2\frac{\partial \ddot{w}(0,t)}{\partial x} = 0. \end{aligned} \tag{6}$$

- Bending moment at $x = L$:

$$EI\frac{\partial^2 w(x,t)}{\partial x^2} + J\frac{\partial \ddot{w}(x,t)}{\partial x} = 0\Big|_{x=L} \quad \text{or} \quad EIw''(L,t) + J\frac{\partial \ddot{w}(L,t)}{\partial x} = 0. \tag{7}$$

- Shear force at $x = L$:

$$\begin{aligned} EI\frac{\partial^3 w(x,t)}{\partial x^3} + P\frac{\partial w(x,t)}{\partial x} - M\ddot{w}(x,t) - mr^2\frac{\partial \ddot{w}(x,t)}{\partial x} = 0\Big|_{x=L} \\ \text{or} \quad EIw'''(L,t) + Pw'(L,t) - M\ddot{w}(L,t) - mr^2\frac{\partial \ddot{w}(L,t)}{\partial x} = 0. \end{aligned} \tag{8}$$

Assuming harmonic solution (the separation of variable) we have

$$w(x,t) = W(\xi)\exp\{\mathrm{i}\omega t\}, \quad \xi = x/L. \tag{9}$$

Substituting this in the equation of motion and the boundary conditions, Eqs. (4) – (8), results

$$\frac{EI}{L^4}\frac{\partial^4 W(\xi)}{\partial \xi^4} + \frac{P}{L^2}\frac{\partial^2 W(\xi)}{\partial \xi^2} - m\omega^2 W(\xi) + \frac{mr^2\omega^2}{L^2}\frac{\partial^2 W(\xi)}{\partial \xi^2} = (M\varpi^2\epsilon)\exp[i\varpi t]\delta(\xi L - L) \tag{10}$$

$$\frac{EI}{L^2}W''(0) - \frac{k_r}{L}W'(0) = 0 \tag{11}$$

$$\frac{EI}{L^3}W'''(0) + \frac{P}{L}W'(0) + k_t W(0) + \frac{mr^2\omega^2}{L}W'(0) = 0 \tag{12}$$

$$\frac{EI}{L^2}W''(1) - \frac{\omega^2 J}{L}W'(1) = 0 \tag{13}$$

$$\frac{EI}{L^3}W'''(1) + \frac{P}{L}W'(1) + \omega^2 M\, W(1) + \frac{mr^2\omega^2}{L}W'(1) = 0. \tag{14}$$

It is convenient to express these equations in terms of non-dimensional parameters by elementary rearrangements as

$$\frac{\partial^4 W(\xi)}{\partial \xi^4} + \widetilde{\nu}\frac{\partial^2 W(\xi)}{\partial \xi^2} - \Omega^2 W(\xi) = p_L \exp[i\varpi t]\delta(\xi L - L) \tag{15}$$

$$W''(0) - \eta_r W'(0) = 0 \tag{16}$$

$$W'''(0) + \widetilde{\nu}W'(0) + \eta_t W(0) = 0 \tag{17}$$

$$W''(1) - \beta\Omega^2 W'(1) = 0 \tag{18}$$

$$W'''(1) + \widetilde{\nu}W'(1) + \alpha\Omega^2 W(1) = 0 \tag{19}$$

where

$$\widetilde{\nu} = \nu + \mu^2\Omega^2 \tag{20}$$

and

$$\nu = \frac{PL^2}{EI} \quad \text{(nondimensional axial force)} \tag{21}$$
$$\eta_r = \frac{k_r L}{EI} \quad \text{(nondimensional rotational end stiffness)} \tag{22}$$
$$\eta_t = \frac{k_t L^3}{EI} \quad \text{(nondimensional translational end stiffness)} \tag{23}$$
$$\Omega^2 = \omega^2 \frac{mL^4}{EI} \quad \text{(nondimensional frequency parameter)} \tag{24}$$
$$\alpha = \frac{M}{mL} \quad \text{(mass ratio)} \tag{25}$$
$$\beta = \frac{J}{mL^3} \quad \text{(nondimensional rotary inertia)} \tag{26}$$
$$\mu = \frac{r}{L} \quad \text{(nondimensional radius of gyration)} \tag{27}$$
$$p_L = \epsilon \frac{ML^4}{EI} \varpi^2 \quad \text{(normalized forcing amplitude at the top end).} \tag{28}$$

For most columns $\mu = r/L \ll 1$ so that $\mu^2 \approx 0$. As a result for low frequency vibration one expects $\widetilde{\nu} \approx \nu$. For notational convenience we define the natural frequency scaling parameter

$$f_0 = \sqrt{\frac{EI}{mL^4}}. \tag{29}$$

Using this, from equation (24) the natural frequencies of the system can be obtained as

$$\omega_j = \Omega_j f_0; \quad j = 1, 2, 3, \cdots \tag{30}$$

3 Equation of the Natural Frequencies

3.1 General Derivation

Natural frequencies of the system can be obtained from the 'free vibration problem' by considering no force on the system. Therefore, we consider $p_L = 0$ in the subsequent analysis. Assuming a solution of the form

$$W(\xi) = \exp\{\lambda\xi\} \tag{31}$$

and substituting in the equation of motion (15) results

$$\lambda^4 + \widetilde{\nu}\lambda^2 - \Omega^2 = 0. \tag{32}$$

This equation is often know as the dispersion relationship. This is the equation governing the natural frequencies of the beam. Solving this equation for λ^2 we have

$$\begin{aligned}\lambda^2 &= -\frac{\widetilde{\nu}}{2} \pm \sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + \Omega^2} \\ &= -\left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + \Omega^2} + \frac{\widetilde{\nu}}{2}\right), \quad \left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + \Omega^2} - \frac{\widetilde{\nu}}{2}\right).\end{aligned} \tag{33}$$

Because $\widetilde{\nu}^2$ and Ω^2 are always positive quantities, both roots are real with one negative and one positive root. Therefore, the four roots can be expressed as

$$\lambda = \pm \mathrm{i}\lambda_1, \quad \pm\lambda_2 \tag{34}$$

where

$$\lambda_1 = \left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + \Omega^2} + \frac{\widetilde{\nu}}{2} \right)^{1/2} \tag{35}$$

$$\text{and} \quad \lambda_2 = \left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + \Omega^2} - \frac{\widetilde{\nu}}{2} \right)^{1/2}. \tag{36}$$

From Eqs. (35) and (36) also note that

$$\lambda_1^2 - \lambda_2^2 = \widetilde{\nu}. \tag{37}$$

In view of the roots in equation (34) the solution $W(\xi)$ can be expressed as

$$\begin{aligned} W(\xi) &= a_1 \sin \lambda_1 \xi + a_2 \cos \lambda_1 \xi + a_3 \sinh \lambda_2 \xi + a_4 \cosh \lambda_2 \xi \\ \text{or} \quad W(\xi) &= \mathbf{s}^T(\xi)\mathbf{a} \end{aligned} \tag{38}$$

where the vectors

$$\mathbf{s}(\xi) = \{\sin \lambda_1 \xi, \cos \lambda_1 \xi, \sinh \lambda_2 \xi, \cosh \lambda_2 \xi\}^T \tag{39}$$

$$\text{and} \quad \mathbf{a} = \{a_1, a_2, a_3, a_4\}^T. \tag{40}$$

Applying the boundary conditions in Eqs. (16) – (19) on the expression of $W(\xi)$ in (38) we have

$$\mathbf{Ra} = \mathbf{0} \tag{41}$$

where the matrix

$$\mathbf{R} = \left[\begin{matrix} s_1''(0) - \eta_r s_1'(0) & s_2''(0) - \eta_r s_2'(0) \\ s_1'''(0) + \widetilde{\nu} s_1'(0) + \eta_t s_1(0) & s_2'''(0) + \widetilde{\nu} s_2'(0) + \eta_t s_2(0) \\ s_1''(1) - \beta\Omega^2 s_1'(1) & s_2''(1) - \beta\Omega^2 s_2'(1) \\ s_1'''(1) + \widetilde{\nu} s_1'(1) + \alpha\Omega^2 s_1(1) & s_2'''(1) + \widetilde{\nu} s_2'(1) + \alpha\Omega^2 s_2(1) \end{matrix} \right. \\ \left. \begin{matrix} s_3''(0) - \eta_r s_3'(0) & s_4''(0) - \eta_r s_4'(0) \\ s_3'''(0) + \widetilde{\nu} s_3'(0) + \eta_t s_3(0) & s_4'''(0) + \widetilde{\nu} s_4'(0) + \eta_t s_4(0) \\ s_3''(1) - \beta\Omega^2 s_3'(1) & s_4''(1) - \beta\Omega^2 s_4'(1) \\ s_3'''(1) + \widetilde{\nu} s_3'(1) + \alpha\Omega^2 s_3(1) & s_3'''(1) + \widetilde{\nu} s_3'(1) + \alpha\Omega^2 s_3(1) \end{matrix} \right]. \tag{42}$$

Substituting functions $s_j(\xi), j = 1, \cdots, 4$ from equation (39) and simplifying we obtain

$$
\mathbf{R} = \left[\begin{array}{cccc}
-\lambda_1 \eta_r & -\lambda_1^2 & -\lambda_2 \eta_r & \lambda_2^2 \\
\lambda_1^3 + \nu \lambda_1 & \eta_t & \lambda_2^3 + \nu \lambda_2 & \eta_t \\
-\sin(\lambda_1) {\lambda_1}^2 - \Omega^2 \beta \cos(\lambda_1) \lambda_1 & -\cos(\lambda_1) {\lambda_1}^2 + \Omega^2 \beta \sin(\lambda_1) \lambda_1 & \sinh(\lambda_2) {\lambda_2}^2 - \Omega^2 \beta \cosh(\lambda_2) \lambda_2 & \cosh(\lambda_2) {\lambda_2}^2 - \Omega^2 \beta \sinh(\lambda_2) \lambda_2 \\
-\cos(\lambda_1) {\lambda_1}^3 + \widetilde{\nu} \cos(\lambda_1) \lambda_1 + \Omega^2 \alpha \sin(\lambda_1) & \sin(\lambda_1) {\lambda_1}^3 - \widetilde{\nu} \sin(\lambda_1) \lambda_1 + \Omega^2 \alpha \cos(\lambda_1) & \cosh(\lambda_2) {\lambda_2}^3 + \widetilde{\nu} \cosh(\lambda_2) \lambda_2 + \Omega^2 \alpha \sinh(\lambda_2) & \sinh(\lambda_2) {\lambda_2}^3 + \widetilde{\nu} \sinh(\lambda_2) \lambda_2 + \Omega^2 \alpha \cosh(\lambda_2)
\end{array} \right]. \tag{43}
$$

The constant vector in equation (41) cannot be zero. Therefore, the equation governing the natural frequencies is given by

$$
|\mathbf{R}| = 0. \tag{44}
$$

This, upon simplification (see Appendix A for the `Maple`® code developed for this purpose) reduces to

$$
\begin{aligned}
&-{\lambda_1}^6 s_1 {\lambda_2}^4 sh_2 + 2\nu^2 {\lambda_1}^3 c_1 {\lambda_2}^3 ch_2 + 2{\lambda_1}^3 \Omega^4 \alpha c_1 {\lambda_2}^3 \beta ch_2 + 2\nu {\lambda_1}^3 c_1 {\lambda_2}^5 ch_2 - \nu {\lambda_1}^3 \Omega^4 \beta \lambda_2 \alpha \\
&+{\lambda_1}^5 \Omega^4 \beta \lambda_2 \alpha - 2{\lambda_1}^5 c_1 {\lambda_2}^5 ch_2 - 2\nu^2 {\lambda_1}^3 {\lambda_2}^3 - 2\nu {\lambda_1}^3 {\lambda_2}^5 + 2{\lambda_1}^5 {\lambda_2}^3 \nu - {\lambda_1}^5 c_1 {\lambda_2}^2 \Omega^2 \alpha sh_2 + {\lambda_1}^4 \Omega^2 \beta s_1 {\lambda_2}^5 ch_2 \\
&+{\lambda_1}^4 \Omega^4 \beta s_1 {\lambda_2}^2 \alpha sh_2 + {\lambda_1}^6 s_1 {\lambda_2}^3 \Omega^2 \beta ch_2 - {\lambda_1}^3 \Omega^2 \alpha c_1 {\lambda_2}^4 sh_2 + \nu {\lambda_1}^3 c_1 {\lambda_2}^2 \Omega^2 \alpha sh_2 - \nu {\lambda_1}^2 \Omega^2 \beta s_1 {\lambda_2}^5 ch_2 \\
&-\nu {\lambda_1}^4 s_1 {\lambda_2}^3 \Omega^2 \beta ch_2 - \nu^2 {\lambda_1}^2 s_1 {\lambda_2}^4 sh_2 + \nu \lambda_1 \Omega^2 \alpha c_1 {\lambda_2}^4 sh_2 - \nu \lambda_1 \Omega^4 \alpha c_1 {\lambda_2}^3 \beta ch_2 \\
&+ s_1 {\lambda_1}^4 {\lambda_2}^3 \Omega^2 \alpha ch_2 + s_1 {\lambda_1}^4 \nu^2 {\lambda_2}^2 sh_2 + \Omega^2 \alpha s_1 {\lambda_1}^2 {\lambda_2}^5 ch_2 - \Omega^4 \alpha s_1 {\lambda_1}^2 {\lambda_2}^4 \beta sh_2 \\
&+c_1 {\lambda_1}^5 {\lambda_2}^4 \Omega^2 \beta sh_2 + c_1 {\lambda_1}^5 \nu {\lambda_2}^2 \Omega^2 \beta sh_2 + \Omega^2 \beta c_1 {\lambda_1}^3 {\lambda_2}^4 \nu sh_2 + \Omega^4 \beta c_1 {\lambda_1}^3 \nu \lambda_2 \alpha ch_2 + s_1 {\lambda_1}^4 \nu \lambda_2 \Omega^2 \alpha ch_2 \\
&+ \Omega^2 \beta c_1 {\lambda_1}^3 {\lambda_2}^6 sh_2 + \Omega^2 \alpha s_1 {\lambda_1}^2 \nu {\lambda_2}^3 ch_2 + 2{\lambda_1}^5 {\lambda_2}^5 + \Big(\big(\lambda_1 \Omega^2 \alpha c_1 sh_2 {\lambda_2}^2 - 2{\lambda_1}^2 \nu s_1 sh_2 {\lambda_2}^2 \\
&+{\lambda_1}^3 c_1 \nu ch_2 \lambda_2 + {\lambda_1}^5 \lambda_2 - {\lambda_1}^4 s_1 \Omega^2 \beta ch_2 \lambda_2 + 2{\lambda_1}^3 c_1 ch_2 {\lambda_2}^3 - 2\lambda_1 \Omega^4 \alpha c_1 \beta ch_2 \lambda_2 - {\lambda_1}^2 \Omega^4 \beta s_1 \alpha sh_2 \\
&- s_1 {\lambda_2}^3 \Omega^2 \alpha ch_2 + 2\lambda_1 \Omega^4 \beta \lambda_2 \alpha + \Omega^4 \beta s_1 {\lambda_2}^2 \alpha sh_2 - s_1 {\lambda_1}^2 {\lambda_2}^4 sh_2 + {\lambda_1}^3 c_1 \Omega^2 \alpha sh_2 - s_1 {\lambda_1}^2 \lambda_2 \Omega^2 \alpha ch_2 \\
&-{\lambda_1}^3 \nu \lambda_2 - c_1 {\lambda_1}^3 {\lambda_2}^2 \Omega^2 \beta sh_2 - \Omega^2 \beta c_1 \lambda_1 {\lambda_2}^4 sh_2 + \lambda_1 {\lambda_2}^3 \nu + {\lambda_1}^4 s_1 sh_2 {\lambda_2}^2 + \lambda_1 {\lambda_2}^5 - {\lambda_1}^2 \Omega^2 \beta s_1 ch_2 {\lambda_2}^3 \\
&-\nu c_1 \lambda_1 {\lambda_2}^3 ch_2 \big) \eta_t + \lambda_1 \Omega^4 \alpha c_1 {\lambda_2}^4 \beta sh_2 - {\lambda_1}^3 c_1 {\lambda_2}^4 \nu sh_2 - \lambda_1 \Omega^2 \alpha c_1 {\lambda_2}^5 ch_2 - {\lambda_1}^5 c_1 {\lambda_2}^2 \nu sh_2 \\
&- 2{\lambda_1}^3 c_1 {\lambda_2}^3 \Omega^2 \alpha ch_2 - {\lambda_1}^3 c_1 {\lambda_2}^6 sh_2 + 2{\lambda_1}^4 s_1 {\lambda_2}^4 \Omega^2 \beta sh_2 + {\lambda_1}^2 \nu s_1 {\lambda_2}^5 ch_2 + {\lambda_1}^6 s_1 {\lambda_2}^2 \Omega^2 \beta sh_2 \\
&+ {\lambda_1}^3 \Omega^4 \alpha c_1 {\lambda_2}^2 \beta sh_2 + {\lambda_1}^2 \Omega^4 \beta s_1 {\lambda_2}^3 \alpha ch_2 + {\lambda_1}^4 s_1 \nu {\lambda_2}^3 ch_2 - {\lambda_1}^4 s_1 {\lambda_2}^5 ch_2 + {\lambda_1}^4 \Omega^4 \beta s_1 \lambda_2 \alpha ch_2 \\
&-{\lambda_1}^5 c_1 {\lambda_2}^4 sh_2 + {\lambda_1}^2 \Omega^2 \beta s_1 {\lambda_2}^6 sh_2 - {\lambda_1}^6 s_1 {\lambda_2}^3 ch_2 - {\lambda_1}^5 c_1 \lambda_2 \Omega^2 \alpha ch_2 \Big) \eta_r + \Omega^4 \alpha \beta \lambda_1 {\lambda_2}^5 + \Omega^4 \alpha \beta \lambda_1 {\lambda_2}^3 \nu \\
&+ \big(-s_1 {\lambda_1}^2 {\lambda_2}^5 ch_2 - \Omega^4 \beta c_1 {\lambda_1}^3 \alpha sh_2 + \Omega^4 \alpha s_1 {\lambda_1}^2 \beta ch_2 \lambda_2 - s_1 {\lambda_1}^4 ch_2 {\lambda_2}^3 - s_1 {\lambda_1}^2 \nu {\lambda_2}^3 ch_2 \\
&- \Omega^2 \beta c_1 \lambda_1 {\lambda_2}^5 ch_2 - \Omega^4 \beta c_1 \lambda_1 {\lambda_2}^2 \alpha sh_2 + \Omega^4 \alpha s_1 {\lambda_2}^3 \beta ch_2 - \Omega^2 \alpha s_1 {\lambda_2}^4 sh_2 - \nu c_1 \lambda_1 {\lambda_2}^4 sh_2 \\
&-2 s_1 {\lambda_1}^2 {\lambda_2}^2 \Omega^2 \alpha sh_2 - s_1 {\lambda_1}^4 \nu ch_2 \lambda_2 - \nu c_1 {\lambda_1}^3 sh_2 {\lambda_2}^2 + c_1 {\lambda_1}^3 {\lambda_2}^4 sh_2 + c_1 {\lambda_1}^5 sh_2 {\lambda_2}^2 - c_1 {\lambda_1}^5 \Omega^2 \beta ch_2 \lambda_2 \\
&- s_1 {\lambda_1}^4 \Omega^2 \alpha sh_2 - 2\Omega^2 \beta c_1 {\lambda_1}^3 ch_2 {\lambda_2}^3 \big) \eta_t + s_1 {\lambda_1}^4 {\lambda_2}^6 sh_2 - 2\nu {\lambda_1}^2 \Omega^4 \beta s_1 {\lambda_2}^2 \alpha sh_2 \\
&+ 4{\lambda_1}^4 \nu s_1 {\lambda_2}^4 sh_2 - 2{\lambda_1}^5 c_1 \nu {\lambda_2}^3 ch_2 = 0.
\end{aligned} \tag{45}
$$

where

$$
s_1 = \sin(\lambda_1), c_1 = \cos(\lambda_1), sh_2 = \sinh(\lambda_2), ch_2 = \cosh(\lambda_2). \tag{46}
$$

The natural frequencies can be obtained by solving equation (45) for Ω. Due to the complexity of this transcendental equation it should be solved numerically. Equation (45) is shown not to scare you, but to show the complicated nature of the frequency equation even under the simplifying assumptions discussed before. If we aim to relax any of the assumptions (e.g., variable bending stiffness), this equation is likely to be even more complicated. Luckily, equation (45) can be translated to `Matlab`® automatically and can be solved numerically.

3.2 Special Cases

Equation (45) is quite general as they consider axial load, elastic end restraints, mass and rotary inertia of the rotor. Several interesting special cases can be obtained from this expression.

- *Standard cantilever column: fixed support without any top mass, rotary inertia and axial force:*
  For this case $\beta = 0$, $\alpha = 0$, $\mu = 0$ and $\nu = 0$. From the dispersion relationship in (32) observe that for this case $\lambda_1 = \lambda_2 = \sqrt{\Omega} = \omega\sqrt{\frac{mL^4}{EI}} = \lambda$ (say). Since the bottom of the column is fixed, the stiffness parameters η_r and η_k approach to infinity. Substituting these in equation (45) and simplifying (see the `Maple`® script in Appendix A) we obtain the frequency equation as
  $$1 + \cos(\lambda)\cosh(\lambda) = 0. \tag{47}$$
  This matches exactly with the frequency equation for a standard cantilever (see the book by Meirovitch [21] or Géradin and Rixen [10]).

- *Cantilever column with a top mass: fixed support without any rotary inertia and axial force:*
  For this case $\eta = 0$, $\beta = 0$, and $\nu = 0$. From the dispersion relationship in (32) observe that for this case again $\lambda_1 = \lambda_2 = \lambda$ (say). Substituting these in equation (45), taking the limit $\eta_r, \eta_t \to \infty$ and simplifying (see the `Maple`® script in Appendix A) we obtain the frequency equation as
  $$(-\sin(\lambda)\cosh(\lambda) + \cos(\lambda)\sinh(\lambda))\,\alpha\,\lambda + \cos(\lambda)\cosh(\lambda) + 1 = 0. \tag{48}$$
  If we substitute $\alpha = 0$ in this equation, we retrieve the standard cantilever case obtained in equation (47).

- *Cantilever column with a top mass and rotary inertia: fixed support without axial force:*
  For this case only $\eta = 0$ and $\nu = 0$ and we also have $\lambda_1 = \lambda_2 = \lambda$ (say). Substituting these in equation (45), taking the limit $\eta_r, \eta_t \to \infty$ and simplifying (see the `Maple`® script in Appendix A) we obtain the frequency equation as
  $$\begin{aligned}(-\cos(\lambda)\cosh(\lambda) + 1)\,\alpha\,\beta\,\lambda^4 + (-\cos(\lambda)\sinh(\lambda) - \sin(\lambda)\cosh(\lambda))\,\beta\,\lambda^3 \\ + (-\sin(\lambda)\cosh(\lambda) + \cos(\lambda)\sinh(\lambda))\,\alpha\,\lambda + \cos(\lambda)\cosh(\lambda) + 1 = 0.\end{aligned} \tag{49}$$
  If we substitute $\beta = 0$ in this equation, we retrieve the case obtained in equation (48).

4 Numerical Example

In this section we aim to understand the analytical expressions developed in the last section. First we determine the relevant non-dimensional parameters in the equations derived here. We focus our attention on the affect of ν and the non-dimensional soil rotational stiffness η_r on the first-natural frequency. For this reason, the numerical results are presented as a function of ν and η_r. These two parameters are considered because the boundary condition and the load of the turbine are the crucial design issues for the overall system.

The non-dimensional mass ratio can be obtained as

$$\alpha = \frac{M}{mL} = \frac{P}{gmL} = \frac{PL^2}{EI}\left(\frac{EI}{gmL^3}\right) = \nu\left(\frac{EI}{mL^4}\right)L/g = \nu f_0^2 L/g \tag{50}$$

We consider the rotary inertia of the blade assembly $J = 0$. This is not a very bad assumption if there are very less misalignment.

In this example we have used the data of a wind turbine given in reference [26]. The numerical values of the main parameters are summarised in Table 1 The moment of inertia of the circular

Turbine Structure Properties	Numerical values
Length (L)	81 m
Average diameter (D)	3.5m
Thickness (t_h)	0.075 mm
Mass density (ρ)	7800 kg/m^3
Young's modulus (E)	2.1×10^{11} Pa
Mass density (ρ_l)	7800 kg/m^3
Rotational speed (ϖ)	22 r.p.m = 0.37 Hz
Top mass (M)	130,000 kg
Rated power	3 MW

Table 1: Material and geometric properties of the turbine structure

cross section can be obtained as

$$I = \frac{\pi}{64}D^4 - \frac{\pi}{64}(D - t_h)^4 \approx \frac{1}{16}\pi D^3 t_h = 0.6314m^4 \tag{51}$$

The mass density per unit length of the system can be obtained as

$$m = \rho A \approx \rho\pi D t_h/2 = 3.1817 \times 10^3 \text{kg/m} \tag{52}$$

Using these, the mass ratio $\alpha = 0.2495$ and the nondimensional axial force $\nu = 0.0652$. We also obtain the natural frequency scaling parameter can be obtained as

$$f_0 = \frac{EI}{mL^4} = 0.9682\,\text{s}^{-1}. \tag{53}$$

Since no information on the rotary inertia of the blade assembly is given, we consider $J = 0$. This approximation is likely cause insignificant error for the first natural frequency. The radius of gyration of the wind turbine is given by

$$r = \sqrt{\frac{I}{A}} = \frac{1}{4}\sqrt{D^2 + (D - t_h)^2} \approx \frac{D}{2\sqrt{2}} = 1.2374m \tag{54}$$

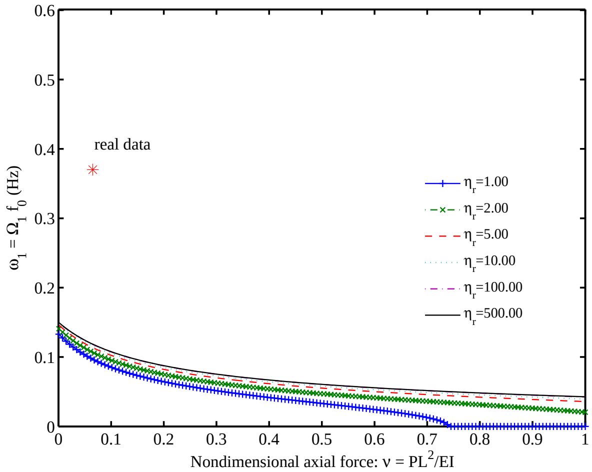
Therefore, the nondimensional radius of gyration $\mu = r/L = 0.0151$. From equation (20) we therefore have

$$\widetilde{\nu} = \nu + 2.2844 \times 10^{-4}\Omega^2 \approx \nu \tag{55}$$

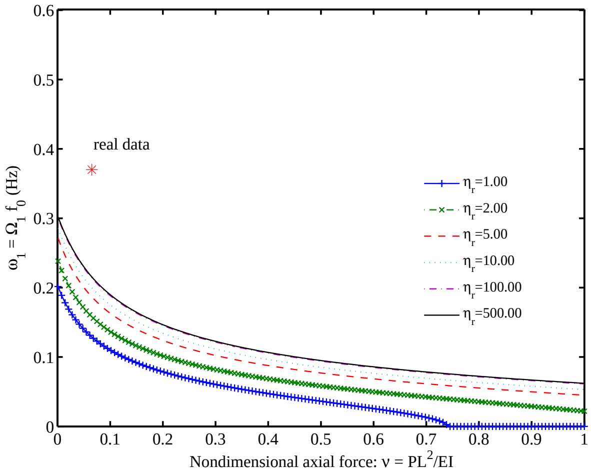
The values of the soil stiffness parameters were not given. For the fixed support, the values of

η_r and η_t will be infinity. Therefore, we will use η_r and η_t as variable parameters and try to understand how they affect the overall behavior of the system.

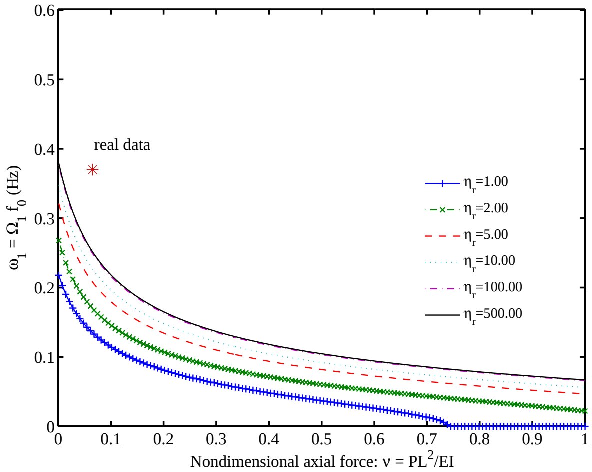
We substitute the derived constants in Eqs. (35) and (36) to obtain λ_1 and λ_2. Substituting them in equation (45) we solve for the nondimensional first natural frequency Ω_1 in `Matlab`®. For most applications the first natural frequency is the most important as the excitation frequency is generally between 1/3-10 Hz and close to the blade passing frequency. Higher natural frequencies can however be obtained by solving equation (45).



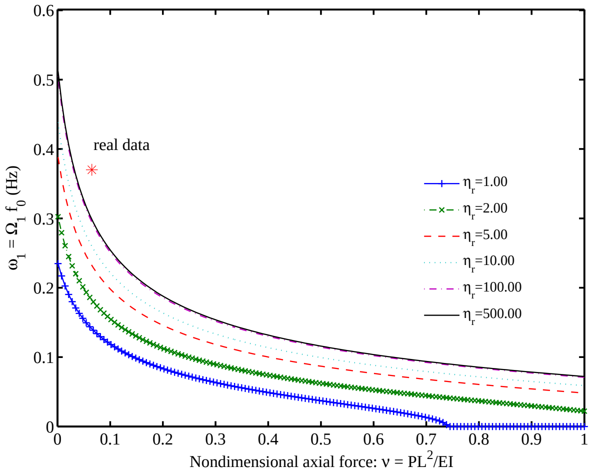
(a) $\eta_t = 1$



(b) $\eta_t = 5$



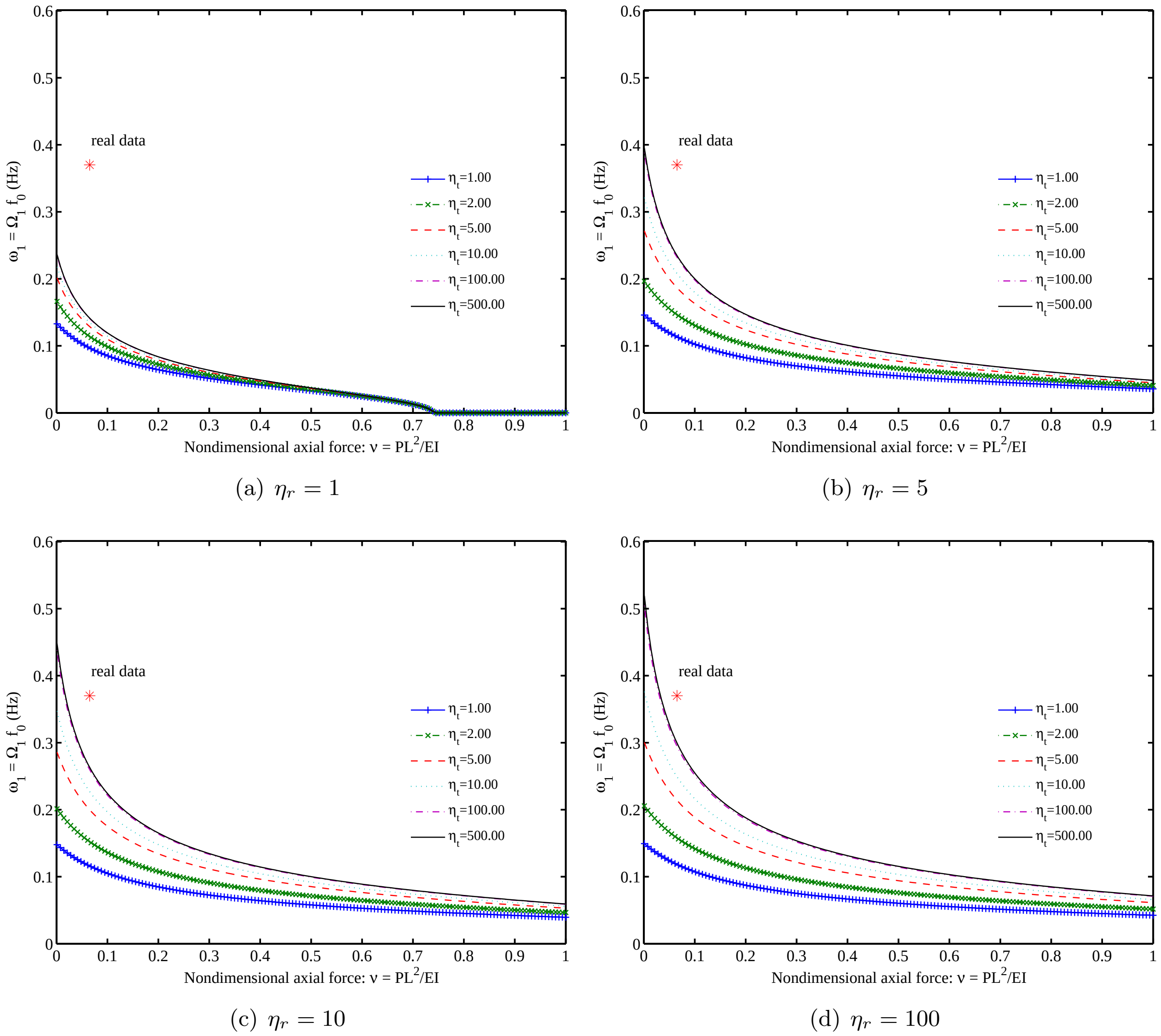
(c) $\eta_t = 10$



(d) $\eta_t = 100$

Figure 2: The variation of the first natural frequency of the wind turbine with respect to the nondimensional axial load ν for different values of nondimensional rotational soil stiffness η_r. Four fixed values of the nondimensional translational stiffness η_t are considered in the four subplots. The data from the example ($\varpi = 0.37$Hz and $\nu = 0.0315$) is shown by a '*' in the diagram.

The variation of the first natural frequency of the wind turbine with respect to the axial load for different values of nondimensional rotational soil stiffness and four fixed values of the nondimensional translational stiffness are shown in Figure 2. A similar plot, but this time for different values of η_r is shown in Figure 3 for furhter understanding. Since these plots are in



(a) $\eta_r = 1$

(b) $\eta_r = 5$

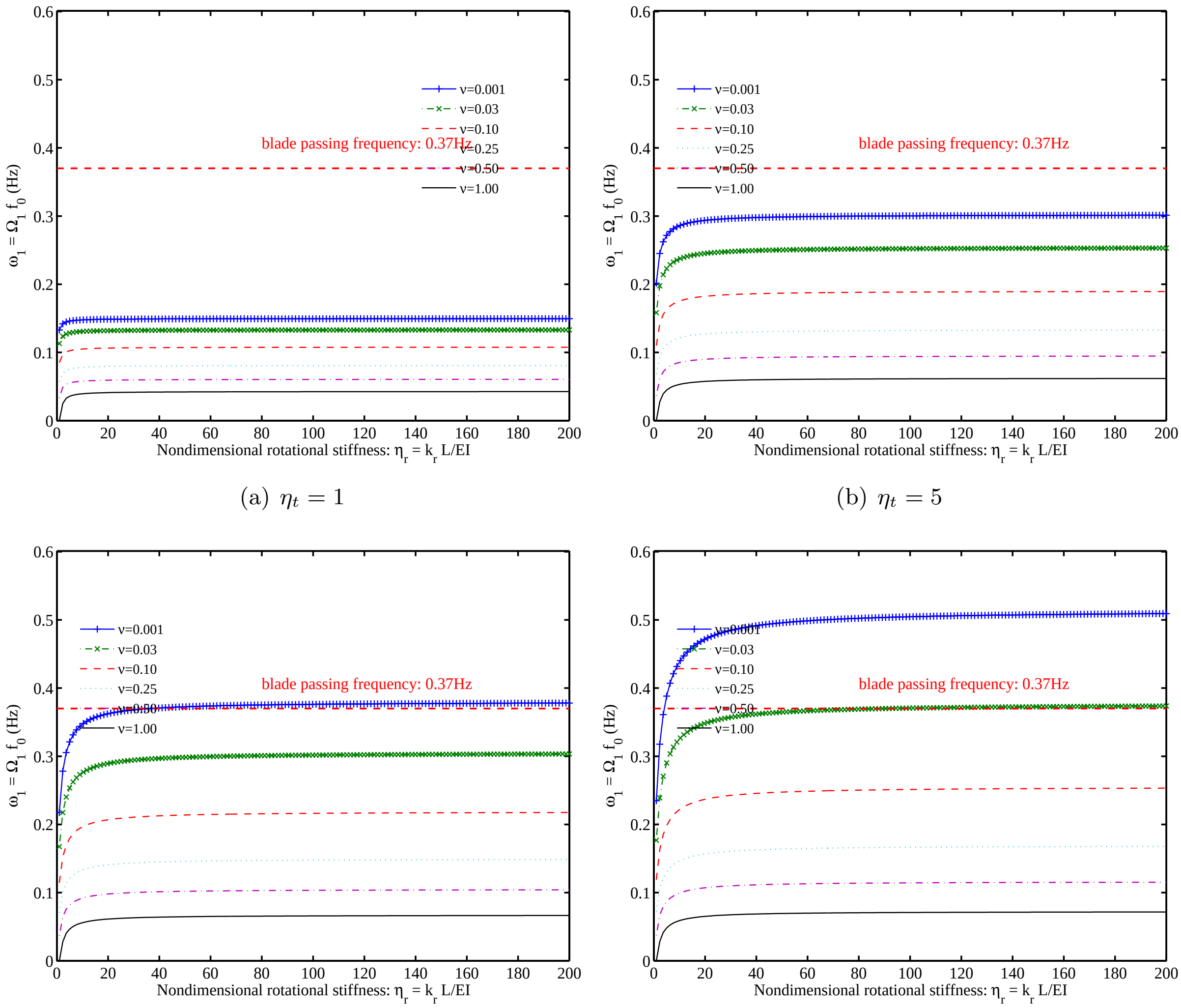
(c) $\eta_r = 10$

(d) $\eta_r = 100$

Figure 3: The variation of the first natural frequency of the wind turbine with respect to the nondimensional axial load ν for different values of nondimensional translational soil stiffness η_t. Four fixed values of the nondimensional rotational stiffness η_r are considered in the four subplots. The data from the example ($\varpi = 0.37$Hz and $\nu = 0.0315$) is shown by a '*' in the diagram.

terms of generalized non-dimensional quantities, they are applicable to *any* turbine structures which can be modeled within the scope of the theory discussed here. The natural frequency of the system increases with the increasing values of the stiffness parameters η_r and η_t. This is due to the fact that the increase in the rotational and translational stiffness properties stabilizes the system. Note that after certain values of η_r and η_t (typically above 100), a further increase in their values do not change the natural frequency. This is because after these values, the soil structure interaction can be essentially considered as fixed so that further increase in η_r and/or η_t has no effect. The natural frequency of the system decreases with the increasing value of ν. This expected as the increase in the downward axial force essentially drives the system closer to buckling. In fact the zero natural frequency corresponding to very small value of the rotational stiffness ($\eta_r \approx 1$) shown in the figures suggests that for that particular configuration the system has actually buckled. In these figures we have plotted the real data corresponding to the system (marked by '*'). The design should be such that the natural frequency of the system should avoid the blade passing frequency.

The variation of the first natural frequency of the wind turbine with respect to the nondimensional rotational soil stiffness for different values of axial force and four fixed values of the nondimensional translational stiffness is shown in Figure 4. A similar plot, showing the variation

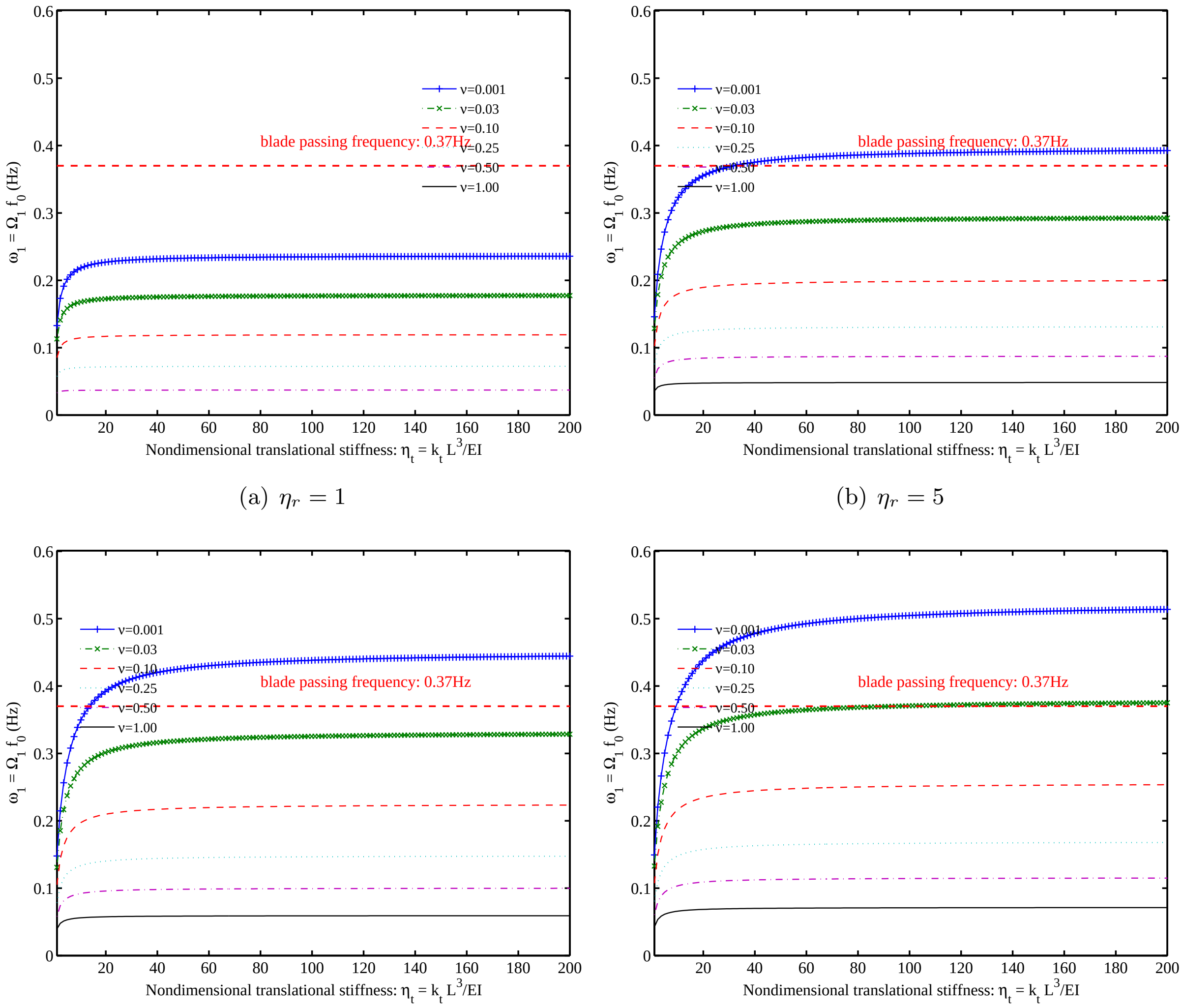


(a) $\eta_t = 1$ (b) $\eta_t = 5$

(c) $\eta_t = 10$ (d) $\eta_t = 100$

Figure 4: The variation of the first natural frequency of the wind turbine with respect to the nondimensional rotational soil stiffness η_r for different values of nondimensional axial load ν. Four fixed values of the nondimensional translational stiffness η_t are considered in the four subplots. The bade passing frequency $\varpi = 0.37$Hz is shown by a dashed line in the diagram.

of the first natural frequency with respect to the nondimensional translational soil stiffness for different values of axial force and four fixed values of the nondimensional rotational stiffness is given by Figure 5. In the same diagrams the bade passing frequency is shown by a dashed line. The design should be such that the natural frequency of the system must always be far from this line. From this criteria one can decide which parameter values should be avoided for a safe



(a) $\eta_r = 1$

(b) $\eta_r = 5$

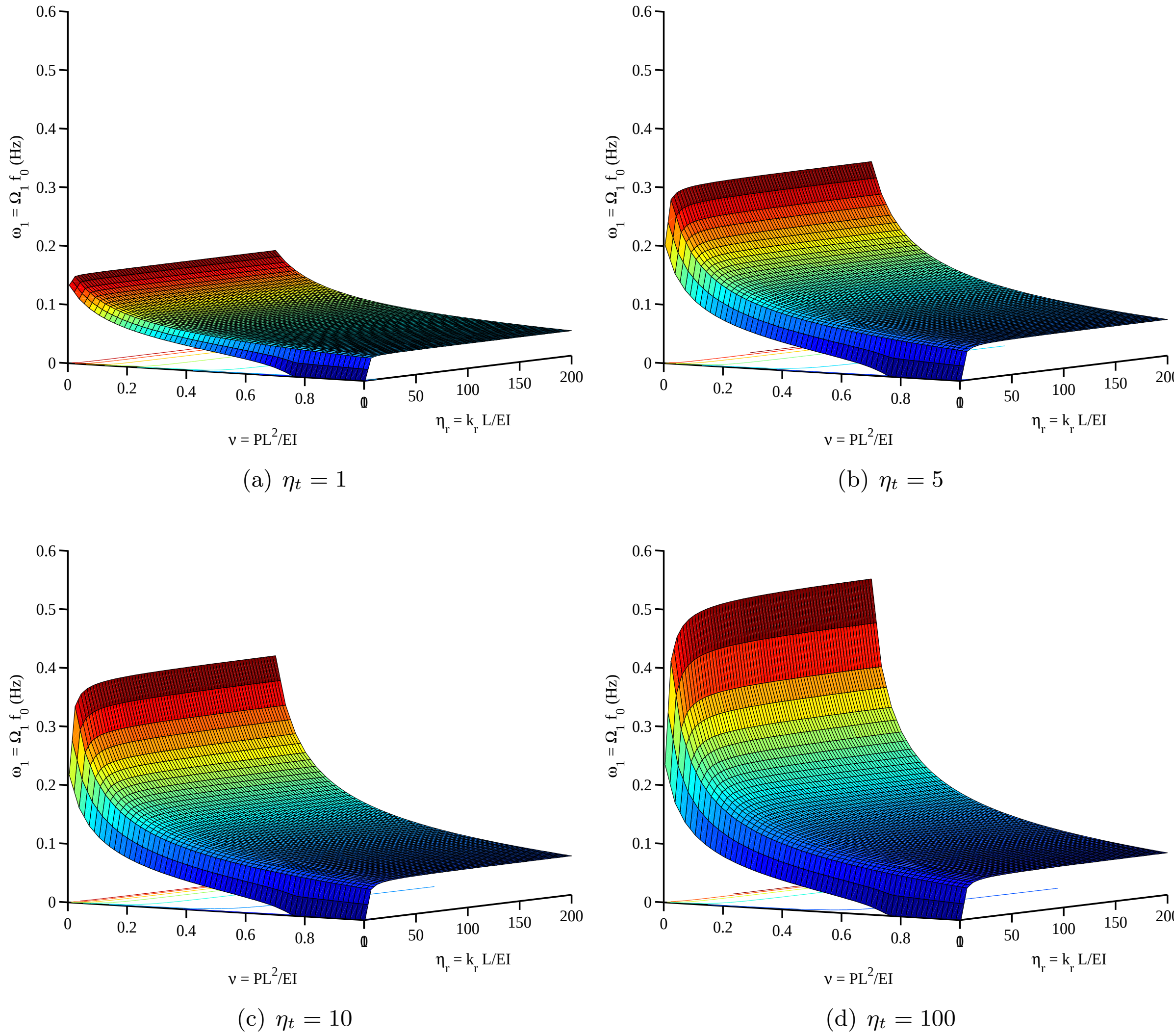
(c) $\eta_r = 10$

(d) $\eta_r = 100$

Figure 5: The variation of the first natural frequency of the wind turbine with respect to the nondimensional translational stiffness η_t for different values of nondimensional axial load ν. Four fixed values of the nondimensional rotational soil stiffness η_r are considered in the four subplots. The bade passing frequency $\varpi = 0.37$Hz is shown by a dashed line in the diagram.

design.

In Figure 6, the overall variation of the first natural frequency of the wind turbine with respect to both nondimensional rotational soil stiffness and axial load is shown in a 3D plot. Four fixed values of the nondimensional translational stiffness η_t are considered in the four subplots. The interesting feature to observe from this plot are (a) the rapid and sharp 'fall' in the natural frequency for small values of η_r and relative flatness for values of η_r approximately over 50, and (b) extremely high sensitivity for lower values of ν. The parameter η_t has an overall 'scaling effect' of the natural frequency. Higher values of the stiffness corresponds to higher values of the natural frequency as expected. In Figure 7, the overall variation of the first natural frequency with respect to both the nondimensional soil stiffness parameters is shown in a 3D plot. Four fixed values of the nondimensional axial load ν are considered in the four subplots. It can be



(a) $\eta_t = 1$ (b) $\eta_t = 5$

(c) $\eta_t = 10$ (d) $\eta_t = 100$

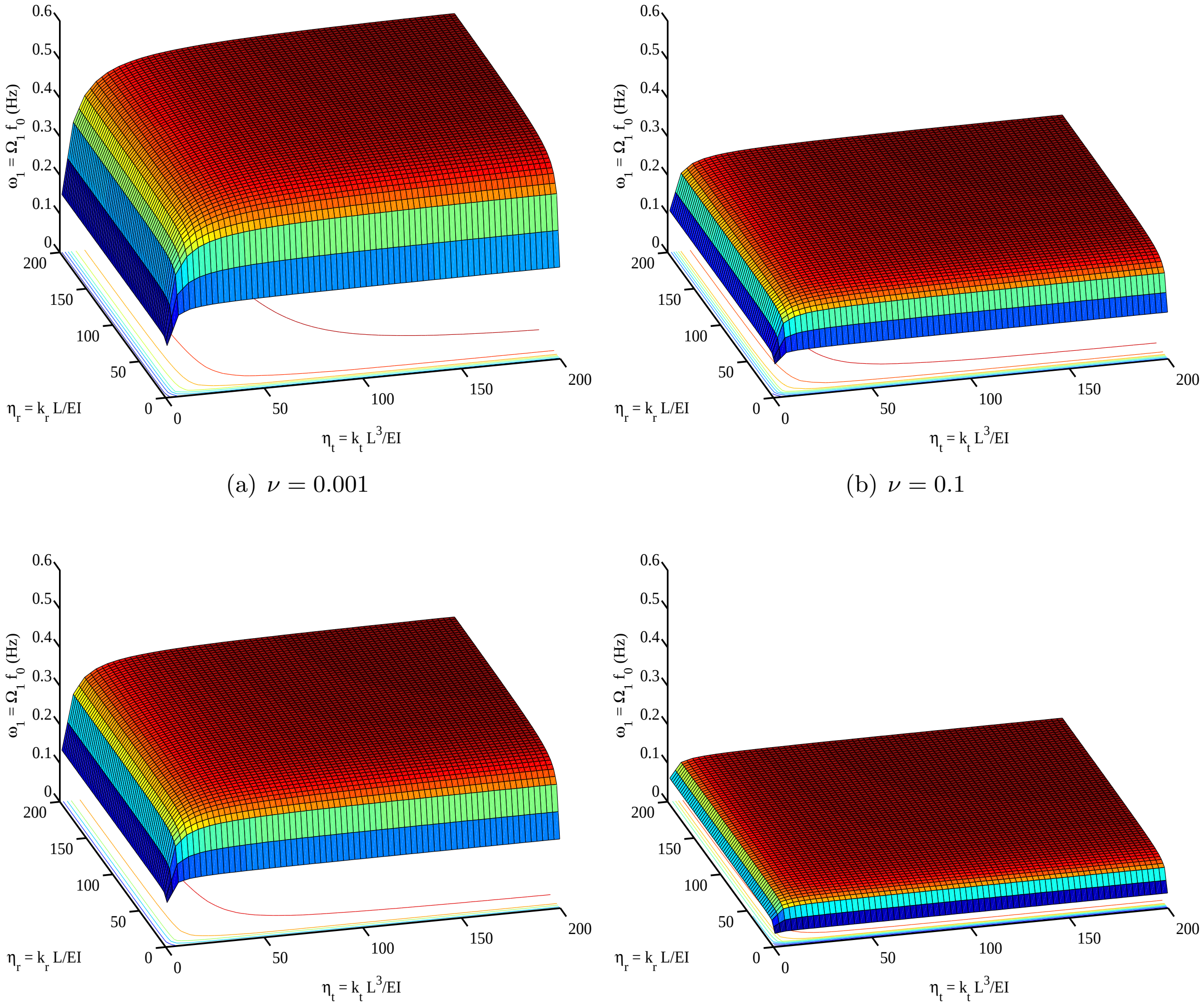
Figure 6: The variation of the first natural frequency of the wind turbine with respect to the nondimensional axial load ν and nondimensional rotational soil stiffness η_r. Four fixed values of the nondimensional translational stiffness η_t are considered in the four subplots.

seen that lower values of ν corresponds to higher values of the natural frequency as seen in the previous figures. These plots can be used to understand the overall design of the system.

5 Approximate Natural Frequency Based on SDOF Assumption

In subsection 3.1 an exact expression for the equation of the natural frequencies of the wind turbine has been derived. Since only numerical solutions to the transcendental equation (45) are available, in this section we aim to derive an approximate closed-form solution with the aim of gaining more physical insights and simplicity. For wind and marine turbines, the first mode of vibration is the most significant. As a result, we focus our attention on the first mode only.

In the first mode we can replace the distributed system by a single-degree-of-freedom (SDOF) system with equivalent stiffness k_e and equivalent mass M_e as shown in Figure 8. The first



(a) $\nu = 0.001$ (b) $\nu = 0.1$

(c) $\nu = 0.03$ (d) $\nu = 0.5$

Figure 7: The variation of the first natural frequency of the wind turbine with respect to the nondimensional translational stiffness η_t and nondimensional rotational soil stiffness η_r. Four fixed values of the nondimensional axial load ν are considered in the four subplots.

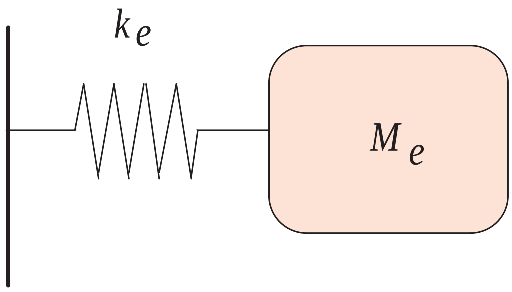


Figure 8: Equivalent single-degree-of-freedom system for the first bending mode of the turbine structure.

natural frequency is given by

$$\omega_1^2 = \frac{k_e}{M_e} \tag{56}$$

Following [Table 8-8, case 1, page 158, 4] for a cantilever column one has

$$M_e = M + 0.24 M_b = (\alpha + 0.24) mL \tag{57}$$

For our case the column is standing on elastic springs and also have an axial force. Therefore the coefficient 0.24 needs to be modified to take these effects into account. We suppose that the equivalent mass can be represented by

$$M_e = (\alpha + \gamma_m) mL \tag{58}$$

where γ_m is the mass correction factor.

It is useful to express k_e normalized by the stiffness term, $k_{CL} = EI/L^3$. Therefore, the first natural frequency can be expressed as

$$\omega_1^2 \approx \frac{k_e}{M_e} = \frac{k_e}{k_{CL}} \frac{EI/L^3}{(\alpha + \gamma_m) mL} = \frac{EI}{mL^4} \frac{\gamma_k}{(\alpha + \gamma_m)} \tag{59}$$

$$\text{or} \quad \omega_1 \approx f_0 \sqrt{\frac{\gamma_k}{\alpha + \gamma_m}} \tag{60}$$

where the stiffness correction factor γ_k is defined as

$$\gamma_k = \frac{k_e}{k_{CL}}. \tag{61}$$

We only need to obtain γ_k and γ_m in order to apply the expression of the first natural frequency in equation (59).

These correction factors can be obtained using the procedure developed in reference [3]. Here we avoid the details of the derivation and give the final results only as:

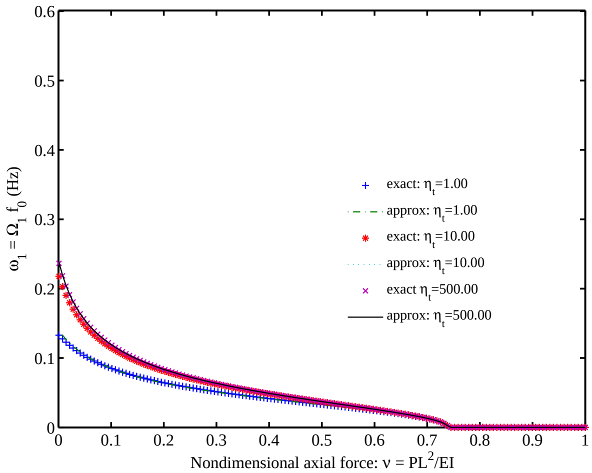
$$\begin{aligned} \gamma_k &= \frac{\lambda^3 \eta_t \left(\eta_r \cos(\lambda) - \lambda \sin(\lambda)\right)}{\eta_r \eta_t \left(\sin(\lambda) - \lambda \cos(\lambda)\right) + \lambda^2 \left(\eta_t \sin(\lambda) + \eta_r \cos(\lambda) - \lambda^2 \sin(\lambda)\right)} \\ \text{and} \quad \gamma_m &= \frac{3}{140} \frac{11\,{\eta_r}^2{\eta_t}^2 + 77\,{\eta_t}^2\eta_r + 105\,{\eta_r}^2\eta_t + 140\,{\eta_t}^2 + 420\,\eta_r\eta_t + 420\,{\eta_r}^2}{9\,{\eta_r}^2 + 6\,{\eta_r}^2\eta_t + 18\,\eta_r\eta_t + {\eta_r}^2{\eta_t}^2 + 6\,{\eta_t}^2\eta_r + 9\,{\eta_t}^2} \end{aligned} \tag{62}$$

where

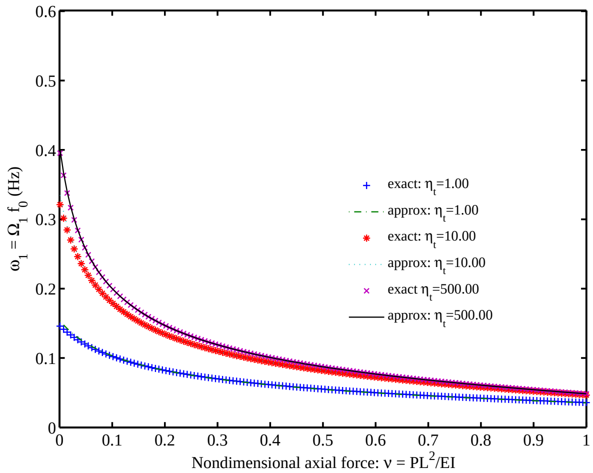
$$\lambda = \sqrt{\nu} \tag{63}$$

Substituting these expressions in the approximate formula (60) gives the complete parametric variation of the first-natural frequency in terms of ν, α, η_r and η_t. The resulting expression is simple enough to program in software like `MS Excel`®.

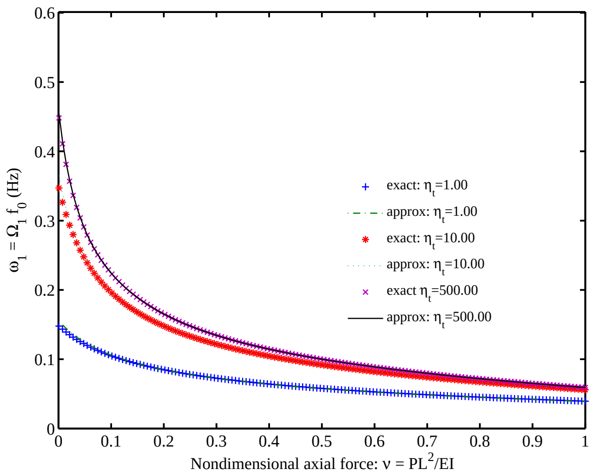
To understand the accuracy of the derived approximate expression, we compare it to the numerical solution of the exact transcendental equation (45). Approximation of the first natural frequency of the wind turbine with respect to the nondimensional axial load for different values of nondimensional translational soil stiffness and four fixed values of the nondimensional rotational stiffness are shown in Figure 9. A similar plot, showing the approximation of the first natural frequency with respect to the nondimensional translational soil stiffness for different values of axial force and four fixed values of the nondimensional rotational stiffness is given by Figure 10. Both these plots show that the results from this simple closed-form expression is in excellent agreement from the results obtained via numerical solution of the complex transcen-



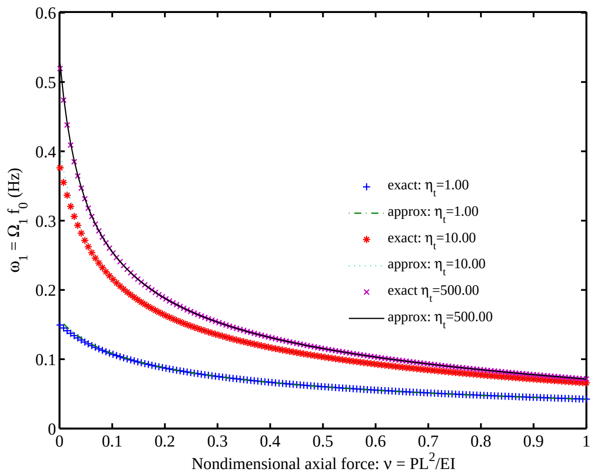
(a) $\eta_r = 1$



(b) $\eta_r = 5$



(c) $\eta_r = 10$



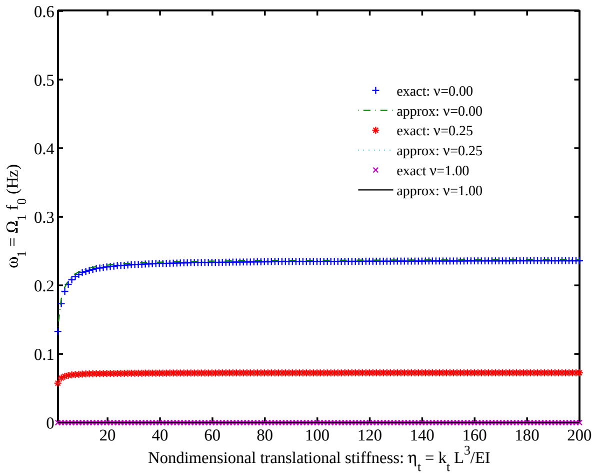
(d) $\eta_r = 100$

Figure 9: Approximation of the first natural frequency of the wind turbine with respect to the nondimensional axial load ν for different values of nondimensional translational soil stiffness η_t. Four fixed values of the nondimensional rotational stiffness η_r are considered in the four subplots.

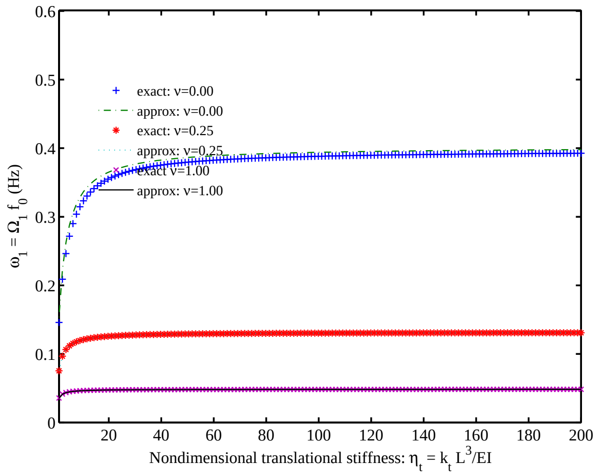
dental frequency equation over the wide range of parameter values considered. Therefore, the formula shown in equation (60) should be used for all practical purposes.

6 Conclusions

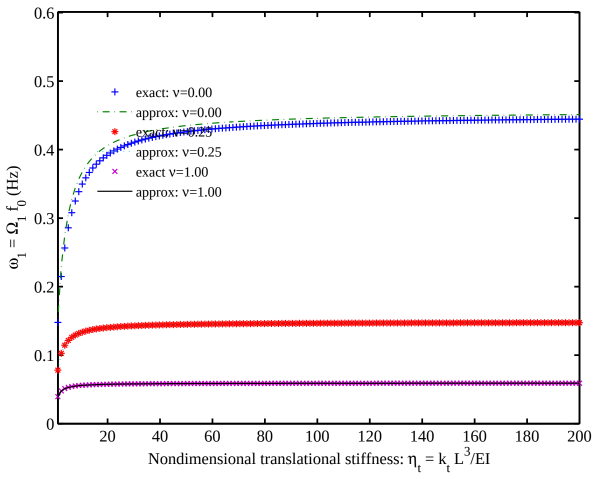
Dynamics of flexible turbine structures on elastic end support has been investigated. A distributed parameter model using the Euler Bernoulli beam theory with axial load, elastic support stiffness and top mass with rotary inertia is considered. The physical assumptions behind the simplified model have been explained in details. The non-dimensional parameters necessary to understand the dynamic behavior have been identified. These parameters are nondimensional axial force (ν), nondimensional rotational soil stiffness, (η_r), nondimensional translational soil stiffness, (η_t), mass ratio between the building and the turbine (α), nondimensional radius of



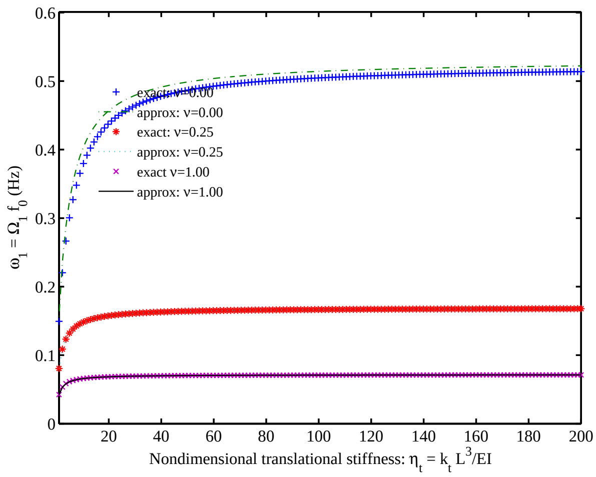
(a) $\eta_r = 1$



(b) $\eta_r = 5$



(c) $\eta_r = 10$



(d) $\eta_r = 100$

Figure 10: Approximation of the first natural frequency of the wind turbine with respect to the nondimensional translational stiffness η_t for different values of nondimensional axial load ν. Four fixed values of the nondimensional rotational soil stiffness η_r are considered in the four subplots.

gyration of the turbine (μ). The characteristic equation governing the natural frequency of the system by solving the associated eigenvalue problem. Necessary computer codes in `Maple`® and `Matlab`® have been developed to solve the equation characteristic equation. Some well recognized special cases arising from the general approach have been discussed for improved understanding.

The analytical results derived here are illustrated by numerical examples. One of the key conclusion is that the first natural frequency of the turbine structure will decrease with the decrease in the stiffness properties of the (soil) support and increase in the axial load in the column. This means that one needs to check the condition of the underlying soil and weight of the rotor-blade assembly. The first natural frequency of the system should be well separated from the the blade passing frequency. Based on an equivalent single-degree-of-freedom system assumption, a simple approximate expression of the first natural frequency is given. Numerical verifications confirm that the results from this simple closed-form expression is in excellent agreement from the results obtained via numerical solution of the complex transcendental frequency equation over a wide range of parameter values. Using this expressions, designers could estimate the first natural frequency for various parameter values and design the turbine structure such that the resulting natural frequency does not come close to the blade passing frequency.

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A Maple® Code for the Derivation of the Frequency Equation

Here we show the code in `Maple`® [22] used to generate the equation for the natural frequencies, that is equation (45). Codes used to obtain the special cases discussed in subsection 3.2 are also given below.

```
#-------------------------------------------------------------------
# Program for calculating the symbolic expression of the eigenvalue
# equation of a beam-column with axial load supported by rotation and
# translational springs. The beam-column has a mass at the free end
# with rotary inertia.
# Developed by Prof S Adhikari: January 2009, Swnasea
#-------------------------------------------------------------------
with(combinat):with(linalg):with(CodeGeneration):

R:=array(1..4,1..4):
s:=array(1..4,[sin(lambda[1]*x),cos(lambda[1]*x),
               sinh(lambda[2]*x),cosh(lambda[2]*x)]);
for jj from 1 to 4 do
    R[1,jj]:=subs(x=0,diff(s[jj],x$2) -eta[r]*diff(s[jj],x));
    R[2,jj]:=subs(x=0,diff(s[jj],x$3) +nu*diff(s[jj],x) +eta[r]*s[jj]);
    R[3,jj]:=subs(x=1,diff(s[jj],x$2) -Omega^2*beta*diff(s[jj],x));
    R[4,jj]:=subs(x=1,diff(s[jj],x$3) +nu*diff(s[jj],x) +Omega^2*alpha*s[jj]);
od:
evalm(R);
feq1:=collect(simplify(det(R)),{alpha,nu});

#                               SPECIAL CASES
# no top mass, rotary inertia & axial force
feqSC1:=subs(lambda[1]=lambda,lambda[2]=lambda,
        beta=0,alpha=0,nu=0,feq1):
collect(simplify(feqSC1/(2*lambda^6)),{eta[r],eta[t]})=0;

# Standard cantilever column: fixed support without any top
# mass, rotary inertia and axial force
t1:=collect(simplify(feqSC1/(2*lambda^6)),{eta[r],eta[t]})=0:
t2:=limit(t1,eta[r]=infinity):
limit(t2,eta[t]=infinity);

# Cantilever column with a top mass: fixed support without any
```

```
# rotary inertia and axial force
feqSC2:=subs(Omega=lambda^2,nu=0,
            lambda[1]=lambda,lambda[2]=lambda,beta=0,feq1):
t1:=collect(subs(simplify(feqSC2/(2*lambda^6))),{alpha,lambda})=0;
t2:=limit(t1,eta[r]=infinity):
limit(t2,eta[t]=infinity);

# Cantilever column with a top mass and rotary inertia: fixed
# support without axial force
feqSC3:=subs(Omega=lambda^2,nu=0,
        lambda[1]=lambda,lambda[2]=lambda,feq1):
t1:=collect(subs(simplify(feqSC3/(2*lambda^6))),{alpha,lambda,beta})=0;
t2:=limit(t1,eta[r]=infinity):
limit(t2,eta[t]=infinity);

# export the frequency function to Matlab with correct notations
feq2:=subs({Omega^2=x,Omega^4=x^2,eta[r]=eta_r,eta[t]=eta_t,
alpha=alpha_p,beta=beta_p},feq1):
Matlab(feq2, resultname="f");
```