AENG M3102: Dynamic analysis of wind and marine turbines

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1 Introduction

Wind and marine turbine structures are long slender columns with a rotor and blade assembly placed on the top. These slender structures vibrate due to dynamic environmental forces and its own dynamics. Analysis of the dynamic behavior of wind and marine turbines is fundamental to the stability, performance, operation and safety of these systems. For the design and analysis of real-life systems, detailed finite-element models are often used. A multi-physics model of a modern wind or marine turbine with (non-linear) structural dynamics, fluid-structure interaction, soil-structure interaction and rotor dynamics can easily lead to computational model consisting several million degrees-of-freedom. While such a detailed analysis can give incredible resolution of the dynamic behavior of the system, the understanding of basic physical principles which govern the overall design may be somewhat difficult to deduce from such a complex analysis. For this reason, in this note we will focus on a simplified analysis with the aim of understanding fundamental physics which underpins the overall dynamic behavior of the system. Our approach involves the following key steps:

- Idealisation of the complex system and related assumptions
- Derivation of the equation of motion
- Derivation of the boundary conditions
- Analytical solution of the eigenvalue problem to obtain natural frequencies of the system

2 Equation of Motion and Boundary Conditions

2.1 Governing Partial Differential Equation

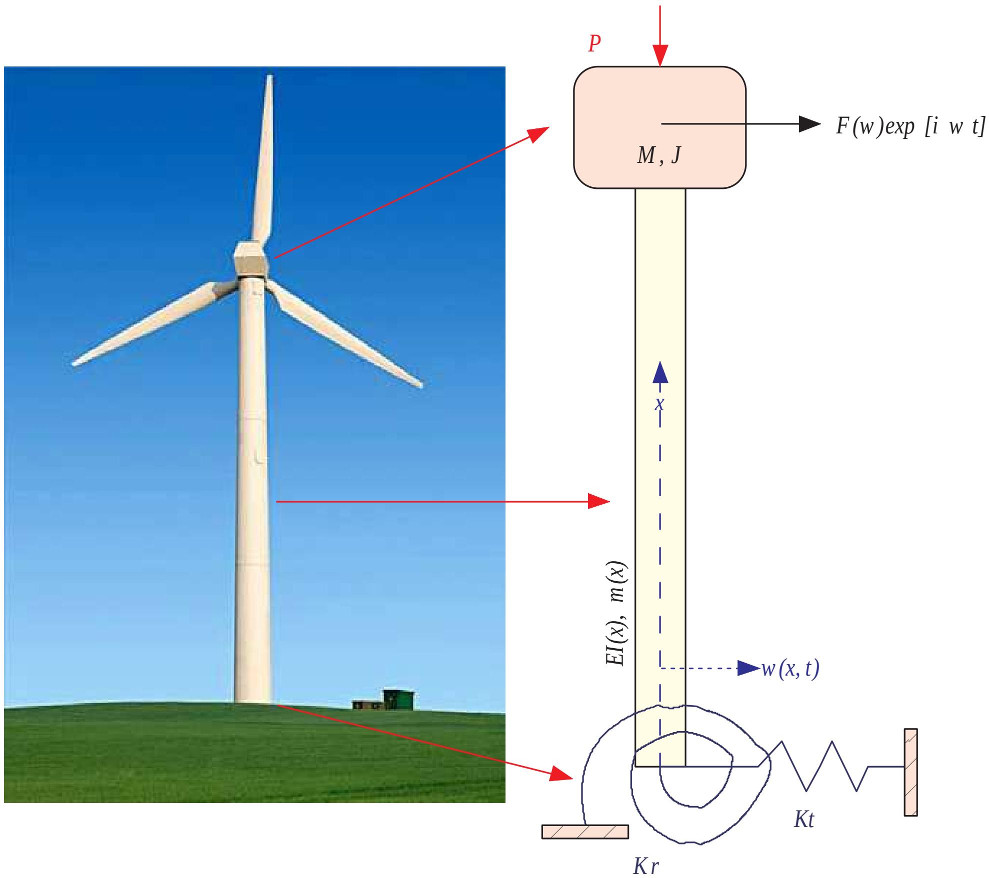


Figure 1: Idealisation of a wind turbine (the first picture is taken from http://www.segen.co.uk) using Euler Bernoulli beam with a top mass. Flexible springs are assumed to model the soil-structure interaction. The weight of the rotor-hub and blades are assumed to be P. The forcing due to the rotation of the blades is assumed to be harmonic in nature with frequency ϖ.

The Equation

The equation of motion of the beam is given by (see the book by Géradin and Rixen [3] for the derivation of this equation):

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(P(x)\frac{\partial w(x,t)}{\partial x}\right)-\frac{\partial}{\partial x}\left(mr^2(x)\frac{\partial \ddot{w}(x,t)}{\partial x}\right) + m\,\ddot{w}(x,t)=f(x,t). \quad (1)$$

Here $w(x,t)$ is the transverse deflection of the beam, t is time, $\dot{(\bullet)}$ denotes derivative with respect to time and $f(x,t)$ is the applied time depended load on the beam. The height of the structure is considered to be L.

The Forcing

The forcing due to the rotation of the blades is assumed to be harmonic in nature with frequency ϖ. This implies that the system is subjected to a forcing

$$F(\varpi)\exp[i\varpi t] \quad (2)$$

at $x=L$. We can assume that there is a slight imbalance between the rotor and the column. Such an imbalance in unavoidable due to construction defects of such complex systems. Assuming the amount of imbalance is ϵ, we have $F(\varpi)=(M\varpi^2\epsilon)$ due to the centrifugal force. Therefore, the dynamic loading on the system at $x=L$ can be idealized as

$$f(x,t)=(M\varpi^2\epsilon)\exp[i\varpi t]\delta(x-L) \quad (3)$$

where $\delta(\bullet)$ is the Dirac delta function. This equation shows that the magnitude of the force acting on the system is proportional to the square of the blade passing frequency.

The Assumptions

- The inertial and the elastic properties of the structure are constant along the height of the structure.
- The effect of soil stiffness is elastic and linear and can be captured by a translational and a rotational spring at the point of contact. In effect, the soil-structure interaction can be completely captured by the boundary condition at the bottom of the structure.
- The end-mass is rigidly attached to the structure.
- The axial force in the structure is constant and remains axial during vibration.
- Deflections due to shear force are negligible and a plain section in the structure remains plane during the bending vibration (standard assumptions in the Euler-Bernoulli beam theory).
- The flexible dynamics of the rotor and the blades above the top point is assumed to be uncoupled with the dynamics of the column.
- Forcing to the system is harmonic and arising to due to the imbalance in the rotor-beam system.
- None of the properties are changing with time. In other words, the system is time invariant.

2.2 Boundary Conditions

Noting that the properties are not changing with x, equation (1) can be simplified as

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+P\frac{\partial^2 w(x,t)}{\partial x^2}-mr^2\frac{\partial^2 \ddot{w}(x,t)}{\partial x^2}+m\ddot{w}(x,t) = (M\varpi^2\epsilon)\exp[i\varpi t]\delta(x-L). \tag{4}$$

The four boundary conditions associated with this equation are

- Bending moment at $x = 0$:

$$EI\frac{\partial^2 w(x,t)}{\partial x^2} - k_r\frac{\partial w(x,t)}{\partial x} = 0\Big|_{x=0} \quad \text{or} \quad EIw''(0,t)-k_r w'(0,t) = 0. \tag{5}$$

- Shear force at $x = 0$:

$$EI\frac{\partial^3 w(x,t)}{\partial x^3} + P\frac{\partial w(x,t)}{\partial x} + k_t w(x,t) - mr^2\frac{\partial \ddot{w}(x,t)}{\partial x} = 0\Big|_{x=0} \tag{6}$$
$$\text{or} \quad EIw'''(0,t) + Pw'(0,t) + k_t w(0,t) - mr^2\frac{\partial \ddot{w}(0,t)}{\partial x} = 0.$$

- Bending moment at $x = L$:

$$EI\frac{\partial^2 w(x,t)}{\partial x^2} + J\frac{\partial \ddot{w}(x,t)}{\partial x} = 0\Big|_{x=L} \quad \text{or} \quad EIw''(L,t) + J\frac{\partial \ddot{w}(L,t)}{\partial x} = 0. \tag{7}$$

● Shear force at $x = L$:

$$EI\frac{\partial^3 w(x,t)}{\partial x^3} + P\frac{\partial w(x,t)}{\partial x} - M\ddot{w}(x,t) - mr^2\frac{\partial \ddot{w}(x,t)}{\partial x} = 0\Big|_{x=L}$$

$$\text{or} \quad EIw'''(L,t) + Pw'(L,t) - M\ddot{w}(L,t) - mr^2\frac{\partial \ddot{w}(L,t)}{\partial x} = 0. \tag{8}$$

Assuming harmonic solution (the separation of variable) we have

$$w(x,t) = W(\xi)\exp\{\mathrm{i}\omega t\}, \quad \xi = x/L. \tag{9}$$

Substituting this in the equation of motion and the boundary conditions, Eqs. (4) – (8), results

$$\frac{EI}{L^4}\frac{\partial^4 W(\xi)}{\partial \xi^4} + \frac{P}{L^2}\frac{\partial^2 W(\xi)}{\partial \xi^2} \tag{10}$$

$$- m\omega^2 W(\xi) + \frac{mr^2\omega^2}{L^2}\frac{\partial^2 W(\xi)}{\partial \xi^2} = (M\varpi^2\epsilon)\exp[i\varpi t]\delta(\xi L - L)$$

$$\frac{EI}{L^2}W''(0) - \frac{k_r}{L}W'(0) = 0 \tag{11}$$

$$\frac{EI}{L^3}W'''(0) + \frac{P}{L}W'(0) + k_t W(0) + \frac{mr^2\omega^2}{L}W'(0) = 0 \tag{12}$$

$$\frac{EI}{L^2}W''(1) - \frac{\omega^2 J}{L}W'(1) = 0 \tag{13}$$

$$\frac{EI}{L^3}W'''(1) + \frac{P}{L}W'(1) + \omega^2 M\,W(1) + \frac{mr^2\omega^2}{L}W'(1) = 0. \tag{14}$$

It is convenient to express these equations in terms of non-dimensional parameters by elementary rearrangements as

$$\frac{\partial^4 W(\xi)}{\partial \xi^4} + \widetilde{\nu}\frac{\partial^2 W(\xi)}{\partial \xi^2} - \Omega^2 W(\xi) = p_L \exp[i\varpi t]\delta(\xi L - L) \tag{15}$$

$$W''(0) - \eta_r W'(0) = 0 \tag{16}$$

$$W'''(0) + \widetilde{\nu} W'(0) + \eta_t W(0) = 0 \tag{17}$$

$$W''(1) - \beta\Omega^2 W'(1) = 0 \tag{18}$$

$$W'''(1) + \widetilde{\nu} W'(1) + \alpha\Omega^2 W(1) = 0 \tag{19}$$

where

$$\widetilde{\nu} = \nu + \mu^2\Omega^2 \tag{20}$$

$$\nu = \frac{PL^2}{EI} \quad \text{(nondimensional axial force)} \tag{21}$$

$$\eta_r = \frac{k_r L}{EI} \quad \text{(nondimensional rotational end stiffness)} \tag{22}$$

$$\eta_t = \frac{k_t L^3}{EI} \quad \text{(nondimensional translational end stiffness)} \tag{23}$$

$$\Omega^2 = \omega^2 \frac{mL^4}{EI} \quad \text{(nondimensional frequency parameter)} \tag{24}$$

$$\alpha = \frac{M}{mL} \quad \text{(mass ratio)} \tag{25}$$

$$\beta = \frac{J}{mL^3} \quad \text{(nondimensional rotary inertia)} \tag{26}$$

$$\mu = \frac{r}{L} \quad \text{(nondimensional radius of gyration)} \tag{27}$$

$$p_L = \epsilon\frac{ML^4}{EI}\varpi^2 \quad \text{(normalized forcing amplitude at the top end)} \tag{28}$$

$$f_0 = \sqrt{\frac{EI}{mL^4}} \quad \text{(natural frequency scaling parameter).} \tag{29}$$

The natural frequencies of the system can be obtained as

$$\omega_j = \Omega_j f_0; \quad j = 1, 2, 3, \cdots \tag{30}$$

3 Equation of the Natural Frequencies

3.1 General Derivation

Natural frequencies of the system can be obtained from the 'free vibration problem' by considering no force on the system. Therefore, we consider $p_L = 0$ in the subsequent analysis. Assuming a solution of the form

$$W(\xi) = \exp\{\lambda\xi\} \tag{31}$$

and substituting in the equation of motion (15) results

$$\lambda^4 + \widetilde{\nu}\lambda^2 - \Omega^2 = 0. \tag{32}$$

This equation is often know as the dispersion relationship. This is the equation governing the natural frequencies of the beam. Solving this equation for λ^2 we have

$$\begin{aligned} \lambda^2 &= -\frac{\widetilde{\nu}}{2} \pm \sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + \Omega^2} \\ &= -\left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + \Omega^2} + \frac{\widetilde{\nu}}{2}\right), \quad \left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + \Omega^2} - \frac{\widetilde{\nu}}{2}\right). \end{aligned} \tag{33}$$

Because $\widetilde{\nu}^2$ and Ω^2 are always positive quantities, both roots are real with one negative and one positive root. Therefore, the four roots can be expressed as

$$\lambda = \pm \mathrm{i}\lambda_1, \quad \pm\lambda_2 \tag{34}$$

where

$$\lambda_1 = \left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + \Omega^2} + \frac{\widetilde{\nu}}{2}\right)^{1/2} \tag{35}$$

$$\text{and} \quad \lambda_2 = \left(\sqrt{\left(\frac{\widetilde{\nu}}{2}\right)^2 + \Omega^2} - \frac{\widetilde{\nu}}{2}\right)^{1/2}. \tag{36}$$

From Eqs. (35) and (36) also note that

$$\lambda_1^2 - \lambda_2^2 = \widetilde{\nu}. \tag{37}$$

In view of the roots in equation (34) the solution $W(\xi)$ can be expressed as

$$\begin{aligned} W(\xi) &= a_1 \sin \lambda_1 \xi + a_2 \cos \lambda_1 \xi + a_3 \sinh \lambda_2 \xi + a_4 \cosh \lambda_2 \xi \\ \text{or} \quad W(\xi) &= \mathbf{s}^T(\xi)\mathbf{a} \end{aligned} \tag{38}$$

where the vectors

$$\mathbf{s}(\xi) = \{\sin \lambda_1 \xi, \cos \lambda_1 \xi, \sinh \lambda_2 \xi, \cosh \lambda_2 \xi\}^T \tag{39}$$

$$\text{and} \quad \mathbf{a} = \{a_1, a_2, a_3, a_4\}^T. \tag{40}$$

Applying the boundary conditions in Eqs. (16) – (19) on the expression of $W(\xi)$ in (38) we have

$$\mathbf{Ra} = \mathbf{0} \tag{41}$$

where the matrix

$$\mathbf{R} = \left[\begin{array}{cc} s_1''(0) - \eta_r s_1'(0) & s_2''(0) - \eta_r s_2'(0) \\ s_1'''(0) + \widetilde{\nu} s_1'(0) + \eta_t s_1(0) & s_2'''(0) + \widetilde{\nu} s_2'(0) + \eta_t s_2(0) \\ s_1''(1) - \beta\Omega^2 s_1'(1) & s_2''(1) - \beta\Omega^2 s_2'(1) \\ s_1'''(1) + \widetilde{\nu} s_1'(1) + \alpha\Omega^2 s_1(1) & s_2'''(1) + \widetilde{\nu} s_2'(1) + \alpha\Omega^2 s_2(1) \end{array} \right.$$
$$\left. \begin{array}{cc} s_3''(0) - \eta_r s_3'(0) & s_4''(0) - \eta_r s_4'(0) \\ s_3'''(0) + \widetilde{\nu} s_3'(0) + \eta_t s_3(0) & s_4'''(0) + \widetilde{\nu} s_4'(0) + \eta_t s_4(0) \\ s_3''(1) - \beta\Omega^2 s_3'(1) & s_4''(1) - \beta\Omega^2 s_4'(1) \\ s_3'''(1) + \widetilde{\nu} s_3'(1) + \alpha\Omega^2 s_3(1) & s_3'''(1) + \widetilde{\nu} s_3'(1) + \alpha\Omega^2 s_3(1) \end{array} \right]. \tag{42}$$

Substituting functions $s_j(\xi), j = 1, \cdots, 4$ from equation (39) and simplifying we obtain

$$\mathbf{R} = \left[\begin{array}{cc} -\lambda_1\eta_r & -\lambda_1^2 \\ \lambda_1^3 + \nu\lambda_1 & \eta_t \\ -\sin(\lambda_1)\,{\lambda_1}^2 - \Omega^2\beta\cos(\lambda_1)\,\lambda_1 & -\cos(\lambda_1)\,{\lambda_1}^2 + \Omega^2\beta\sin(\lambda_1)\,\lambda_1 \\ -\cos(\lambda_1)\,{\lambda_1}^3 + \widetilde{\nu}\cos(\lambda_1)\,\lambda_1 + \Omega^2\alpha\sin(\lambda_1) & \sin(\lambda_1)\,{\lambda_1}^3 - \widetilde{\nu}\sin(\lambda_1)\,\lambda_1 + \Omega^2\alpha\cos(\lambda_1) \end{array} \right.$$
$$\left. \begin{array}{cc} -\lambda_2\eta_r & \lambda_2^2 \\ \lambda_2^3 + \nu\lambda_2 & \eta_t \\ \sinh(\lambda_2)\,{\lambda_2}^2 - \Omega^2\beta\cosh(\lambda_2)\,\lambda_2 & \cosh(\lambda_2)\,{\lambda_2}^2 - \Omega^2\beta\sinh(\lambda_2)\,\lambda_2 \\ \cosh(\lambda_2)\,{\lambda_2}^3 + \widetilde{\nu}\cosh(\lambda_2)\,\lambda_2 + \Omega^2\alpha\sinh(\lambda_2) & \sinh(\lambda_2)\,{\lambda_2}^3 + \widetilde{\nu}\sinh(\lambda_2)\,\lambda_2 + \Omega^2\alpha\cosh(\lambda_2) \end{array} \right]. \tag{43}$$

The constant vector in equation (41) cannot be zero. Therefore, the equation governing the natural frequencies is given by

$$|\mathbf{R}| = 0. \tag{44}$$

This, upon simplification (a`Maple`® code developed for this purpose) reduces to

$$
\begin{aligned}
&-{\lambda_1}^6\mathrm{s}_1\,{\lambda_2}^4\mathrm{sh}_2+2\,\nu^2{\lambda_1}^3\mathrm{c}_1\,{\lambda_2}^3\mathrm{ch}_2+2\,{\lambda_1}^3\Omega^4\alpha\,\mathrm{c}_1\,{\lambda_2}^3\beta\,\mathrm{ch}_2+2\,\nu\,{\lambda_1}^3\mathrm{c}_1\,{\lambda_2}^5\mathrm{ch}_2-\nu\,{\lambda_1}^3\Omega^4\beta\,\lambda_2\alpha\\
&+{\lambda_1}^5\Omega^4\beta\,\lambda_2\alpha-2\,{\lambda_1}^5\mathrm{c}_1\,{\lambda_2}^5\mathrm{ch}_2-2\,\nu^2{\lambda_1}^3{\lambda_2}^3-2\,\nu\,{\lambda_1}^3{\lambda_2}^5+2\,{\lambda_1}^5{\lambda_2}^3\nu-{\lambda_1}^5\mathrm{c}_1\,{\lambda_2}^2\Omega^2\alpha\,\mathrm{sh}_2+{\lambda_1}^4\Omega^2\beta\,\mathrm{s}_1\,{\lambda_2}^5\mathrm{ch}_2\\
&+{\lambda_1}^4\Omega^4\beta\,\mathrm{s}_1\,{\lambda_2}^2\alpha\,\mathrm{sh}_2+{\lambda_1}^6\mathrm{s}_1\,{\lambda_2}^3\Omega^2\beta\,\mathrm{ch}_2-{\lambda_1}^3\Omega^2\alpha\,\mathrm{c}_1\,{\lambda_2}^4\mathrm{sh}_2+\nu\,{\lambda_1}^3\mathrm{c}_1\,{\lambda_2}^2\Omega^2\alpha\,\mathrm{sh}_2-\nu\,{\lambda_1}^2\Omega^2\beta\,\mathrm{s}_1\,{\lambda_2}^5\mathrm{ch}_2\\
&-\nu\,{\lambda_1}^4\mathrm{s}_1\,{\lambda_2}^3\Omega^2\beta\,\mathrm{ch}_2-\nu^2{\lambda_1}^2\mathrm{s}_1\,{\lambda_2}^4\mathrm{sh}_2+\nu\,\lambda_1\Omega^2\alpha\,\mathrm{c}_1\,{\lambda_2}^4\mathrm{sh}_2-\nu\,\lambda_1\Omega^4\alpha\,\mathrm{c}_1\,{\lambda_2}^3\beta\,\mathrm{ch}_2\\
&+\mathrm{s}_1\,{\lambda_1}^4{\lambda_2}^3\Omega^2\alpha\,\mathrm{ch}_2+\mathrm{s}_1\,{\lambda_1}^4\nu^2{\lambda_2}^2\mathrm{sh}_2+\Omega^2\alpha\,\mathrm{s}_1\,{\lambda_1}^2{\lambda_2}^5\mathrm{ch}_2-\Omega^4\alpha\,\mathrm{s}_1\,{\lambda_1}^2{\lambda_2}^4\beta\,\mathrm{sh}_2\\
&+\mathrm{c}_1\,{\lambda_1}^5{\lambda_2}^4\Omega^2\beta\,\mathrm{sh}_2+\mathrm{c}_1\,{\lambda_1}^5\nu\,{\lambda_2}^2\Omega^2\beta\,\mathrm{sh}_2+\Omega^2\beta\,\mathrm{c}_1\,{\lambda_1}^3{\lambda_2}^4\nu\,\mathrm{sh}_2+\Omega^4\beta\,\mathrm{c}_1\,{\lambda_1}^3\nu\,\lambda_2\alpha\,\mathrm{ch}_2+\mathrm{s}_1\,{\lambda_1}^4\nu\,\lambda_2\Omega^2\alpha\,\mathrm{ch}_2\\
&+\Omega^2\beta\,\mathrm{c}_1\,{\lambda_1}^3{\lambda_2}^6\mathrm{sh}_2+\Omega^2\alpha\,\mathrm{s}_1\,{\lambda_1}^2\nu\,{\lambda_2}^3\mathrm{ch}_2+2\,{\lambda_1}^5{\lambda_2}^5+\big(\big(\lambda_1\Omega^2\alpha\,\mathrm{c}_1\,\mathrm{sh}_2\,{\lambda_2}^2-2\,{\lambda_1}^2\nu\,\mathrm{s}_1\,\mathrm{sh}_2\,{\lambda_2}^2\\
&+{\lambda_1}^3\mathrm{c}_1\,\nu\,\mathrm{ch}_2\,\lambda_2+{\lambda_1}^5\lambda_2-{\lambda_1}^4\mathrm{s}_1\,\Omega^2\beta\,\mathrm{ch}_2\,\lambda_2+2\,{\lambda_1}^3\mathrm{c}_1\,\mathrm{ch}_2\,{\lambda_2}^3-2\,\lambda_1\Omega^4\alpha\,\mathrm{c}_1\,\beta\,\mathrm{ch}_2\,\lambda_2-{\lambda_1}^2\Omega^4\beta\,\mathrm{s}_1\,\alpha\,\mathrm{sh}_2\\
&-\mathrm{s}_1\,{\lambda_2}^3\Omega^2\alpha\,\mathrm{ch}_2+2\,\lambda_1\Omega^4\beta\,\lambda_2\alpha+\Omega^4\beta\,\mathrm{s}_1\,{\lambda_2}^2\alpha\,\mathrm{sh}_2-\mathrm{s}_1\,{\lambda_1}^2{\lambda_2}^4\mathrm{sh}_2+{\lambda_1}^3\mathrm{c}_1\,\Omega^2\alpha\,\mathrm{sh}_2-\mathrm{s}_1\,{\lambda_1}^2\lambda_2\Omega^2\alpha\,\mathrm{ch}_2\\
&-{\lambda_1}^3\nu\,\lambda_2-\mathrm{c}_1\,{\lambda_1}^3{\lambda_2}^2\Omega^2\beta\,\mathrm{sh}_2-\Omega^2\beta\,\mathrm{c}_1\,\lambda_1{\lambda_2}^4\mathrm{sh}_2+\lambda_1{\lambda_2}^3\nu+{\lambda_1}^4\mathrm{s}_1\,\mathrm{sh}_2\,{\lambda_2}^2+\lambda_1{\lambda_2}^5-{\lambda_1}^2\Omega^2\beta\,\mathrm{s}_1\,\mathrm{ch}_2\,{\lambda_2}^3\\
&-\nu\,\mathrm{c}_1\,\lambda_1{\lambda_2}^3\mathrm{ch}_2\big)\,\eta_t+\lambda_1\Omega^4\alpha\,\mathrm{c}_1\,{\lambda_2}^4\beta\,\mathrm{sh}_2-{\lambda_1}^3\mathrm{c}_1\,{\lambda_2}^4\nu\,\mathrm{sh}_2-\lambda_1\Omega^2\alpha\,\mathrm{c}_1\,{\lambda_2}^5\mathrm{ch}_2-{\lambda_1}^5\mathrm{c}_1\,{\lambda_2}^2\nu\,\mathrm{sh}_2\\
&-2\,{\lambda_1}^3\mathrm{c}_1\,{\lambda_2}^3\Omega^2\alpha\,\mathrm{ch}_2-{\lambda_1}^3\mathrm{c}_1\,{\lambda_2}^6\mathrm{sh}_2+2\,{\lambda_1}^4\mathrm{s}_1\,{\lambda_2}^4\Omega^2\beta\,\mathrm{sh}_2+{\lambda_1}^2\nu\,\mathrm{s}_1\,{\lambda_2}^5\mathrm{ch}_2+{\lambda_1}^6\mathrm{s}_1\,{\lambda_2}^2\Omega^2\beta\,\mathrm{sh}_2\\
&+{\lambda_1}^3\Omega^4\alpha\,\mathrm{c}_1\,{\lambda_2}^2\beta\,\mathrm{sh}_2+{\lambda_1}^2\Omega^4\beta\,\mathrm{s}_1\,{\lambda_2}^3\alpha\,\mathrm{ch}_2+{\lambda_1}^4\mathrm{s}_1\,\nu\,{\lambda_2}^3\mathrm{ch}_2-{\lambda_1}^4\mathrm{s}_1\,{\lambda_2}^5\mathrm{ch}_2+{\lambda_1}^4\Omega^4\beta\,\mathrm{s}_1\,\lambda_2\alpha\,\mathrm{ch}_2\\
&-{\lambda_1}^5\mathrm{c}_1\,{\lambda_2}^4\mathrm{sh}_2+{\lambda_1}^2\Omega^2\beta\,\mathrm{s}_1\,{\lambda_2}^6\mathrm{sh}_2-{\lambda_1}^6\mathrm{s}_1\,{\lambda_2}^3\mathrm{ch}_2-{\lambda_1}^5\mathrm{c}_1\,\lambda_2\Omega^2\alpha\,\mathrm{ch}_2\big)\,\eta_r+\Omega^4\alpha\,\beta\,\lambda_1{\lambda_2}^5+\Omega^4\alpha\,\beta\,\lambda_1{\lambda_2}^3\nu\\
&+\big(-\mathrm{s}_1\,{\lambda_1}^2{\lambda_2}^5\mathrm{ch}_2-\Omega^4\beta\,\mathrm{c}_1\,{\lambda_1}^3\alpha\,\mathrm{sh}_2+\Omega^4\alpha\,\mathrm{s}_1\,{\lambda_1}^2\beta\,\mathrm{ch}_2\,\lambda_2-\mathrm{s}_1\,{\lambda_1}^4\mathrm{ch}_2\,{\lambda_2}^3-\mathrm{s}_1\,{\lambda_1}^2\nu\,{\lambda_2}^3\mathrm{ch}_2\\
&-\Omega^2\beta\,\mathrm{c}_1\,\lambda_1{\lambda_2}^5\mathrm{ch}_2-\Omega^4\beta\,\mathrm{c}_1\,\lambda_1{\lambda_2}^2\alpha\,\mathrm{sh}_2+\Omega^4\alpha\,\mathrm{s}_1\,{\lambda_2}^3\beta\,\mathrm{ch}_2-\Omega^2\alpha\,\mathrm{s}_1\,{\lambda_2}^4\mathrm{sh}_2-\nu\,\mathrm{c}_1\,\lambda_1{\lambda_2}^4\mathrm{sh}_2\\
&-2\,\mathrm{s}_1\,{\lambda_1}^2{\lambda_2}^2\Omega^2\alpha\,\mathrm{sh}_2-\mathrm{s}_1\,{\lambda_1}^4\nu\,\mathrm{ch}_2\,\lambda_2-\nu\,\mathrm{c}_1\,{\lambda_1}^3\mathrm{sh}_2\,{\lambda_2}^2+\mathrm{c}_1\,{\lambda_1}^3{\lambda_2}^4\mathrm{sh}_2+\mathrm{c}_1\,{\lambda_1}^5\mathrm{sh}_2\,{\lambda_2}^2-\mathrm{c}_1\,{\lambda_1}^5\Omega^2\beta\,\mathrm{ch}_2\,\lambda_2\\
&-\mathrm{s}_1\,{\lambda_1}^4\Omega^2\alpha\,\mathrm{sh}_2-2\,\Omega^2\beta\,\mathrm{c}_1\,{\lambda_1}^3\mathrm{ch}_2\,{\lambda_2}^3\big)\,\eta_t+\mathrm{s}_1\,{\lambda_1}^4{\lambda_2}^6\mathrm{sh}_2-2\,\nu\,{\lambda_1}^2\Omega^4\beta\,\mathrm{s}_1\,{\lambda_2}^2\alpha\,\mathrm{sh}_2\\
&+4\,{\lambda_1}^4\nu\,\mathrm{s}_1\,{\lambda_2}^4\mathrm{sh}_2-2\,{\lambda_1}^5\mathrm{c}_1\,\nu\,{\lambda_2}^3\mathrm{ch}_2=0. \qquad (45)
\end{aligned}
$$

where

$$\mathrm{s}_1=\sin(\lambda_1),\,\mathrm{c}_1=\cos(\lambda_1),\,\mathrm{sh}_2=\sinh(\lambda_2),\,\mathrm{ch}_2=\cosh(\lambda_2). \qquad (46)$$

The natural frequencies can be obtained by solving equation (45) for Ω. Due to the complexity of this transcendental equation it should be solved numerically. Equation (45) is shown not to scare you, but to show the complicated nature of the frequency equation even under the simplifying assumptions discussed before. If we aim to relax any of the assumptions (e.g., variable bending stiffness), this equation is likely to be even more complicated. Luckily, equation (45) can be translated to `Matlab`® automatically and can be solved numerically.

4 Numerical Example

Turbine Structure Properties	Numerical values
Length (L)	81 m
Average diameter (D)	3.5m
Thickness (t_h)	0.075 mm
Mass density (ρ)	7800 kg/m^3
Young's modulus (E)	2.1×10^{11} Pa
Mass density (ρ_l)	7800 kg/m^3
Rotational speed (ϖ)	22 r.p.m = 0.37 Hz
Top mass (M)	130,000 kg
Rated power	3 MW

Table 1: Material and geometric properties of the turbine structure [4]

- The non-dimensional mass ratio can be obtained as

$$\alpha = \frac{M}{mL} = \frac{P}{gmL} = \frac{PL^2}{EI}\left(\frac{EI}{gmL^3}\right) = \nu\left(\frac{EI}{mL^4}\right)L/g = \nu f_0^2 L/g \qquad (47)$$

- We consider the rotary inertia of the blade assembly $J = 0$. This is not a very bad assumption because the 'point of contact' is very close to the center of gravity of the rotor-blade assembly.

- The moment of inertia of the circular cross section can be obtained as

$$I = \frac{\pi}{64}D^4 - \frac{\pi}{64}(D - t_h)^4 \approx \frac{1}{16}\pi D^3 t_h = 0.6314m^4 \qquad (48)$$

- The mass density per unit length of the system can be obtained as

$$m = \rho A \approx \rho\pi D t_h/2 = 3.1817 \times 10^3 \mathrm{kg/m} \qquad (49)$$

- Using these, the mass ratio $\alpha = 0.2495$ and the nondimensional axial force $\nu = 0.0652$. We also obtain the natural frequency scaling parameter can be obtained as

$$f_0 = \frac{EI}{mL^4} = 0.9682\,\mathrm{s}^{-1}. \qquad (50)$$

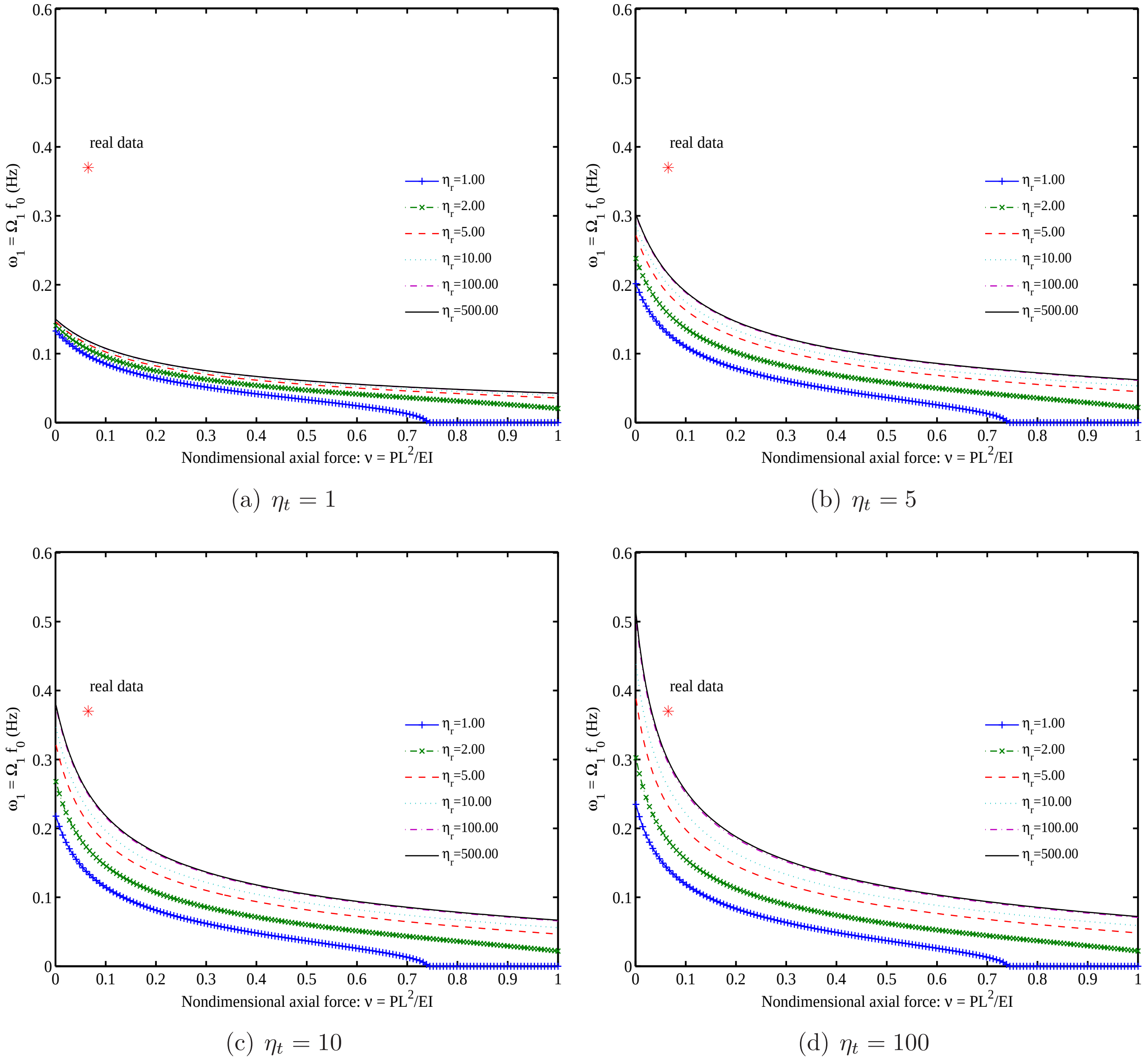
- The radius of gyration of the wind turbine is given by

$$r = \sqrt{\frac{I}{A}} = \frac{1}{4}\sqrt{D^2 + (D - t_h)^2} \approx \frac{D}{2\sqrt{2}} = 1.2374m \qquad (51)$$

- Therefore, the nondimensional radius of gyration $\mu = r/L = 0.0151$. From equation (20) we therefore have

$$\widetilde{\nu} = \nu + 2.2844 \times 10^{-4}\Omega^2 \approx \nu \qquad (52)$$

We use η_r and η_t as variable parameters and try to understand how they affect the overall behavior of the system.



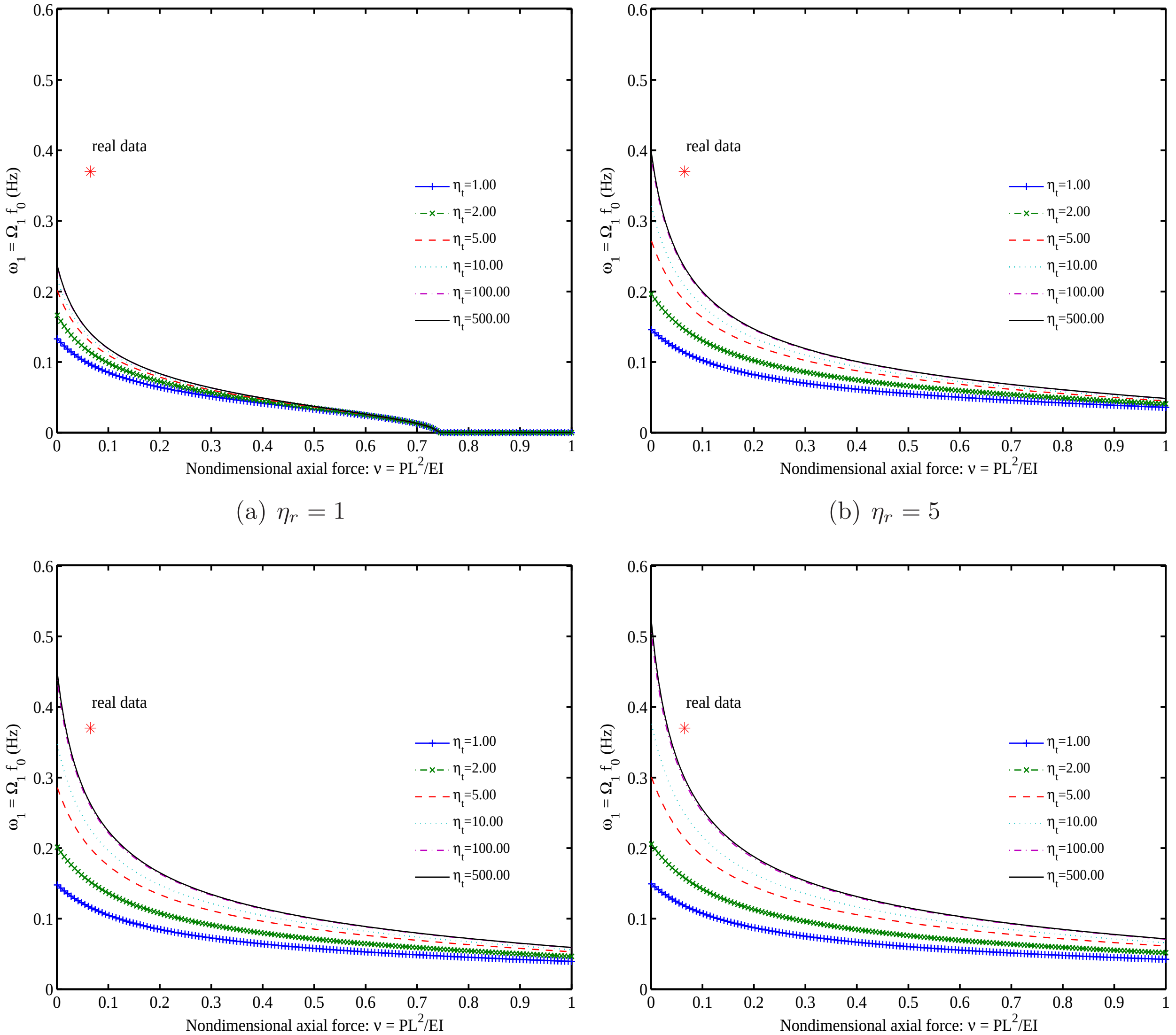
(a) $\eta_t = 1$

(b) $\eta_t = 5$

(c) $\eta_t = 10$

(d) $\eta_t = 100$

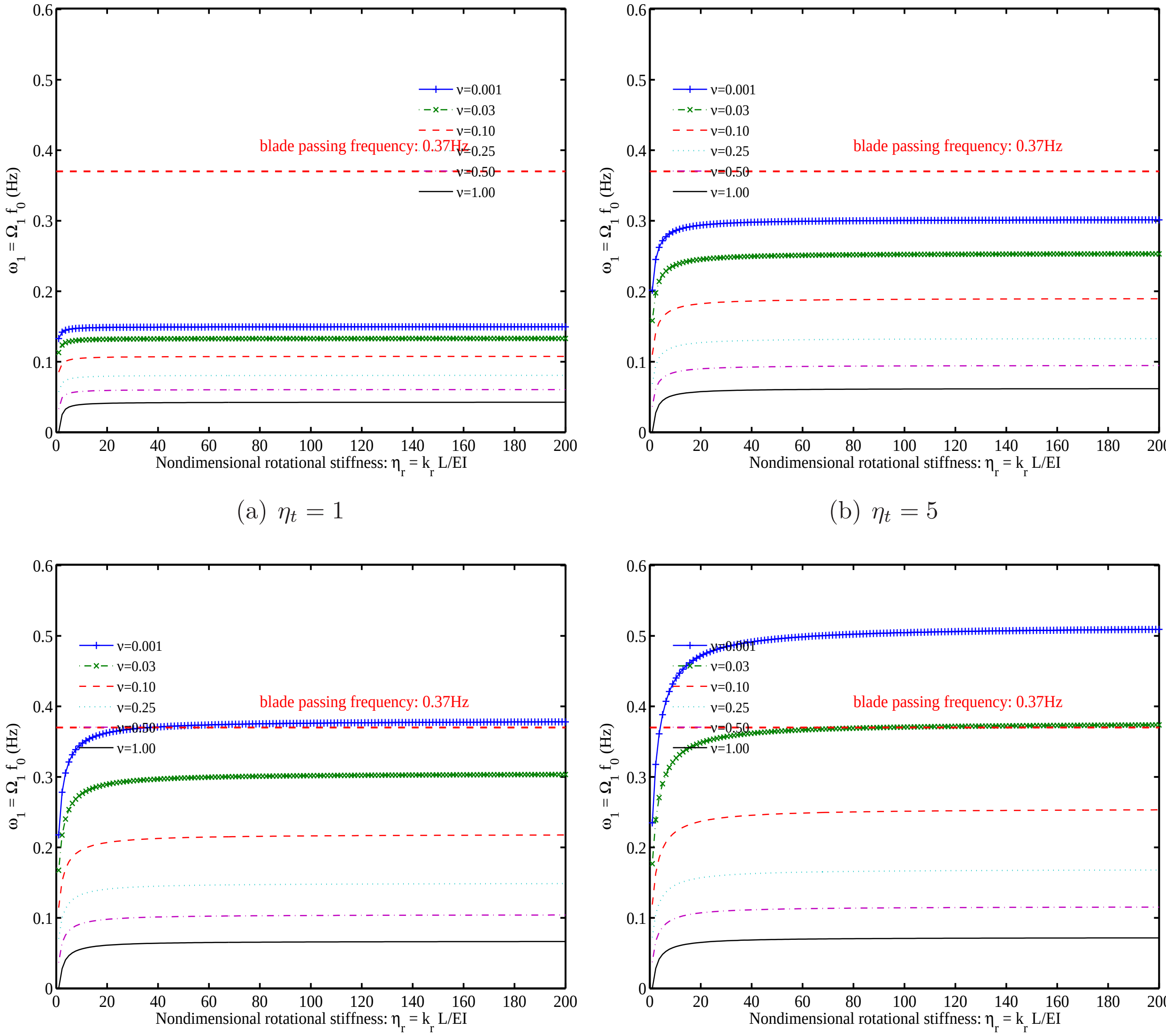
Figure 2: The variation of the first natural frequency of the wind turbine with respect to the nondimensional axial load ν for different values of nondimensional rotational soil stiffness η_r. Four fixed values of the nondimensional translational stiffness η_t are considered in the four subplots. The data from the example ($\varpi = 0.37$Hz and $\nu = 0.0315$) is shown by a '*' in the diagram.



(a) $\eta_r = 1$ (b) $\eta_r = 5$

(c) $\eta_r = 10$ (d) $\eta_r = 100$

Figure 3: The variation of the first natural frequency of the wind turbine with respect to the nondimensional axial load ν for different values of nondimensional translational soil stiffness η_t. Four fixed values of the nondimensional rotational stiffness η_r are considered in the four subplots. The data from the example ($\varpi = 0.37$Hz and $\nu = 0.0315$) is shown by a '*' in the diagram.



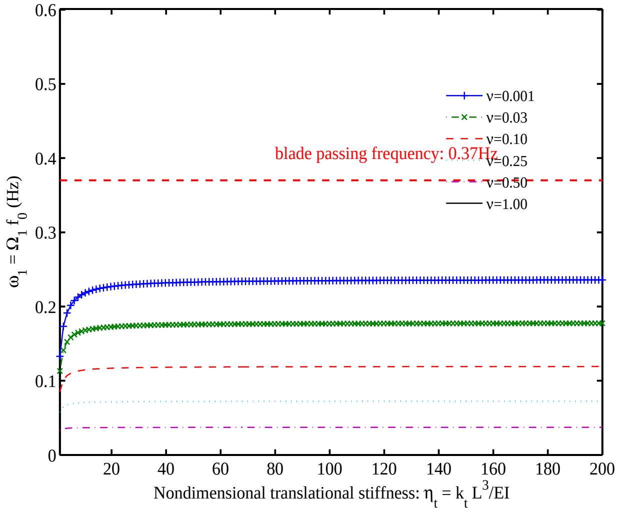
(a) $\eta_t = 1$

(b) $\eta_t = 5$

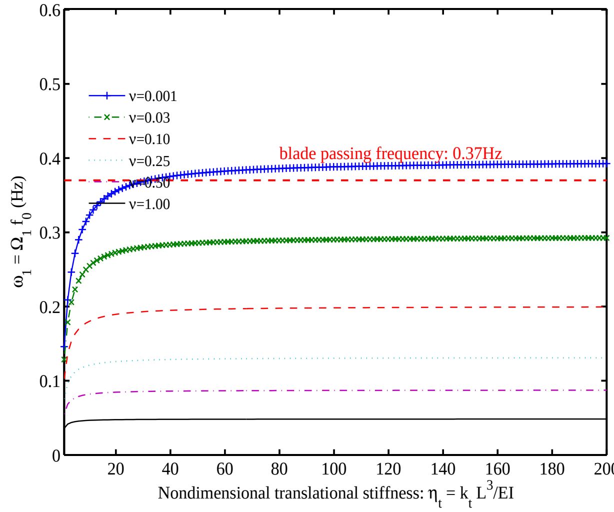
(c) $\eta_t = 10$

(d) $\eta_t = 100$

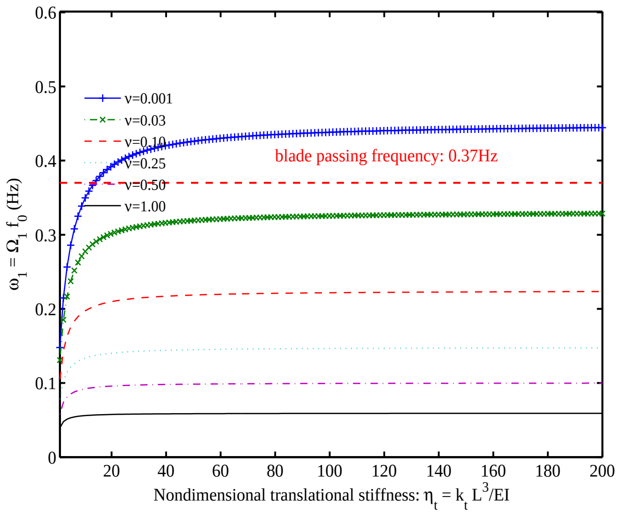
Figure 4: The variation of the first natural frequency of the wind turbine with respect to the nondimensional rotational soil stiffness η_r for different values of nondimensional axial load ν. Four fixed values of the nondimensional translational stiffness η_t are considered in the four subplots. The bade passing frequency $\varpi = 0.37$Hz is shown by a dashed line in the diagram.



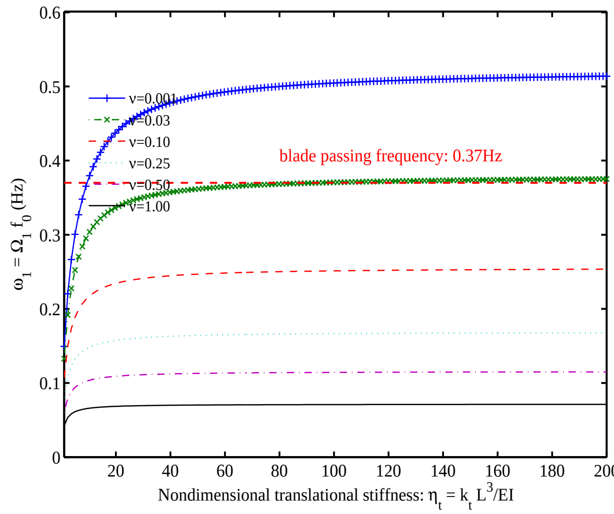
(a) $\eta_r = 1$



(b) $\eta_r = 5$

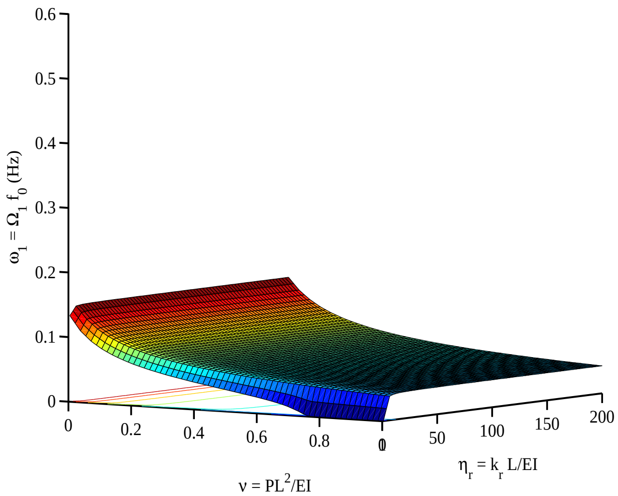


(c) $\eta_r = 10$

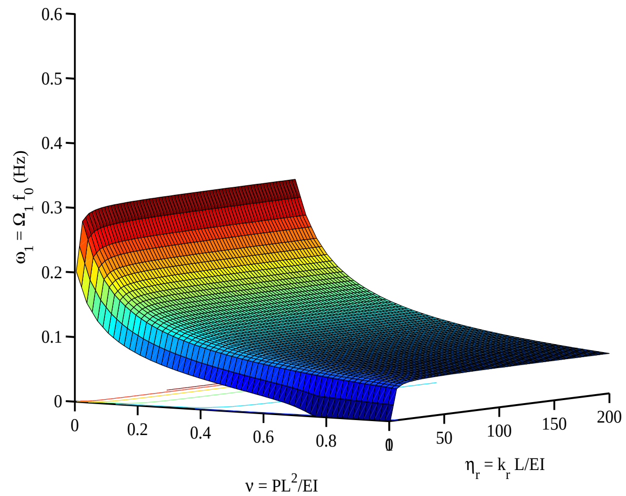


(d) $\eta_r = 100$

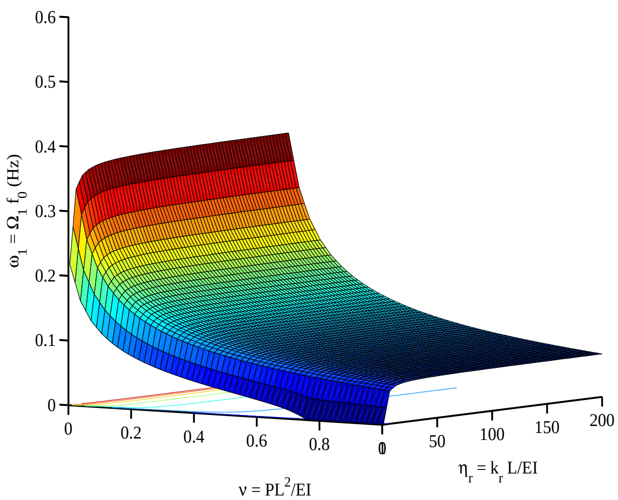
Figure 5: The variation of the first natural frequency of the wind turbine with respect to the nondimensional translational stiffness η_t for different values of nondimensional axial load ν. Four fixed values of the nondimensional rotational soil stiffness η_r are considered in the four subplots. The bade passing frequency $\varpi = 0.37$Hz is shown by a dashed line in the diagram.



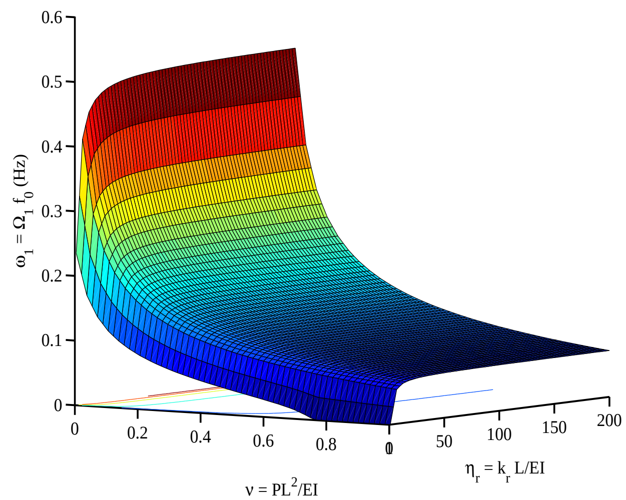
(a) $\eta_t = 1$



(b) $\eta_t = 5$

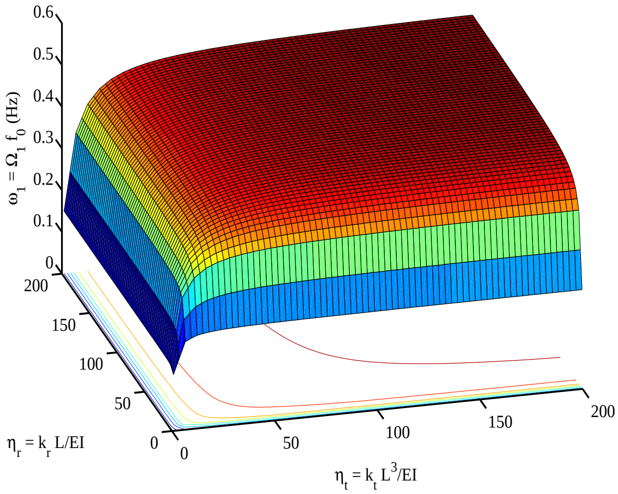


(c) $\eta_t = 10$

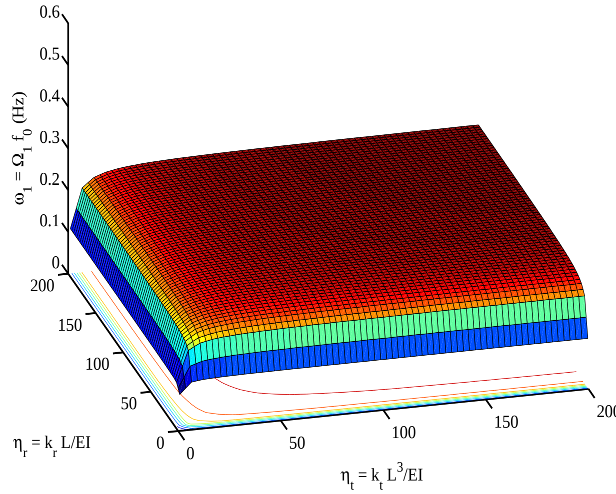


(d) $\eta_t = 100$

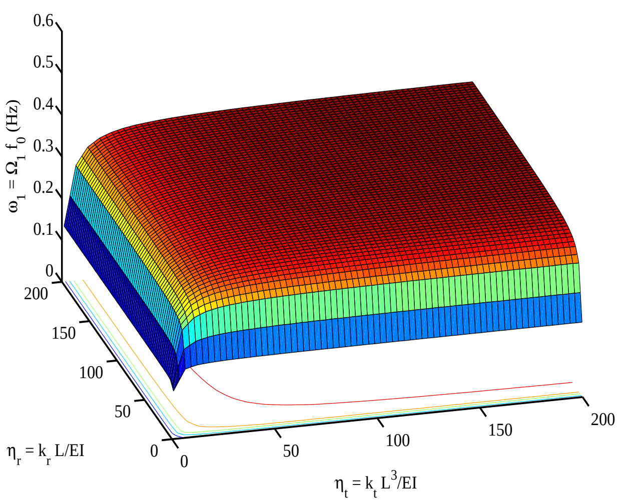
Figure 6: The variation of the first natural frequency of the wind turbine with respect to the nondimensional axial load ν and nondimensional rotational soil stiffness η_r. Four fixed values of the nondimensional translational stiffness η_t are considered in the four subplots.



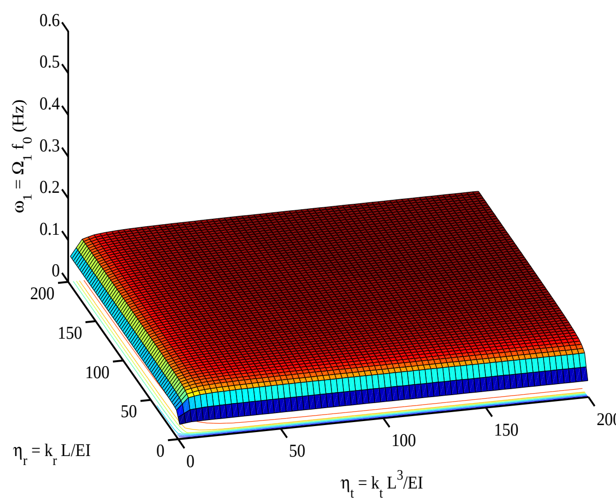
(a) $\nu = 0.001$



(b) $\nu = 0.1$



(c) $\nu = 0.03$



(d) $\nu = 0.5$

Figure 7: The variation of the first natural frequency of the wind turbine with respect to the nondimensional translational stiffness η_t and nondimensional rotational soil stiffness η_r. Four fixed values of the nondimensional axial load ν are considered in the four subplots.

5 Approximate Natural Frequency Based on SDOF Assumption

In the first mode, we can replace the distributed system by a single-degree-of-freedom (SDOF) system with equivalent stiffness k_e and equivalent mass M_e: The first natural frequency is given by

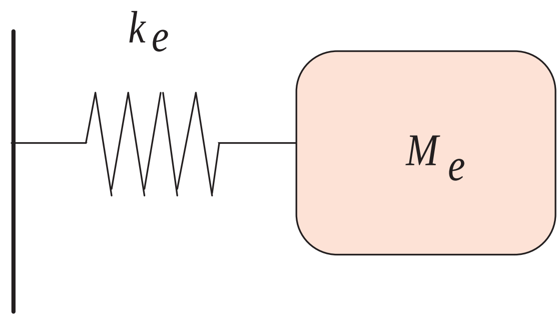


Figure 8: Equivalent single-degree-of-freedom system for the first bending mode of the turbine structure.

$$\omega_1^2 = \frac{k_e}{M_e} \tag{53}$$

Following Blevins [Table 8-8, case 1, page 158, 2] for a perfectly cantilever column one has

$$M_e = M + 0.24M_b = (\alpha + 0.24)mL \tag{54}$$

For our case the column is standing on elastic springs and also have an axial force. Therefore the coefficient 0.24 needs to be modified to take these effects into account. We suppose that the equivalent mass can be represented by

$$M_e = (\alpha + \gamma_m)mL \tag{55}$$

where γ_m is the mass correction factor.

It is useful to express k_e normalized by the stiffness term, $k_{CL} = EI/L^3$.

Therefore, the first natural frequency can be expressed as

$$\omega_1^2 \approx \frac{k_e}{M_e} = \frac{k_e}{k_{CL}} \frac{EI/L^3}{(\alpha+\gamma_m)mL} = \frac{EI}{mL^4} \frac{\gamma_k}{(\alpha+\gamma_m)} \tag{56}$$

$$\text{or} \quad \omega_1 \approx f_0 \sqrt{\frac{\gamma_k}{\alpha+\gamma_m}} \tag{57}$$

where the stiffness correction factor γ_k is defined as

$$\gamma_k = \frac{k_e}{k_{CL}}. \tag{58}$$

We only need to obtain γ_k and γ_m in order to apply the expression of the first natural frequency in equation (56).

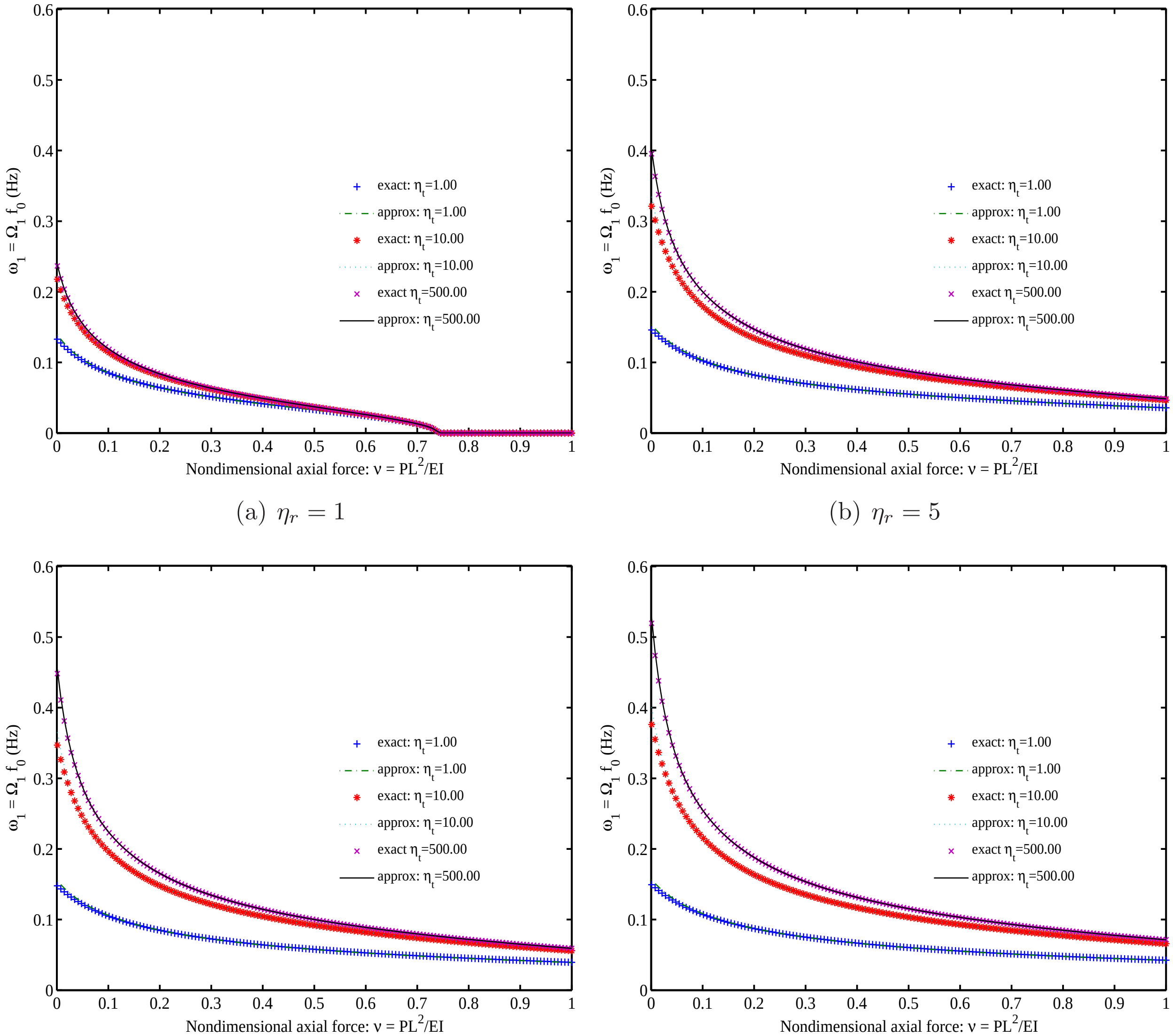
These correction factors can be obtained (using the procedure developed in reference [1]) as:

$$\begin{aligned} \gamma_k &= \frac{\lambda^3 \eta_t \left(\eta_r \cos(\lambda) - \lambda \sin(\lambda)\right)}{\eta_r \eta_t \left(\sin(\lambda) - \lambda \cos(\lambda)\right) + \lambda^2 \left(\eta_t \sin(\lambda) + \eta_r \cos(\lambda) - \lambda^2 \sin(\lambda)\right)} \\ \text{and} \quad \gamma_m &= \frac{3}{140} \frac{11\,{\eta_r}^2{\eta_t}^2 + 77\,{\eta_t}^2\eta_r + 105\,{\eta_r}^2\eta_t + 140\,{\eta_t}^2 + 420\,\eta_r\eta_t + 420\,{\eta_r}^2}{9\,{\eta_r}^2 + 6\,{\eta_r}^2\eta_t + 18\,\eta_r\eta_t + {\eta_r}^2{\eta_t}^2 + 6\,{\eta_t}^2\eta_r + 9\,{\eta_t}^2} \end{aligned} \tag{59}$$

where

$$\lambda = \sqrt{\nu} \tag{60}$$

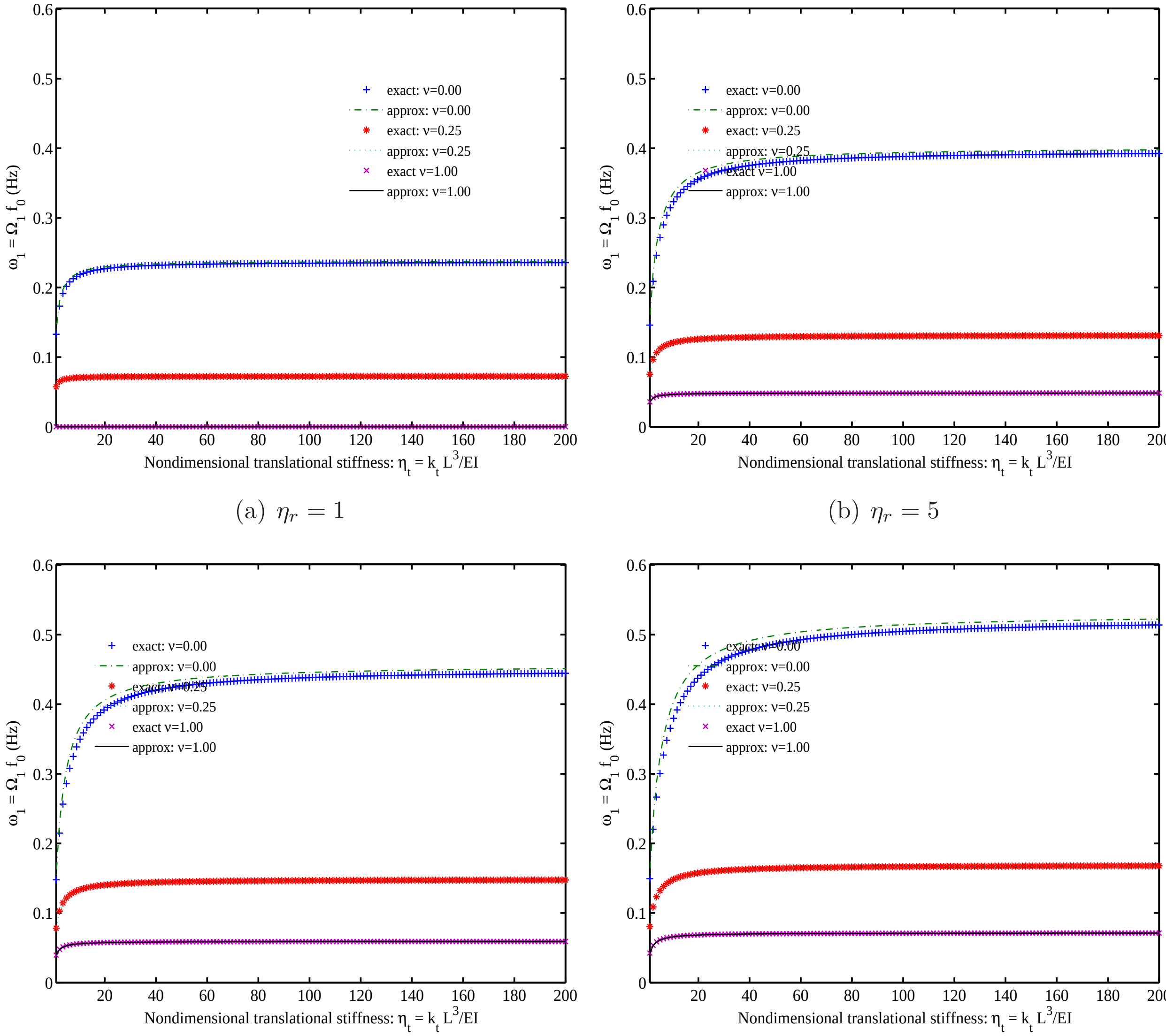
Substituting these expressions in the approximate formula (57) gives the complete parametric variation of the first-natural frequency in terms of ν, α, η_r and η_t.



(a) $\eta_r = 1$ (b) $\eta_r = 5$

(c) $\eta_r = 10$ (d) $\eta_r = 100$

Figure 9: Approximation of the first natural frequency of the wind turbine with respect to the nondimensional axial load ν for different values of nondimensional translational soil stiffness η_t. Four fixed values of the nondimensional rotational stiffness η_r are considered in the four subplots.



(a) $\eta_r = 1$ (b) $\eta_r = 5$

(c) $\eta_r = 10$ (d) $\eta_r = 100$

Figure 10: Approximation of the first natural frequency of the wind turbine with respect to the nondimensional translational stiffness η_t for different values of nondimensional axial load ν. Four fixed values of the nondimensional rotational soil stiffness η_r are considered in the four subplots.

Both these plots show that the results from this simple closed-form expression is in excellent agreement with the exact results. Therefore, the formula shown in equation (57) should be used for all practical purposes.

6 Summary & Conclusions

- Dynamics of flexible (marine and wind) turbine structures on elastic end supports has been investigated using the Euler Bernoulli beam theory with axial load, elastic support stiffness and top mass with rotary inertia.

- The physical assumptions behind the simplified model have been explained in details.

- The non-dimensional parameters necessary to understand the dynamic behavior are: nondimensional axial force (ν), nondimensional rotational soil stiffness, (η_r), nondimensional translational soil stiffness, (η_t), mass ratio between the building and the turbine (α), nondimensional radius of gyration of the turbine (μ).

- The characteristic equation governing the natural frequency of the system is obtained by solving the associated eigenvalue problem.

- One of the key conclusion is that the first natural frequency of the turbine structure will decrease with the decrease in the stiffness properties of the (soil) support and increase in the axial load in the column. This means that one needs to check the condition of the underlying soil and weight of the rotor-blade assembly. The first natural frequency of the system should be well separated from the the blade passing frequency.

- Based on an equivalent single-degree-of-freedom system assumption, a simple approximate expression of the first natural frequency is given.

- Numerical verifications confirm that the results from this simple closed-form expression is in excellent agreement from the results obtained via numerical solution of the complex transcendental frequency equation over a wide range of parameter values.

- Using this expression, designers could estimate the first natural frequency for various parameter values and design the turbine structure such that the resulting natural frequency does not come close to the blade passing frequency.

Suggested Reading

[1] S. Bhattacharya, S. Adhikari and N. A. Alexander, A simplified method for unified buckling and dynamic analysis of pile-supported structures in seismically liquefiable soils, Soil Dynamics and Earthquake Engineering (2009), under review. 22

[2] R. D. Blevins, *Formulas for Natural Frequency and Mode Shape*, Krieger Publishing Company, Malabar, FL, USA, 1984. 21

[3] M. Géradin and D. Rixen, *Mechanical Vibrations*, John Wiely & Sons, New York, NY, second edition, 1997, translation of: Théorie des Vibrations. 4

[4] J. van der Tempel and D.-P. Molenaar, Wind turbine structural dynamics - A review of the principles for modern power generation, onshore and offshore, Wind Engineering 26 (2002), 211–220. 13