VIBRATION OF DAMPED SYSTEMS

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Objective

To develop methodologies for free and forced vibration analysis of damped linear MDOF systems in a generalized and unified manner.

Contents

1 Introduction

- Problems involving vibration occur in many areas of mechanical, civil and aerospace engineering. Quite often vibration is not desirable (except some cases like musical instruments) and our interest lies in reducing it by dissipation of vibration energy or *damping*.
- In spite of a large amount of research, understanding of damping mechanisms is quite primitive.
- A well known method for damping modelling is to use the so called 'viscous damping', first introduced by Rayleigh (1877). A further idealization, also pointed out by Rayleigh, is to assume the damping matrix to be a linear combination of the mass and stiffness matrices ('proportional damping' or 'classical damping'). With such a damping model, the *modal analysis* procedure, originally developed for undamped systems, can be used to analyze damped systems in a very similar manner.
- Viscous damping is not the only linear damping model.

What you need to know for this part

- Basic understanding of matrix algebra - matrix product, transpose, inverse, eigenvalue problem
- Basic understanding of Laplace and Fourier transforms (including inverse Laplace/Fourier transforms)

Some References

- Meirovitch (1967, 1980, 1997)

  - ✱Meirovitch, L. (1967), *Analytical Methods in Vibrations*, Macmillan Publishing Co., Inc., New York.
  - ✱Meirovitch, L. (1980), *Computational Methods in Structural Dynamics*, Sijthoff & Noordohoff, Netherlands.
  - ✱Meirovitch, L. (1997), *Principles and Techniques of Vibrations*, Prentice-Hall International, Inc., New Jersey.

- Newland (1989)

  - ✱Newland, D. E. (1989), *Mechanical Vibration Analysis and Computation*, Longman, Harlow and John Wiley, New York.

- Géradin and Rixen (1997)

  - ✱Géradin, M. and Rixen, D. (1997), *Mechanical Vibrations*, John Wiely & Sons, New York, NY, second edition, translation of: Théorie des Vibrations.

- Bathe (1982)

  - ✱Bathe, K. (1982), *Finite Element Procedures in Engineering Analysis*, Prentice-Hall Inc, New Jersey.

2 Dynamics of Undamped Systems

The equations of motion of an undamped non-gyroscopic system with N degrees of freedom:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{2.1}$$

$\mathbf{M} \in \mathbb{R}^{N\times N}$ is the mass matrix, $\mathbf{K} \in \mathbb{R}^{N\times N}$ is the stiffness matrix, $\mathbf{q}(t) \in \mathbb{R}^N$ is the vector of generalized coordinates and $\mathbf{f}(t) \in \mathbb{R}^N$ is the forcing vector. Solution of (2.1) requires the *initial conditions*:

$$\mathbf{q}(0) = \mathbf{q_0} \in \mathbb{R}^N \quad \text{and} \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_\mathbf{0} \in \mathbb{R}^N. \tag{2.2}$$

2.1 Modal Analysis

The natural frequencies (ω_j) and the mode shapes ($\mathbf{x}_j$) can be obtained by solving the associated matrix eigenvalue problem

$$\mathbf{K}\mathbf{x}_j = \omega_j^2 \mathbf{M}\mathbf{x}_j, \quad \forall\, j = 1, \cdots, N. \tag{2.3}$$

The eigenvalues and the eigenvectors are real, *i.e.*, $\omega_j \in \mathbb{R}$ and $\mathbf{x}_j \in \mathbb{R}^N$ since $\mathbf{M}$ and $\mathbf{K}$ are real symmetric and non-negative definite.

Premultiplying equation (2.3) by $\mathbf{x}_k^T$ we have

$$\mathbf{x}_k^T \mathbf{K}\mathbf{x}_j = \omega_j^2 \mathbf{x}_k^T \mathbf{M}\mathbf{x}_j \tag{2.4}$$

Taking transpose of the above equation and noting that $\mathbf{M}$ and $\mathbf{K}$ are symmetric matrices one has

$$\mathbf{x}_j^T \mathbf{K}\mathbf{x}_k = \omega_j^2 \mathbf{x}_j^T \mathbf{M}\mathbf{x}_k \tag{2.5}$$

Now consider the eigenvalue equation for the kth mode:

$$\mathbf{K}\mathbf{x}_k = \omega_k^2 \mathbf{M}\mathbf{x}_k \tag{2.6}$$

Premultiplying equation (2.6) by $\mathbf{x}_j^T$ we have

$$\mathbf{x}_j^T \mathbf{K} \mathbf{x}_k = \omega_k^2 \mathbf{x}_j^T \mathbf{M} \mathbf{x}_k \tag{2.7}$$

Subtracting equation (2.5) from (2.7) we have

$$\left(\omega_k^2 - \omega_j^2\right) \mathbf{x}_j^T \mathbf{M} \mathbf{x}_k = 0 \tag{2.8}$$

Since we assumed the natural frequencies are not repeated when $j \neq k$, $\omega_j \neq \omega_k$. Therefore, from equation (2.8) it follows that

$$\mathbf{x}_k^T \mathbf{M} \mathbf{x}_j = 0 \tag{2.9}$$

Using this in equation (2.5) we can also obtain

$$\mathbf{x}_k^T \mathbf{K} \mathbf{x}_j = 0 \tag{2.10}$$

Normalize $\mathbf{x}_j$ such that

$$\mathbf{x}_j \mathbf{M} \mathbf{x}_j = 1$$

From equation (2.5) it follows

$$\mathbf{x}_j \mathbf{K} \mathbf{x}_j = \omega_j^2.$$

- This normalization is known as unity mass normalization, a convention often used in practice.
- Equations (2.9) and (2.10) are known as orthogonality relationships.

Orthogonality and normalization relationships can be combined as

$$\mathbf{x}_l^T \mathbf{M} \mathbf{x}_j = \delta_{lj} \tag{2.11}$$

$$\text{and} \quad \mathbf{x}_l^T \mathbf{K} \mathbf{x}_j = \omega_j^2 \delta_{lj}, \quad \forall\, l, j = 1, \cdots, N \tag{2.12}$$

Kroneker delta function: $\delta_{lj} = 1$ for $l = j$ and $\delta_{lj} = 0$ otherwise.

Construct

$$\mathbf{\Omega} = \operatorname{diag}\left[\omega_1, \omega_2, \cdots, \omega_N\right] \in \mathbb{R}^{N \times N} \quad (\omega_1 < \omega_2 < \cdots \omega_k < \omega_{k+1}) \tag{2.13}$$

$$\text{and} \quad \mathbf{X} = \left[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\right] \in \mathbb{R}^{N \times N} \tag{2.14}$$

With these, (2.11) and (2.12) $\Rightarrow$

$$\mathbf{X}^T \mathbf{M} \mathbf{X} = \mathbf{I} \tag{2.15}$$

$$\text{and} \quad \mathbf{X}^T \mathbf{K} \mathbf{X} = \mathbf{\Omega}^2 \tag{2.16}$$

The modal transformation:

$$\mathbf{q}(t) = \mathbf{X}\mathbf{y}(t). \tag{2.17}$$

Substituting $\mathbf{q}(t)$ in equation (2.1), premultiplying by $\mathbf{X}^T$ and using the orthogonality relationships in (2.15) and (2.16), the equations of motion in the modal coordinates may be obtained as

$$\begin{aligned} \ddot{\mathbf{y}}(t) + \mathbf{\Omega}^2 \mathbf{y}(t) &= \tilde{\mathbf{f}}(t) \\ \text{or} \quad \ddot{y}_j(t) + \omega_j^2 y_j(t) &= \tilde{f}_j(t) \quad \forall\, j = 1, \cdots, N \end{aligned} \tag{2.18}$$

where $\tilde{\mathbf{f}}(t) = \mathbf{X}^T \mathbf{f}(t)$.

The set of equations (2.18) are uncoupled!!

'It is true that the grasping of truth is not possible without empirical basis. However, the deeper we penetrate and the more extensive and embracing our theories become, the less empirical knowledge is needed to determine those theories.'

Albert Einstein, December 1952

2.2 Dynamic Response

2.2.1 Frequency Domain Analysis

Taking the Laplace transform of (2.1) and considering the initial conditions in (2.2)

$$s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q_0} - \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) \tag{2.19}$$

$$\text{or} \quad \left[s^2\mathbf{M} + \mathbf{K}\right]\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\mathbf{M}\mathbf{q_0} = \bar{\mathbf{p}}(s) \;\; \text{(say)}. \tag{2.20}$$

Using the modal transformation

$$\bar{\mathbf{q}}(s) = \mathbf{X}\bar{\mathbf{y}}(s) \tag{2.21}$$

and premultiplying (2.20) by $\mathbf{X}^T$:

$$\begin{aligned} \left[s^2\mathbf{M} + \mathbf{K}\right]\mathbf{X}\bar{\mathbf{y}}(s) &= \bar{\mathbf{p}}(s) \\ \text{or} \quad \left\{\mathbf{X}^T\left[s^2\mathbf{M} + \mathbf{K}\right]\mathbf{X}\right\}\bar{\mathbf{y}}(s) &= \mathbf{X}^T\bar{\mathbf{p}}(s). \end{aligned} \tag{2.22}$$

Using the orthogonality relationships in (2.15) and (2.16), this equation reduces to

$$\left[s^2\mathbf{I} + \mathbf{\Omega}^2\right]\bar{\mathbf{y}}(s) = \mathbf{X}^T\bar{\mathbf{p}}(s) \tag{2.23}$$

$$\text{or} \quad \bar{\mathbf{y}}(s) = \left[s^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T\bar{\mathbf{p}}(s) \tag{2.24}$$

$$\text{or} \quad \mathbf{X}\bar{\mathbf{y}}(s) = \mathbf{X}\left[s^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T\bar{\mathbf{p}}(s) \quad \text{(premultiplying by } \mathbf{X}\text{)} \tag{2.25}$$

$$\text{or} \quad \bar{\mathbf{q}}(s) = \mathbf{X}\left[s^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T\bar{\mathbf{p}}(s) \quad \text{(using (2.21))} \tag{2.26}$$

$$\text{or} \quad \bar{\mathbf{q}}(s) = \mathbf{X}\left[s^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T\left\{\bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\mathbf{M}\mathbf{q_0}\right\} \quad \text{(using (2.20))}. \tag{2.27}$$

Equation (2.27) is the complete solution of the undamped dynamic response. In the frequency domain, substitute $s = i\omega$:

$$\begin{aligned} \bar{\mathbf{q}}(i\omega) &= \mathbf{X}\left[-\omega^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T\left\{\bar{\mathbf{f}}(i\omega) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + i\omega\mathbf{M}\mathbf{q_0}\right\} \\ &= \mathbf{H}(i\omega)\left\{\bar{\mathbf{f}}(i\omega) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + i\omega\mathbf{M}\mathbf{q_0}\right\}. \end{aligned} \tag{2.28}$$

The term

$$\mathbf{H}(i\omega) = \mathbf{X}\left[-\omega^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T \tag{2.29}$$

is the transfer function matrix or the receptance matrix.

$$\left[-\omega^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1} = \text{diag}\left[\frac{1}{\omega_1^2 - \omega^2}, \frac{1}{\omega_2^2 - \omega^2}, \cdots, \frac{1}{\omega_N^2 - \omega^2}\right]. \tag{2.30}$$

Therefore

$$\mathbf{X}\left[-\omega^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T =$$

$$[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N]\,\text{diag}\left[\frac{1}{\omega_1^2 - \omega^2}, \frac{1}{\omega_2^2 - \omega^2}, \cdots, \frac{1}{\omega_N^2 - \omega^2}\right]\begin{bmatrix}\mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_N^T\end{bmatrix} \tag{2.31}$$

$$= [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N]\begin{bmatrix}\frac{\mathbf{x}_1^T}{\omega_1^2 - \omega^2} \\ \frac{\mathbf{x}_2^T}{\omega_2^2 - \omega^2} \\ \vdots \\ \frac{\mathbf{x}_N^T}{\omega_N^2 - \omega^2}\end{bmatrix} = \left[\frac{\mathbf{x}_1\mathbf{x}_1^T}{\omega_1^2 - \omega^2} + \frac{\mathbf{x}_2\mathbf{x}_2^T}{\omega_2^2 - \omega^2} + \cdots + \frac{\mathbf{x}_N\mathbf{x}_N^T}{\omega_N^2 - \omega^2}\right]. \tag{2.32}$$

From this we obtain the familiar expression of the receptance matrix

$$\mathbf{H}(i\omega) = \sum_{j=1}^{N} \frac{\mathbf{x}_j\mathbf{x}_j^T}{\omega_j^2 - \omega^2}. \tag{2.33}$$

Substituting $\mathbf{H}(i\omega)$ in (2.28)

$$\begin{aligned}\bar{\mathbf{q}}(i\omega) &= \sum_{j=1}^{N} \frac{\mathbf{x}_j\mathbf{x}_j^T\left\{\bar{\mathbf{f}}(i\omega) + \mathbf{M}\dot{\mathbf{q}}_0 + i\omega\mathbf{M}\mathbf{q}_0\right\}}{\omega_j^2 - \omega^2} \\ &= \sum_{j=1}^{N} \frac{\mathbf{x}_j^T\bar{\mathbf{f}}(i\omega) + \mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_0 + i\omega\mathbf{x}_j^T\mathbf{M}\mathbf{q}_0}{\omega_j^2 - \omega^2}\mathbf{x}_j.\end{aligned} \tag{2.34}$$

2.2.2 Time Domain Analysis

Rewrite (2.34) in the Laplace domain:

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{N} \left\{ \frac{\mathbf{x}_j^T \bar{\mathbf{f}}(s)}{s^2 + \omega_j^2} + \frac{\mathbf{x}_j^T \mathbf{M} \dot{\mathbf{q}}_0}{s^2 + \omega_j^2} + \frac{s}{s^2 + \omega_j^2} \mathbf{x}_j^T \mathbf{M} \mathbf{q}_0 \right\} \mathbf{x}_j. \tag{2.35}$$

Need to take the inverse Laplace transform for the time-domain response:

$$\mathbf{q}(t) = \mathcal{L}^{-1}\left[\bar{\mathbf{q}}(s)\right] = \sum_{j=1}^{N} a_j(t) \mathbf{x}_j \tag{2.36}$$

where

$$a_j(t) = \mathcal{L}^{-1}\left[\frac{\mathbf{x}_j^T \bar{\mathbf{f}}(s)}{s^2 + \omega_j^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2 + \omega_j^2}\right] \mathbf{x}_j^T \mathbf{M} \dot{\mathbf{q}}_0 + \mathcal{L}^{-1}\left[\frac{s}{s^2 + \omega_j^2}\right] \mathbf{x}_j^T \mathbf{M} \mathbf{q}_0. \tag{2.37}$$

For the second and third parts

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + \omega_j^2}\right] = \frac{\sin(\omega_j t)}{\omega_j} \tag{2.38}$$

$$\text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2 + \omega_j^2}\right] = \cos(\omega_j t). \tag{2.39}$$

For the first part we will use the 'convolution theorem':

$$\mathcal{L}^{-1}\left[\bar{f}(s)\bar{g}(s)\right] = \int_0^t f(\tau) g(t-\tau) d\tau. \tag{2.40}$$

Considering $\bar{g}(s) = \dfrac{1}{s^2 + \omega_j^2}$

$$\mathcal{L}^{-1}\left[\mathbf{x}_j^T \bar{\mathbf{f}}(s) \frac{1}{s^2 + \omega_j^2}\right] = \int_0^t \frac{1}{\omega_j} \mathbf{x}_j^T \mathbf{f}(\tau) \sin(\omega_j (t-\tau))\, d\tau. \tag{2.41}$$

Combining (2.41), (2.38) and (2.39):

$$a_j(t) = \int_0^t \frac{1}{\omega_j}\mathbf{x}_j^T\mathbf{f}(\tau)\sin\left(\omega_j(t-\tau)\right)d\tau + \frac{1}{\omega_j}\sin(\omega_j\, t)\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \cos(\omega_j\, t)\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}. \tag{2.42}$$

Collect the terms associated with $\sin\left(\omega_{d_j}\, t\right)$ and $\cos\left(\omega_{d_j}\, t\right)$:

$$a_j(t) = \int_0^t \frac{1}{\omega_j}\mathbf{x}_j^T\mathbf{f}(\tau)\sin\left(\omega_j(t-\tau)\right)d\tau + B_j\cos\left(\omega_j t + \theta_j\right) \tag{2.43}$$

where

$$B_j = \sqrt{\left(\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}\right)^2 + \left(\frac{\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}}{\omega_j}\right)^2} \tag{2.44}$$

$$\text{and} \quad \tan\theta_j = -\frac{\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0}}{\omega_j\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}}. \tag{2.45}$$

From Equations (2.34) [frequency domain] and (2.36) [time domain]

Dynamic response of a linear undamped MDOF dynamic system can be expressed by a linear combination of the mode shapes

3 Models of Damping

- Damping is the dissipation of energy from a vibrating structure. In this context, the term dissipate is used to mean the transformation of energy into the other form of energy and, therefore, a removal of energy from the vibrating system. The type of energy into which the mechanical energy is transformed is dependent on the system and the physical mechanism that cause the dissipation. For most vibrating system, a significant part of the energy is converted into heat.

- The specific ways in which energy is dissipated in vibration are dependent upon the physical mechanisms active in the structure. These physical mechanisms are complicated physical process that are not totally understood. The types of damping that are present in the structure will depend on which mechanisms predominate in the given situation. Thus, any mathematical representation of the physical damping mechanisms in the equations of motion of a vibrating system will have to be a generalization and approximation of the true physical situation. Any mathematical damping model is really only a crutch which does not give a detailed explanation of the underlying physics.

3.1 Viscous Damping

The most popular approach to model damping in the context of multiple degrees-of-freedom (MDOF) systems – first introduced by Rayleigh (1877). By analogy with the potential energy and the kinetic energy, Rayleigh assumed the *dissipation function*:

$$\mathcal{F}(\mathbf{q}) = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} C_{jk} \dot{q}_j \dot{q}_k = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{C} \dot{\mathbf{q}}. \tag{3.1}$$

$\mathbf{C} \in \mathbb{R}^{N \times N}$ is a non-negative definite symmetric matrix – the viscous damping matrix. Viscous damping matrices can be further divided into classical and non-classical damping.

3.2 Non-viscous Damping Models

- It is important to avoid the widespread misconception that viscous damping is the *only* linear model of vibration damping in the context of MDOF systems. Any causal model which makes the energy dissipation functional non-negative is a possible candidate for a damping model.

- Fractional Derivative Model:

One popular approach is to model damping in terms of fractional derivatives of the displacements. The damping force:

$$\mathbf{F}_d = \sum_{j=1}^{l} \mathbf{g}_j D^{\nu_j}[\mathbf{q}(t)]. \tag{3.2}$$

$\mathbf{g}_j$ are complex constant matrices and the fractional derivative operator

$$D^{\nu_j}[\mathbf{q}(t)] = \frac{\mathrm{d}^{\nu_j}\mathbf{q}(t)}{\mathrm{d}t^{\nu_j}} = \frac{1}{\Gamma(1-\nu_j)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\mathbf{q}(t)}{(t-\tau)^{\nu_j}}\mathrm{d}\tau \tag{3.3}$$

where ν_j is a fraction and $\Gamma(\bullet)$ is the Gamma function.

- The familiar viscous damping appears as a special case when $\nu_j = 1$.
- Although this model might fit experimental data quite well, the physical justification for such models, however, is far from clear at the present time.

- Convolution Integration Model:

Here damping forces depend on the past history of motion via convolution integrals over some kernel functions. A *modified dissipation function* for such damping model can be defined as

$$\mathcal{F}(\mathbf{q}) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\dot{q}_k\int_0^t \mathcal{G}_{jk}(t-\tau)\dot{q}_j(\tau)\mathrm{d}\tau = \frac{1}{2}\dot{\mathbf{q}}^T\int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{q}}(\tau)\mathrm{d}\tau. \tag{3.4}$$

Here $\boldsymbol{\mathcal{G}}(t) \in \mathbb{R}^{N\times N}$ is a symmetric matrix of the damping kernel functions, $\mathcal{G}_{jk}(t)$.

* The familiar viscous damping appears as a special case when $\boldsymbol{\mathcal{G}}(t-\tau) = \mathbf{C}\,\delta(t-\tau)$, where $\delta(t)$ is the Dirac-delta function.
* By choosing suitable kernel functions, it can also be shown that the fractional derivative model discussed before is also a special case of this damping model. It is therefore, possibly the most general way to model damping.
* For further discussions see Woodhouse (1998), Adhikari (2000, 2002)

The damping kernel functions are commonly defined in the frequency/Laplace domain. Several authors have proposed several damping models and they are summarized below:

Table 1: Summary of damping functions in the Laplace domain

Damping functions	Author, Year
$G(s) = \sum_{k=1}^{n} \frac{a_k s}{s + b_k}$	Biot (1955, 1958)
$G(s) = \frac{E_1 s^\alpha - E_0 b s^\beta}{1 + b s^\beta}$ $0 < \alpha < 1, \quad 0 < \beta < 1$	Bagley and Torvik (1983)
$sG(s) = G^\infty \left[1 + \sum_k \alpha_k \frac{s^2 + 2\zeta_k \omega_k s}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}\right]$	Golla and Hughes (1985) and McTavish and Hughes (1993)
$G(s) = 1 + \sum_{k=1}^{n} \frac{\Delta_k s}{s + \beta_k}$	Lesieutre and Mingori (1990)
$G(s) = c \frac{1 - e^{-st_0}}{st_0}$	Adhikari (1998)
$G(s) = c \frac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}$	Adhikari (1998)

4 Proportionally Damped Systems

- The non-conservative forces in Lagrange's equation

$$Q_{\text{nc}_k} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_k}, \quad k = 1, \cdots, N \tag{4.1}$$

The equations of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t). \tag{4.2}$$

The aim is to solve this equation (together with the initial conditions) by modal analysis as described in Section 2.1.

- Equations of motion of a damped system in the modal coordinates

$$\ddot{\mathbf{y}}(t) + \mathbf{X}^T\mathbf{C}\mathbf{X}\dot{\mathbf{y}}(t) + \boldsymbol{\Omega}^2\mathbf{y}(t) = \tilde{\mathbf{f}}(t). \tag{4.3}$$

Unless $\mathbf{X}^T\mathbf{C}\mathbf{X}$ is a diagonal matrix, no advantage can be gained by employing modal analysis because the equations of motion will still be coupled.

- Proportional damping assumptions is required.

The proportional damping model expresses the damping matrix as a linear combination of the mass and stiffness matrices, that is

$$\mathbf{C} = \alpha_1 \mathbf{M} + \alpha_2 \mathbf{K} \tag{4.4}$$

where α_1, α_2 are real scalars. This damping model is also known as 'Rayleigh damping' or 'classical damping'. Modes of classically damped systems preserve the simplicity of the real normal modes as in the undamped case.

With proportional damping assumption:

- The damping matrix $\mathbf{C}$ is simultaneously diagonalizable with $\mathbf{M}$ and $\mathbf{K}$ *i.e.*, the damping matrix in the modal coordinate

$$\mathbf{C}' = \mathbf{X}^T \mathbf{C} \mathbf{X} \tag{4.5}$$

  is a diagonal matrix.

- The damping ratios ζ_j are defined from the diagonal elements of the modal damping matrix as

$$C'_{jj} = 2\zeta_j \omega_j \quad \forall j = 1, \cdots, N. \tag{4.6}$$

- It allows to analyze damped systems in very much the same manner as undamped systems since the equations of motion in the modal coordinate can be decoupled as

$$\ddot{y}_j(t) + 2\zeta_j \omega_j \dot{y}_j(t) + \omega_j^2 y_j(t) = \tilde{f}_j(t) \quad \forall\, j = 1, \cdots, N. \tag{4.7}$$

4.1 Condition for Proportional Damping

Classical damping can exist in more general situation. Caughey and O'Kelly (1965) have derived the condition which the system matrices must satisfy so that viscously damped linear systems possess classical normal modes.

Theorem 1 *Viscously damped system (4.2) possesses classical normal modes if and only if* $\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$.

Outline of the Proof. Assuming $\mathbf{M}$ is not singular, premultiplying equation (4.2) by $\mathbf{M}^{-1}$ we have

$$\mathbf{I}\ddot{\mathbf{q}}(t) + [\mathbf{M}^{-1}\mathbf{C}]\dot{\mathbf{q}}(t) + [\mathbf{M}^{-1}\mathbf{K}]\mathbf{q}(t) = \mathbf{M}^{-1}\mathbf{f}(t). \tag{4.8}$$

For classical normal modes, (4.8) must be diagonalized by an orthogonal transformation. Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be diagonalized by an orthogonal transformation if and only if the commute in product, *i.e.*, $\mathbf{AB} = \mathbf{BA}$. Using this condition in (4.8)

$$[\mathbf{M}^{-1}\mathbf{C}][\mathbf{M}^{-1}\mathbf{K}] = [\mathbf{M}^{-1}\mathbf{K}][\mathbf{M}^{-1}\mathbf{C}], \quad \text{or} \quad \mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}. \tag{4.9}$$

This theorem was originally proved by Caughey and O'Kelly (1965). A modified and more general version of this theorem was proved by Adhikari (2001).

Example 1: Assume that

$$\mathbf{M} = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 1.0 & 2.0 & 2.0 \\ 1.0 & 2.0 & 3.0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1.2 & 0.4 \\ 0.5 & 0.4 & 1.8 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 15.25 & -9.8 & 3.4 \\ -9.8 & 6.48 & -1.84 \\ 3.4 & -1.84 & 2.22 \end{bmatrix}.$$

All the system matrices are positive definite. The mass-normalized undamped modal matrix

$$\mathbf{X} = \begin{bmatrix} 0.4027 & -0.5221 & -1.2511 \\ 0.5845 & -0.4888 & 1.1914 \\ -0.1127 & 0.9036 & -0.4134 \end{bmatrix}. \tag{4.10}$$

Since Caughey and O'Kelly's condition

$$\mathbf{K}\mathbf{M}^{-1}\mathbf{C} = \mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} 125.45 & -80.92 & 28.61 \\ -80.92 & 52.272 & -18.176 \\ 28.61 & -18.176 & 7.908 \end{bmatrix}$$

is satisfied, the system possess classical normal modes and that $\mathbf{X}$ given in equation (4.10) is the modal matrix.

Verify using MATLAB!!

4.2 Generalized Proportional Damping

- We want to find $\mathbf{C}$ in terms of $\mathbf{M}$ and $\mathbf{K}$ such that the system still possesses classical normal modes. Caughey (1960) proposed that a *sufficient* condition for the existence of classical normal modes is: if $\mathbf{M}^{-1}\mathbf{C}$ can be expressed in a series involving powers of $\mathbf{M}^{-1}\mathbf{K}$. Later, Caughey and O'Kelly (1965) proved that the series ('Caughey series') representation of damping

$$\mathbf{C} = \mathbf{M} \sum_{j=0}^{N-1} \alpha_j \left[\mathbf{M}^{-1}\mathbf{K}\right]^j \tag{4.11}$$

is the *necessary and sufficient* condition for existence of classical normal modes. This generalized Rayleigh's proportional damping, which turns out to be the first two terms of the series.

- A further generalized and useful form of proportional damping was proposed by Adhikari (2001):

$$\mathbf{C} = \mathbf{M}\,\beta_1\left(\mathbf{M}^{-1}\mathbf{K}\right) + \mathbf{K}\,\beta_2\left(\mathbf{K}^{-1}\mathbf{M}\right) \tag{4.12}$$

$$\text{or} \quad \mathbf{C} = \beta_3\left(\mathbf{K}\mathbf{M}^{-1}\right)\mathbf{M} + \beta_4\left(\mathbf{M}\mathbf{K}^{-1}\right)\mathbf{K} \tag{4.13}$$

The functions β_i can be any analytical functions. This kind of damping model is known as *generalized proportional damping*. Equation (4.12) is *right-functional form* and equation (4.13) is *left-functional form*.

- Rayleigh's proportional damping is a special case:

$$\beta_i(\bullet) = \alpha_i \mathbf{I}. \tag{4.14}$$

The functions $\beta_i(\bullet)$ are called *proportional damping functions*.

Theorem 2 *A viscously damped positive definite linear system possesses classical normal modes if and only if* $\mathbf{C}$ *can be represented by*

(a) $\mathbf{C} = \mathbf{M}\,\beta_1\left(\mathbf{M}^{-1}\mathbf{K}\right) + \mathbf{K}\,\beta_2\left(\mathbf{K}^{-1}\mathbf{M}\right)$, *or*

(b) $\mathbf{C} = \beta_3\left(\mathbf{K}\mathbf{M}^{-1}\right)\mathbf{M} + \beta_4\left(\mathbf{M}\mathbf{K}^{-1}\right)\mathbf{K}$

for any $\beta_i(\bullet), i = 1, \cdots, 4$.

4.3 Dynamic Response

4.3.1 Frequency Domain Analysis

Taking the Laplace transform of (4.2) and considering the initial conditions

$$s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q_0} - \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\mathbf{C}\bar{\mathbf{q}} - \mathbf{C}\mathbf{q_0} + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) \tag{4.15}$$

$$\text{or} \quad \left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{C}\mathbf{q_0} + s\mathbf{M}\mathbf{q_0}. \tag{4.16}$$

Consider the modal damping matrix

$$\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X} = 2\boldsymbol{\zeta}\boldsymbol{\Omega} \tag{4.17}$$

$$\text{where} \quad \boldsymbol{\zeta} = \text{diag}\left[\zeta_1, \zeta_2, \cdots, \zeta_N\right] \in \mathbb{R}^{N\times N} \tag{4.18}$$

Using the mode orthogonality relationships and following the procedure similar to undamped systems

$$\bar{\mathbf{q}}(s) = \mathbf{X}\left[s^2\mathbf{I} + 2s\boldsymbol{\zeta}\boldsymbol{\Omega} + \boldsymbol{\Omega}^2\right]^{-1}\mathbf{X}^T\left\{\bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{C}\mathbf{q_0} + s\mathbf{M}\mathbf{q_0}\right\}. \tag{4.19}$$

In the frequency domain can be obtained by substitute $s = i\omega$. The transfer function matrix or the receptance matrix:

$$\mathbf{H}(i\omega) = \mathbf{X}\left[-\omega^2\mathbf{I} + 2i\omega\boldsymbol{\zeta}\boldsymbol{\Omega} + \boldsymbol{\Omega}^2\right]^{-1}\mathbf{X}^T = \sum_{j=1}^{N}\frac{\mathbf{x}_j\mathbf{x}_j^T}{-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2}. \tag{4.20}$$

Using this, the dynamic response in the frequency domain can be conveniently represented from equation (4.19) as

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{N}\frac{\mathbf{x}_j^T\bar{\mathbf{f}}(i\omega) + \mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{x}_j^T\mathbf{C}\mathbf{q_0} + i\omega\mathbf{x}_j^T\mathbf{M}\mathbf{q_0}}{-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2}\mathbf{x}_j. \tag{4.21}$$

Therefore, like undamped systems, the dynamic response of proportionally damped system can also be expressed as a linear combination of the undamped mode shapes.

4.3.2 Time Domain Analysis

Rewrite equation (4.21) in the Laplace domain as

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{N} \left\{ \frac{\mathbf{x}_j^T \bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} + \frac{\mathbf{x}_j^T \mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{x}_j^T \mathbf{C}\mathbf{q}_0}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} + \frac{s}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} \mathbf{x}_j^T \mathbf{M}\mathbf{q}_0 \right\} \mathbf{x}_j. \tag{4.22}$$

Reorganize the denominator as

$$s^2 + 2s\zeta_j\omega_j + \omega_j^2 = (s + \zeta_j\omega_j)^2 - (\zeta_j\omega_j)^2 + \omega_j^2 = (s + \zeta_j\omega_j)^2 + \omega_{d_j}^2 \tag{4.23}$$

$$\text{where} \quad \omega_{d_j} = \omega_j \sqrt{1 - \zeta_j^2} \tag{4.24}$$

is known as the damped natural frequency. From the table of Laplace transforms

$$\mathcal{L}^{-1}\left[\frac{1}{(s+\alpha)^2 + \beta^2}\right] = \frac{e^{-\alpha t} \sin(\beta t)}{\beta} \tag{4.25}$$

$$\text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{(s+\alpha)^2 + \beta^2}\right] = e^{-\alpha t} \cos(\beta t) - \frac{\alpha e^{-\alpha t} \sin(\beta t)}{\beta}. \tag{4.26}$$

Taking the inverse Laplace transform of (4.22)

$$\mathbf{q}(t) = \mathcal{L}^{-1}\left[\bar{\mathbf{q}}(s)\right] = \sum_{j=1}^{N} a_j(t)\mathbf{x}_j \tag{4.27}$$

$$\begin{aligned} a_j(t) =& \mathcal{L}^{-1}\left[\frac{\mathbf{x}_j^T\bar{\mathbf{f}}(s)}{(s+\zeta_j\omega_j)^2+\omega_{d_j}^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+\zeta_j\omega_j)^2+\omega_{d_j}^2}\right]\left(\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0}+\mathbf{x}_j^T\mathbf{C}\mathbf{q}_\mathbf{0}\right) \\ &+ \mathcal{L}^{-1}\left[\frac{s}{(s+\zeta_j\omega_j)^2+\omega_{d_j}^2}\right]\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0} \\ =& \int_0^t \frac{1}{\omega_{d_j}}\mathbf{x}_j^T\mathbf{f}(\tau)e^{-\zeta_j\omega_j(t-\tau)}\sin\left(\omega_{d_j}(t-\tau)\right)d\tau \\ &+ \frac{e^{-\zeta_j\omega_j t}}{\omega_{d_j}}\sin(\omega_{d_j}\,t)\left(\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0}+\mathbf{x}_j^T\mathbf{C}\mathbf{q}_\mathbf{0}\right) \\ &+ \left\{e^{-\zeta_j\omega_j t}\cos\left(\omega_{d_j}t\right) - \frac{\zeta_j\omega_j\,e^{-\zeta_j\omega_j t}\sin\left(\omega_{d_j}\,t\right)}{\omega_{d_j}}\right\}\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0} \end{aligned} \tag{4.28}$$

Collecting the terms associated with $\sin\left(\omega_{d_j}\,t\right)$ and $\cos\left(\omega_{d_j}\,t\right)$:

$$a_j(t) = \int_0^t \frac{1}{\omega_{d_j}}\mathbf{x}_j^T\mathbf{f}(\tau)e^{-\zeta_j\omega_j(t-\tau)}\sin\left(\omega_{d_j}(t-\tau)\right)d\tau + e^{-\zeta_j\omega_j t}B_j\cos\left(\omega_{d_j}t+\theta_j\right) \tag{4.29}$$

$$\text{where}\quad B_j = \sqrt{\left(\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}\right)^2 + \frac{1}{\omega_{d_j}^2}\left(\zeta_j\omega_j\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0} - \mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} - \mathbf{x}_j^T\mathbf{C}\mathbf{q}_\mathbf{0}\right)^2} \tag{4.30}$$

$$\text{and}\quad \tan\theta_j = \frac{1}{\omega_{d_j}}\left(\zeta_j\omega_j - \frac{\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0}+\mathbf{x}_j^T\mathbf{C}\mathbf{q}_\mathbf{0}}{\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}}\right) \tag{4.31}$$

Exercise:

1. Verify that when damping is zero ($i.e.$, $\zeta_j = 0, \forall j$) these expressions reduce to the corresponding expressions for undamped systems obtained before.
2. Verify that equations (4.22) and (4.27) have dimensions of lengths.
3. Check that dynamic response (in the frequency and time domain) is linear with respect to the applied loading and initial conditions.

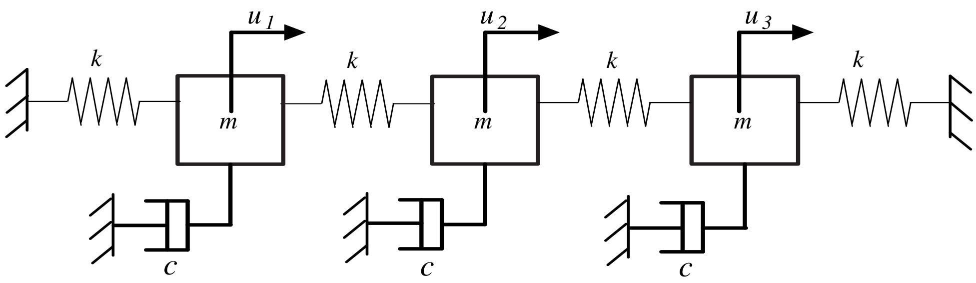

Figure 1: There DOF damped spring-mass system with dampers attached to the ground

Example 2: Figure 1 shows a three DOF spring-mass system. The mass of each block is m Kg and the stiffness of each spring is k N/m. The viscous damping constant of the damper associated with each block is c Ns/m. The aim is to obtain the dynamic response for the following load cases:

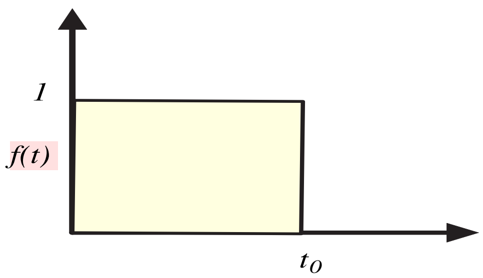

Figure 2: Unit step forcing, $t_0 = \frac{2\pi}{\omega_1}$

1. When only the first mass (DOF 1) is subjected to an unit step input (see Figure 2) so that $\mathbf{f}(t) = \{f(t), 0, 0\}^T$ and $f(t) = 1 - U(t - t_0)$ with $t_0 = \frac{2\pi}{\omega_1}$ where ω_1 is the first undamped natural frequency of the system and $U(\bullet)$ is the unit step function.

2. When only the second mass (DOF 2) is subjected to unit initial displacement, *i.e.*, $\mathbf{q_0} = \{0, 1, 0\}^T$.

3. When only the second and the third masses (DOF 3) are subjected to unit initial velocities, *i.e.*, $\dot{\mathbf{q}}_{\mathbf{0}} = \{0, 1, 1\}^T$.

4. When all three of the above loading are acting together on the system.

You are encouraged to use the associated Matlab programs for better understanding

I. *Obtain the System Matrices:* The mass, stiffness and the damping matrices are given by

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}. \tag{4.32}$$

Note that the damping matrix is mass proportional, so that the system is proportionally damped.

II. *Obtain the undamped natural frequencies:* For notational convenience assume that the eigenvalues $\lambda_j = \omega_j^2$. The three DOF system has three eigenvalues and they are the roots of the following characteristic equation

$$\det\left[\mathbf{K} - \lambda \mathbf{M}\right] = 0. \tag{4.33}$$

Using the mass and the stiffness matrices from equation (4.32), this can be simplified as

$$\begin{aligned} &\det \begin{bmatrix} 2k - \lambda m & -k & 0 \\ -k & 2k - \lambda m & -k \\ 0 & -k & 2k - \lambda m \end{bmatrix} = 0 \\ \text{or} \quad &m \det \begin{bmatrix} 2\alpha - \lambda & -\alpha & 0 \\ -\alpha & 2\alpha - \lambda & -\alpha \\ 0 & -\alpha & 2\alpha - \lambda \end{bmatrix} = 0 \quad \text{where} \quad \alpha = \frac{k}{m}. \end{aligned} \tag{4.34}$$

Expanding the determinant in (4.34) we have

$$\begin{aligned}
& (2\alpha-\lambda)\left\{(2\alpha-\lambda)^2-\alpha^2\right\}-\alpha\alpha\,(2\alpha-\lambda)=0 \\
\text{or} \quad & (2\alpha-\lambda)\left\{(2\alpha-\lambda)^2-2\alpha^2\right\}=0 \\
\text{or} \quad & (2\alpha-\lambda)\left\{(2\alpha-\lambda)^2-\left(\sqrt{2}\alpha\right)^2\right\}=0 \\
\text{or} \quad & (2\alpha-\lambda)\left(2\alpha-\lambda-\sqrt{2}\alpha\right)\left(2\alpha-\lambda+\sqrt{2}\alpha\right)=0 \\
\text{or} \quad & (2\alpha-\lambda)\left(\left(2-\sqrt{2}\right)\alpha-\lambda\right)\left(\left(2+\sqrt{2}\right)\alpha-\lambda\right)=0.
\end{aligned} \tag{4.35}$$

It implies that the three roots (in the increasing order) are

$$\lambda_1=\left(2-\sqrt{2}\right)\alpha, \quad \lambda_2=2\alpha, \quad \text{and} \quad \lambda_3=\left(2+\sqrt{2}\right)\alpha. \tag{4.36}$$

Since $\lambda_j=\omega_j^2$, the natural frequencies are

$$\omega_1=\sqrt{\left(2-\sqrt{2}\right)}\sqrt{\alpha}, \quad \omega_2=\sqrt{2}\sqrt{\alpha}, \quad \text{and} \quad \omega_3=\sqrt{\left(2+\sqrt{2}\right)}\sqrt{\alpha}. \tag{4.37}$$

III. *Obtain the undamped mode shapes:* From equation (2.3) the eigenvalue equation can be written as

$$[\mathbf{K}-\lambda_j\mathbf{M}]\,\mathbf{x}_j=0. \tag{4.38}$$

Substituting **K** and **M** from equation (4.32) and dividing by m

$$\begin{bmatrix} 2\alpha-\lambda & -\alpha & 0 \\ -\alpha & 2\alpha-\lambda & -\alpha \\ 0 & -\alpha & 2\alpha-\lambda \end{bmatrix}\begin{Bmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \end{Bmatrix}=0. \tag{4.39}$$

Here x_{1j}, x_{2j} and x_{3j} are the three components of jth eigenvector corresponding to the three masses. To obtain $\mathbf{x}_j$ we need to substitute λ_j from equation (4.36) in the above equation and solve for each component of $\mathbf{x}_j$ for every j.

The first eigenvector, $j = 1$:

Substituting $\lambda = \lambda_1 = \left(2 - \sqrt{2}\right) \alpha$ in (4.39) we have

$$\begin{bmatrix} 2\alpha - \left(2-\sqrt{2}\right)\alpha & -\alpha & 0 \\ -\alpha & 2\alpha - \left(2-\sqrt{2}\right)\alpha & -\alpha \\ 0 & -\alpha & 2\alpha - \left(2-\sqrt{2}\right)\alpha \end{bmatrix} \begin{Bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{Bmatrix} = 0 \tag{4.40}$$

or

$$\begin{bmatrix} \sqrt{2}\alpha & -\alpha & 0 \\ -\alpha & \sqrt{2}\alpha & -\alpha \\ 0 & -\alpha & \sqrt{2}\alpha \end{bmatrix} \begin{Bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{Bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{bmatrix} \begin{Bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{Bmatrix} = 0. \tag{4.41}$$

This can be separated into three equations as

$$\sqrt{2}x_{11} - x_{21} = 0, \quad -x_{11} + \sqrt{2}x_{21} - x_{31} = 0 \quad \text{and} \quad -x_{21} + \sqrt{2}x_{31} = 0. \tag{4.42}$$

These three equations cannot be solved uniquely but once we fix one element, the other two elements can be expressed in terms of it. This implies that the ratios between the modal amplitudes are unique. Solving the system of linear equations (4.42)

$$x_{21} = \sqrt{2}x_{11}, \quad x_{21} = \sqrt{2}x_{31}, \text{ that is } x_{11} = x_{31} = \gamma_1(\text{say}).$$

The first eigenvector

$$\mathbf{x}_1 = \begin{Bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{Bmatrix} = \gamma_1 \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}. \tag{4.43}$$

The constant γ_1 can be obtained from the mass normalization condition

$$\mathbf{x}_1^T \mathbf{M} \mathbf{x}_1 = 1 \quad \text{or} \quad \gamma_1 \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \gamma_1 \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} = 1 \tag{4.44}$$

$$\text{or} \quad \gamma_1^2 m \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} = 1 \tag{4.45}$$

$$\text{or} \quad \gamma_1^2 m \left(1 + \sqrt{2}\sqrt{2} + 1\right) = 1 \quad \text{that is} \quad \gamma_1 = \frac{1}{2\sqrt{m}}. \tag{4.46}$$

Thus the mass normalized first eigenvector is given by

$$\mathbf{x}_1 = \frac{1}{2\sqrt{m}} \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}. \tag{4.47}$$

The second eigenvector, $j = 2$:

Substituting $\lambda = \lambda_2 = 2\alpha$ in (4.39) we have

$$\begin{bmatrix} 2\alpha - 2\alpha & -\alpha & 0 \\ -\alpha & 2\alpha - 2\alpha & -\alpha \\ 0 & -\alpha & 2\alpha - 2\alpha \end{bmatrix} \begin{Bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{Bmatrix} = 0 \tag{4.48}$$

$$\text{or} \quad \begin{bmatrix} 0 & -\alpha & 0 \\ -\alpha & 0 & -\alpha \\ 0 & -\alpha & 0 \end{bmatrix} \begin{Bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{Bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{Bmatrix} = 0. \tag{4.49}$$

This implies that $x_{22} = 0$ and $x_{12} = -x_{32} = \gamma_2$ (say). The second eigenvector

$$\mathbf{x}_2 = \begin{Bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{Bmatrix} = \gamma_2 \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}. \tag{4.50}$$

Using the mass normalization condition

$$\mathbf{x}_2^T \mathbf{M} \mathbf{x}_2 = 1 \quad \text{or} \quad \gamma_2 \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \gamma_2 \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = 1 \tag{4.51}$$

$$\text{or} \quad \gamma_2^2 m \,(1+1) = 1 \quad \text{that is} \quad \gamma_2 = \frac{1}{\sqrt{2m}} = \frac{\sqrt{2}}{2\sqrt{m}}. \tag{4.52}$$

Thus, the mass normalized second eigenvector is given by

$$\mathbf{x}_2 = \frac{1}{2\sqrt{m}} \begin{Bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{Bmatrix}. \tag{4.53}$$

The third eigenvector, $j = 3$:

Substituting $\lambda = \lambda_3 = \left(2+\sqrt{2}\right)\alpha$ in (4.39) we have

$$\begin{bmatrix} 2\alpha - \left(2+\sqrt{2}\right)\alpha & -\alpha & 0 \\ -\alpha & 2\alpha - \left(2+\sqrt{2}\right)\alpha & -\alpha \\ 0 & -\alpha & 2\alpha - \left(2+\sqrt{2}\right)\alpha \end{bmatrix} \begin{Bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{Bmatrix} = 0 \tag{4.54}$$

$$\text{or} \quad \begin{bmatrix} -\sqrt{2}\alpha & -\alpha & 0 \\ -\alpha & -\sqrt{2}\alpha & -\alpha \\ 0 & -\alpha & -\sqrt{2}\alpha \end{bmatrix} \begin{Bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{Bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \begin{Bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{Bmatrix} = 0. \tag{4.55}$$

This implies that

$$\begin{aligned} \sqrt{2}x_{13} + x_{23} = 0 &\Rightarrow x_{23} = -\sqrt{2}x_{13} \\ x_{13} + \sqrt{2}x_{23} + x_{33} = 0 & \\ \text{and} \quad x_{23} + \sqrt{2}x_{33} = 0 &\Rightarrow x_{23} = -\sqrt{2}x_{33} \end{aligned} \tag{4.56}$$

that is $x_{13} = x_{33} = \gamma_3$ (say). Therefore the third eigenvector is given by

$$\mathbf{x}_3 = \begin{Bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{Bmatrix} = \gamma_3 \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}. \tag{4.57}$$

Using the mass normalization condition

$$\mathbf{x}_3^T \mathbf{M} \mathbf{x}_3 = 1 \quad \text{or} \quad \gamma_3 \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \gamma_3 \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix} = 1 \tag{4.58}$$

$$\text{or} \quad \gamma_3^2 m \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix} = 1 \tag{4.59}$$

$$\text{or} \quad \gamma_3^2 m \left(1 + \sqrt{2}\sqrt{2} + 1\right) = 1, \quad \text{that is,} \quad \gamma_3 = \frac{1}{2\sqrt{m}}. \tag{4.60}$$

Thus the mass normalized third eigenvector is given by

$$\mathbf{x}_3 = \frac{1}{2\sqrt{m}} \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}. \tag{4.61}$$

Combining the three eigenvectors, the mass normalized undamped modal matrix is given by

$$\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = \frac{1}{2\sqrt{m}} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}. \tag{4.62}$$

The modal matrix turns out to be symmetric. But in general this is not the case.

IV. *Obtain the modal damping ratios:* The damping matrix in the modal coordinate can be obtained from (4.5) as

$\mathbf{C}' = \mathbf{X}^T \mathbf{C} \mathbf{X} =$

$$\frac{1}{2\sqrt{m}} = \begin{bmatrix} 1 & \sqrt{2} & 1 \\ & \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}^T \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \frac{1}{2\sqrt{m}} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} = \frac{c}{m}\mathbf{I}. \tag{4.63}$$

Therefore

$$2\zeta_j \omega_j = \frac{c}{m} \quad \text{or} \quad \zeta_j = \frac{c}{2m\omega_j}. \tag{4.64}$$

Since ω_j becomes bigger for higher modes, modal damping gets smaller, *i.e.*, higher modes are less damped.

V. *Response due to applied loading:* The applied loading $\mathbf{f}(t) = \{f(t), 0, 0\}^T$ where $f(t) = 1 - U(t - t_0)$ with $t_0 = \frac{2\pi}{\omega_1}$. In the Laplace domain

$$\bar{f}(s) = \mathcal{L}\left[1 - U(t - t_0)\right] = 1 - \frac{e^{-st_0}}{s}. \tag{4.65}$$

Therefore

$$\mathbf{x}_j^T \bar{\mathbf{f}}(s) = \begin{Bmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \end{Bmatrix}^T \begin{Bmatrix} \bar{f}(s) \\ 0 \\ 0 \end{Bmatrix} = x_{1j} \left(1 - \frac{e^{-st_0}}{s}\right) \quad \forall j. \tag{4.66}$$

Since the initial conditions are zero, the dynamic response in the Laplace domain can be obtained from equation (4.22) as

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{3} \left\{ \frac{x_{1j}\left(1 - \dfrac{e^{-st_0}}{s}\right)}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} \right\} \mathbf{x}_j = \sum_{j=1}^{3} \left\{ \frac{x_{1j}\left(s - e^{-st_0}\right)}{s\left(s^2 + 2s\zeta_j\omega_j + \omega_j^2\right)} \right\} \mathbf{x}_j. \tag{4.67}$$

In the frequency domain, the response is given by

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{3} \left\{ \frac{x_{1j}\left(i\omega - e^{-i\omega t_0}\right)}{i\omega\left(-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2\right)} \right\} \mathbf{x}_j. \tag{4.68}$$

For the numerical calculations we assume $m = 1$, $k = 1$ and $c = 0.2$.

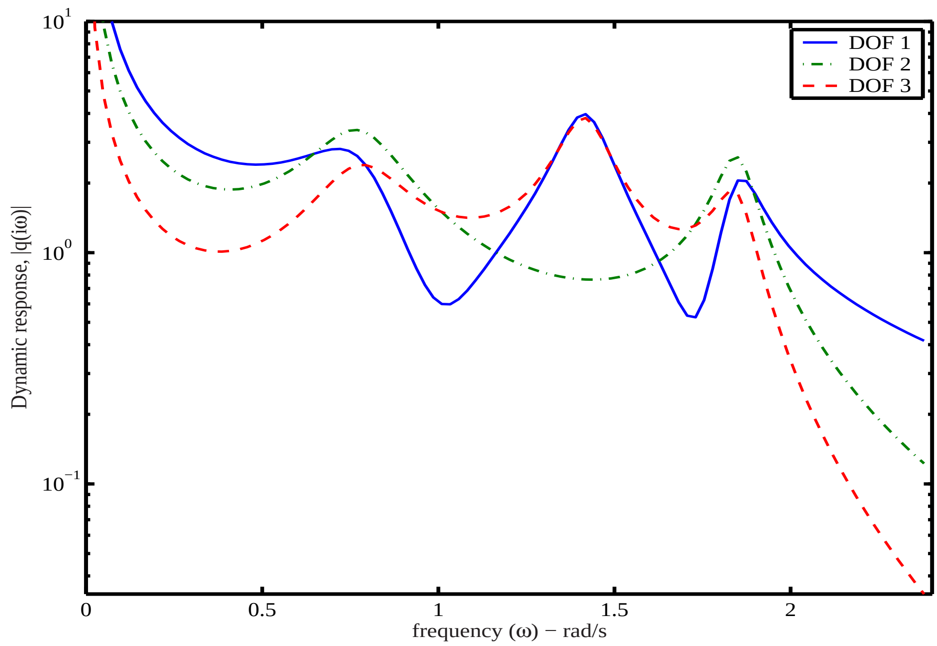

Figure 3: Absolute value of the frequency domain response of the three masses due to applied step loading at first DOF

The time domain response can be obtained by evaluating the convolution integral in (4.29) and substituting $a_j(t)$ in equation (4.27). In practice, usually numerical integration methods are used to evaluate this integral. For this problem a closed-from expression can be obtained. We have

$$\mathbf{x}_j^T \mathbf{f}(\tau) = x_{1j} f(\tau). \tag{4.69}$$

From Figure 2 it can be noted that

$$f(\tau) = \begin{cases} 1 & \text{if} \quad \tau < t_0, \\ 0 & \text{if} \quad \tau > t_0. \end{cases} \tag{4.70}$$

Because of this, the limit of the integral in (4.29) can be changed as

$$\begin{aligned} a_j(t) &= \int_0^t \frac{1}{\omega_{d_j}} \mathbf{x}_j^T \mathbf{f}(\tau) e^{-\zeta_j \omega_j (t-\tau)} \sin\left(\omega_{d_j}(t-\tau)\right) d\tau \\ &= \int_0^{t_0} \frac{1}{\omega_{d_j}} x_{1j} e^{-\zeta_j \omega_j (t-\tau)} \sin\left(\omega_{d_j}(t-\tau)\right) d\tau \\ &= \frac{x_{1j}}{\omega_{d_j}} \int_0^{t_0} e^{-\zeta_j \omega_j (t-\tau)} \sin\left(\omega_{d_j}(t-\tau)\right) d\tau. \end{aligned} \tag{4.71}$$

By making a substitution $\tau' = t - \tau$, this integral can be evaluated as

$$a_j(t) = \frac{x_{1j}}{\omega_{d_j}} \frac{e^{-\zeta_j \omega_j t}}{\omega_j^2} \left\{ \alpha_j \sin\left(\omega_{d_j} t\right) + \beta_j \cos\left(\omega_{d_j} t\right) \right\} \tag{4.72}$$

$$\text{where} \quad \alpha_j = \left\{ \omega_{d_j} \sin\left(\omega_{d_j} t_0\right) + \zeta_j \omega_j \cos\left(\omega_{d_j} t_0\right) \right\} e^{\zeta_j \omega_j t_0} - \zeta_j \omega_j \tag{4.73}$$

$$\text{and} \quad \beta_j = \left\{ \omega_{d_j} \cos\left(\omega_{d_j} t_0\right) - \zeta_j \omega_j \sin\left(\omega_{d_j} t_0\right) \right\} e^{\zeta_j \omega_j t_0} - \omega_{d_j}. \tag{4.74}$$

VI. *Response due to initial displacement:* When $\mathbf{q_0} = \{0, 1, 0\}^T$ we have

$$\mathbf{x}_j^T \mathbf{C} \mathbf{q_0} = \begin{Bmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \end{Bmatrix}^T \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = x_{2j} c \quad \forall j. \tag{4.75}$$

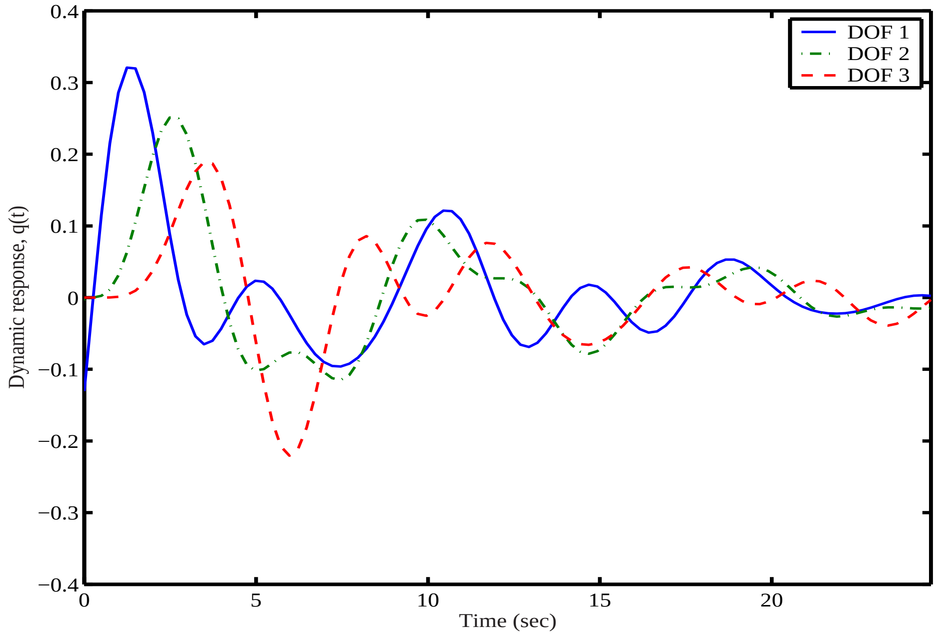

Figure 4: Time domain response of the three masses due to applied step loading at first DOF

Similarly

$$\mathbf{x}_j^T \mathbf{M} \mathbf{q_0} = x_{2j} m \quad \forall j. \tag{4.76}$$

The dynamic response in the Laplace domain can be obtained from equation (4.22) as

$$\begin{aligned}\bar{\mathbf{q}}(s) &= \sum_{j=1}^{3} \left\{ \frac{x_{2j} c}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} + \frac{x_{2j} m s}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} \right\} \mathbf{x}_j \\ &= \sum_{j=1}^{3} x_{2j} \left\{ \frac{c + ms}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} \right\} \mathbf{x}_j.\end{aligned} \tag{4.77}$$

From equation (4.64) note that $c = 2\zeta_j\omega_j m$. Substituting this in the above equation we have

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{3} x_{2j} m \left\{ \frac{2\zeta_j\omega_j + s}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} \right\} \mathbf{x}_j. \tag{4.78}$$

In the frequency domain the response is given by

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{3} x_{2j} m \left\{ \frac{2\zeta_j\omega_j + i\omega}{-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2} \right\} \mathbf{x}_j. \tag{4.79}$$

The time domain response can be obtained by directly taking the inverse

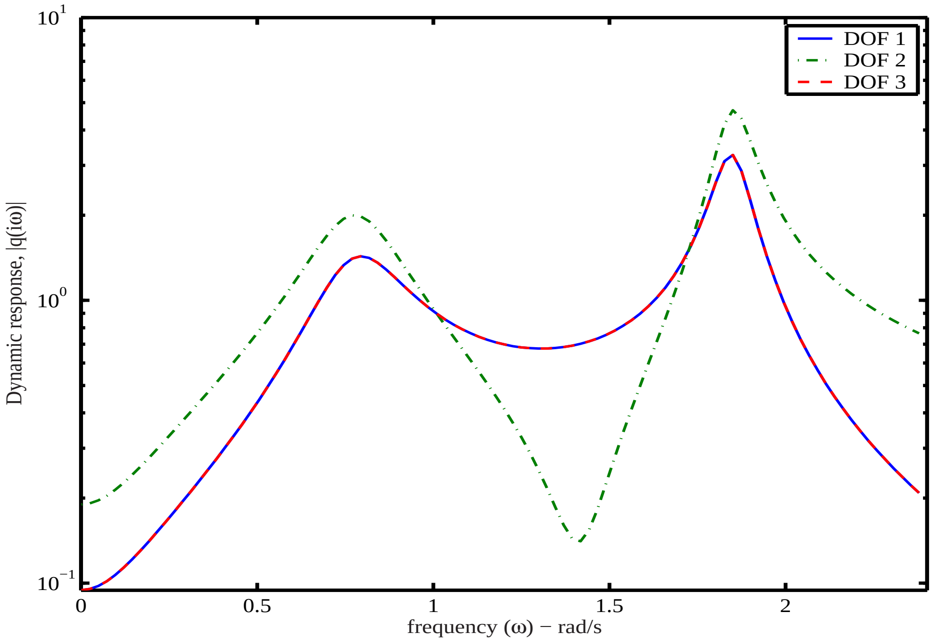

Figure 5: Absolute value of the frequency domain response of the three masses due to unit initial displacement at second DOF

Laplace transform of (4.78) as

$$\begin{aligned} \mathbf{q}(t) &= \mathcal{L}^{-1}\left[\bar{\mathbf{q}}(s)\right] = \sum_{j=1}^{3} x_{2j} m \mathcal{L}^{-1}\left[\frac{2\zeta_j\omega_j + s}{s^2 + 2s\zeta_j\omega_j + \omega_j^2}\right]\mathbf{x}_j \\ &= \sum_{j=1}^{3} x_{2j} m e^{-\zeta_j\omega_j t} \cos\left(\omega_{d_j} t\right)\mathbf{x}_j. \end{aligned} \tag{4.80}$$

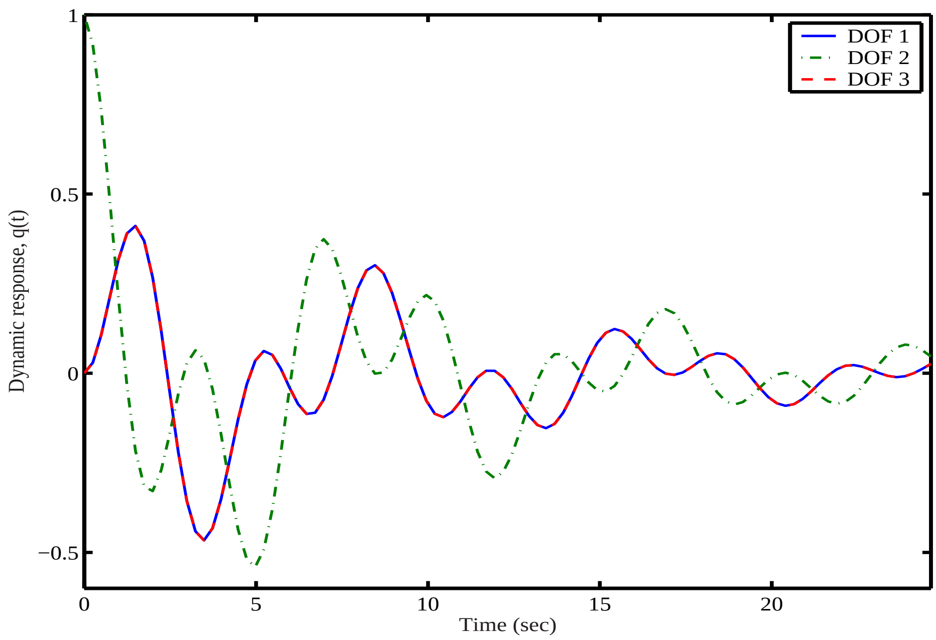

Figure 6: Time domain response of the three masses due to unit initial displacement at second DOF

VII. *Response due to initial velocity:* When $\dot{\mathbf{q}}_0 = \{0, 1, 1\}^T$ we have

$$\mathbf{x}_j^T \mathbf{M} \dot{\mathbf{q}}_0 = \begin{Bmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = (x_{2j} + x_{3j})\, m \quad \forall j. \tag{4.81}$$

The dynamic response in the Laplace domain can be obtained from equation (4.22) as

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{3} \left\{ \frac{(x_{2j} + x_{3j})\, m}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} \right\} \mathbf{x}_j. \tag{4.82}$$

The time domain response can be obtained using the inverse Laplace transform as

$$\mathbf{q}(t) = \sum_{j=1}^{3} (x_{2j} + x_{3j}) \frac{m}{\omega_{d_j}} e^{-\zeta_j\omega_j t} \sin\left(\omega_{d_j} t\right) \mathbf{x}_j. \tag{4.83}$$

The responses of the three masses in frequency domain and in the time domain are respectively shown in Figures 7 and 8. In this case all the

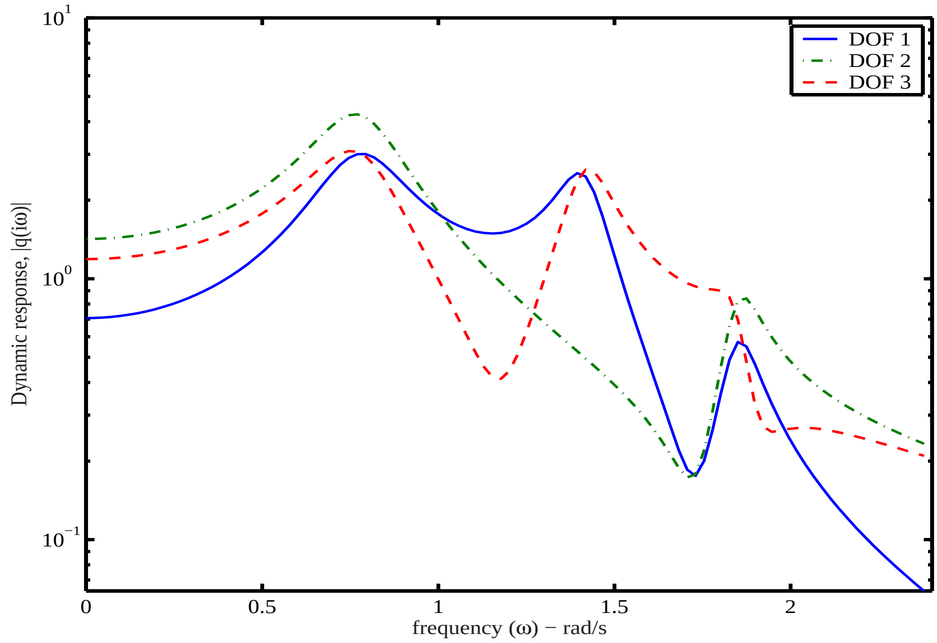

Figure 7: Absolute value of the frequency domain response of the three masses due to unit initial velocity at the second and third DOF

modes of the system can be observed. Because the initial conditions of the second and the third masses are the same, their initial displacements are close to each other. However, as the time passes the displacements of these two masses start differing from each other.

VIII. *Combined Response:* Exercise.

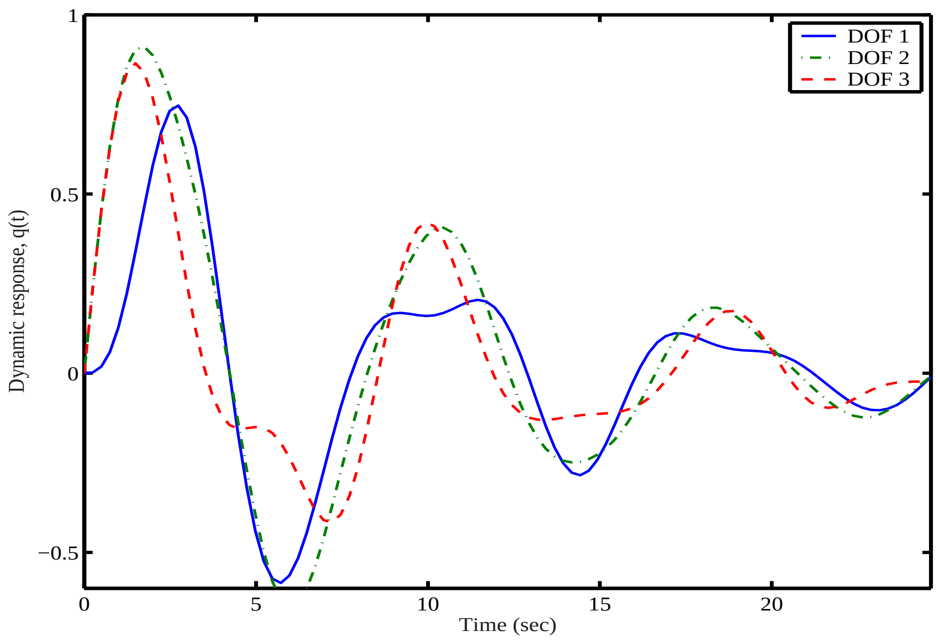

Figure 8: Time domain response of the three masses due to unit initial velocity at the second and third DOF

Exercise problem: Redo this example (a) for undamped system, and (b) for the 3DOF system shown in Figure 9.

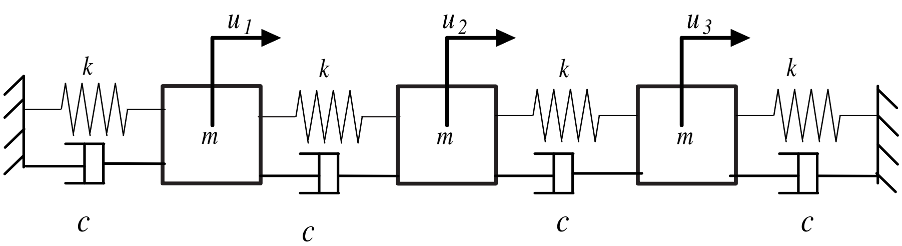

Figure 9: There DOF damped spring-mass system with dampers attached to each other

Hint: The damping is stiffness proportional.

5 Non-proportionally Damped Systems

- There is no physical reason why a general system should have proportional damping.
- Practical experience in modal testing shows that most real-life structures possess complex modes instead of real normal modes.
- In general linear systems are non-classically damped.
- When the system is non-classically damped, some or all of the N differential equations of motion are coupled through the $\mathbf{X}^T\mathbf{C}\mathbf{X}$ term and cannot be reduced to N second-order uncoupled equation.
- This coupling brings several complication in the system dynamics – the eigenvalues and the eigenvectors no longer remain real and also the eigenvectors do not satisfy the classical orthogonality relationships.

5.1 Free Vibration and Complex Modes

The complex eigenvalue problem:

$$s_j^2 \mathbf{M}\mathbf{u}_j + s_j \mathbf{C}\mathbf{u}_j + \mathbf{K}\mathbf{u}_j = \mathbf{0} \tag{5.1}$$

where $s_j \in \mathbb{C}$ is the jth eigenvalue and $\mathbf{u}_j \in \mathbb{C}^N$ is the jth eigenvector. The eigenvalues, s_j, are the roots of the characteristic polynomial

$$\det\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right] = 0. \tag{5.2}$$

The order of the polynomial is $2N$ and if the roots are complex they appear in complex conjugate pairs. The methods for solving this kind of complex problem follow mainly two routes:

I. The state-space method

II. Methods in the configuration space or 'N-space'.

5.1.1 The State-Space Method

- The state-space method is based on transforming the N second-order coupled equations into a set of $2N$ first-order coupled equations by augmenting the displacement response vectors with the velocities of the corresponding coordinates.

- The trick is to write equation (4.2) together with a trivial equation $\mathbf{M}\dot{\mathbf{q}}(t) - \mathbf{M}\dot{\mathbf{q}}(t) = 0$ in a matrix form as

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}(t) \\ \ddot{\mathbf{q}}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & -\mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{Bmatrix} \tag{5.3}$$

$$\text{or} \quad \mathbf{A}\,\dot{\mathbf{z}}(t) + \mathbf{B}\,\mathbf{z}(t) = \mathbf{r}(t) \tag{5.4}$$

where

$$\begin{gathered} \mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \in \mathbb{R}^{2N\times 2N}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & -\mathbf{M} \end{bmatrix} \in \mathbb{R}^{2N\times 2N}, \\ \mathbf{z}(t) = \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix} \in \mathbb{R}^{2N}, \quad \text{and} \quad \mathbf{r}(t) = \begin{Bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{Bmatrix} \in \mathbb{R}^{2N}. \end{gathered} \tag{5.5}$$

In the above equation $\mathbf{O}$ is the $N \times N$ null matrix. This form of equations of motion is also known as the 'Duncan form'.

- The eigenvalue problem associated with equation (5.4):

$$s_j \mathbf{A} \mathbf{z}_j + \mathbf{B} \mathbf{z}_j = \mathbf{0}, \quad \forall j = 1, \cdots, 2N \tag{5.6}$$

where $s_j \in \mathbb{C}$ is the jth eigenvalue and $\mathbf{z}_j \in \mathbb{C}^{2N}$ is the jth eigenvector.

- This eigenvalue problem is similar to the undamped eigenvalue problem (2.3) except:

  1. the dimension of the matrices are $2N$ as opposed to N
  2. the matrices are not positive definite. $\mathbf{z}_j$ can be related to the jth eigenvector of the second-order system:

$$\mathbf{z}_j = \left\{ \begin{array}{c} \mathbf{u}_j \\ s_j \mathbf{u}_j \end{array} \right\}. \tag{5.7}$$

- Since $\mathbf{A}$ and $\mathbf{B}$ are real matrices, taking complex conjugate of the eigenvalue equation::

$$s_j^* \mathbf{A} \mathbf{z}_j^* + \mathbf{B} \mathbf{z}_j^* = \mathbf{0}. \tag{5.8}$$

This implies that the eigensolutions must appear in complex conjugate pairs.

- For convenience arrange the eigenvalues and the eigenvectors so that

$$s_{j+N} = s_j^* \tag{5.9}$$

$$\mathbf{z}_{j+N} = \mathbf{z}_j^*, \quad j = 1, 2, \cdots, N \tag{5.10}$$

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Mode-Orthogonality in State-space

Like real normal modes, complex modes in the state-space also satisfy orthogonal relationships over the **A** and **B** matrices. For distinct eigenvalues:

$$\mathbf{z}_j^T \mathbf{A} \mathbf{z}_k = 0 \quad \text{and} \quad \mathbf{z}_j^T \mathbf{B} \mathbf{z}_k = 0; \quad \forall j \neq k. \tag{5.11}$$

Premultiplying equation (5.6) by $\mathbf{y}_j^T$ one obtains

$$\mathbf{z}_j^T \mathbf{B} \mathbf{z}_j = -s_j \mathbf{z}_j^T \mathbf{A} \mathbf{z}_j. \tag{5.12}$$

The eigenvectors may be normalized so that

$$\mathbf{z}_j^T \mathbf{A} \mathbf{z}_j = \frac{1}{\gamma_j} \tag{5.13}$$

where $\gamma_j \in \mathbb{C}$ is the normalization constant. In view of the expressions of $\mathbf{z}_j$ in equation (5.7) the above relationship can be expressed as

$$\mathbf{u}_j^T \left[2 s_j \mathbf{M} + \mathbf{C} \right] \mathbf{u}_j = \frac{1}{\gamma_j}. \tag{5.14}$$

There are several ways in which the normalization constants can be selected. The one that is most consistent with traditional modal analysis practice, is to choose $\gamma_j = 1/2s_j$. Observe that this degenerates to the familiar mass normalization relationship $\mathbf{u}_j^T \mathbf{M} \mathbf{u}_j = 1$ when the damping is zero.

5.1.2 Approximate Methods in the Configuration Space

- State-space method computationally expensive as the 'size' of the problem doubles.
- Fails to provide the physical insight which modal analysis in the configuration space or N-space offers.

Assuming that the damping is light, a simple *first-order perturbation method* is described to obtain complex modes and frequencies in terms of undamped modes and frequencies.

The undamped modes form a *complete* set of vectors so that each complex mode $\mathbf{u}_j$ can be expressed as a linear combination of $\mathbf{x}_j$:

$$\mathbf{u}_j = \sum_{k=1}^{N} \alpha_k^{(j)} \mathbf{x}_k \tag{5.15}$$

where $\alpha_k^{(j)}$ are complex constants which we want to determine. Since damping is assumed light,

$$\alpha_k^{(j)} \ll 1, \forall j \neq k$$
$$\text{and} \qquad \alpha_j^{(j)} = 1, \forall j.$$

Suppose the complex natural frequencies are denoted by λ_j, which are related to complex eigenvalues s_j through

$$s_j = i\lambda_j. \tag{5.16}$$

Substituting s_j and $\mathbf{u}_j$ in the eigenvalue equation (5.1):

$$\left[-\lambda_j^2 \mathbf{M} + i\lambda_j \mathbf{C} + \mathbf{K}\right] \sum_{k=1}^{N} \alpha_k^{(j)} \mathbf{x}_k = \mathbf{0}. \tag{5.17}$$

Premultiplying by $\mathbf{x}_j^T$ and using the orthogonality conditions (2.11) and (2.12):

$$-\lambda_j^2 + i\lambda_j \sum_{k=1}^{N} \alpha_k^{(j)} C'_{jk} + \omega_j^2 = 0 \tag{5.18}$$

where $C'_{jk} = \mathbf{x}_j^T \mathbf{C} \mathbf{x}_k$, is the jkth element of the modal damping matrix $\mathbf{C}'$. Due to small damping assumption we can neglect the product $\alpha_k^{(j)} C'_{jk}, \forall j \neq k$ since they are small compared to $\alpha_j^{(j)} C'_{jj}$:

$$-\lambda_j^2 + i\lambda_j \alpha_j^{(j)} C'_{jj} + \omega_j^2 \approx 0. \tag{5.19}$$

Solving this quadratic equation

$$\lambda_j \approx \pm\omega_j + iC'_{jj}/2. \tag{5.20}$$

This is the expression of approximate complex natural frequencies. Premultiplying equation (5.17) by $\mathbf{x}_k^T$, using the orthogonality conditions and light damping assumption, it can be shown that

$$\alpha_k^{(j)} \approx \frac{i\omega_j C'_{kj}}{\omega_j^2 - \omega_k^2}, \quad k \neq j. \tag{5.21}$$

Substituting this in (5.15), approximate complex modes are

$$\mathbf{u}_j \approx \mathbf{x}_j + \sum_{k \neq j}^{N} \frac{i\omega_j C'_{kj} \mathbf{x}_k}{\omega_j^2 - \omega_k^2}. \tag{5.22}$$

This expression shows that:

- The imaginary parts of complex modes are approximately orthogonal to the real parts
- The 'complexity' of the modes will be more if ω_j and ω_k are close, *i.e.*, modes will be significantly complex when the natural frequencies of a system are closely spaced.

5.2 Dynamic Response

5.2.1 Frequency Domain Analysis

Taking the Laplace transform of equation (5.4) we have

$$s\mathbf{A}\bar{\mathbf{z}}(s) - \mathbf{A}\mathbf{z_0} + \mathbf{B}\bar{\mathbf{z}}(s) = \bar{\mathbf{r}}(s) \tag{5.23}$$

where $\bar{\mathbf{z}}(s)$ is the Laplace transform of $\mathbf{z}(t)$, $\mathbf{z_0}$ is the vector of initial conditions in the state-space and $\bar{\mathbf{r}}(s)$ is the Laplace transform of $\mathbf{r}(t)$. From the expressions of $\mathbf{z}(t)$ and $\mathbf{r}(t)$ in equation (5.5) it is obvious that

$$\bar{\mathbf{z}}(s) = \begin{Bmatrix} \bar{\mathbf{q}}(s) \\ s\bar{\mathbf{z}}(s) \end{Bmatrix} \in \mathbb{C}^{2N}, \quad \bar{\mathbf{r}}(s) = \begin{Bmatrix} \bar{\mathbf{f}}(s) \\ \mathbf{0} \end{Bmatrix} \in \mathbb{C}^{2N} \quad \text{and} \quad \mathbf{z_0} = \begin{Bmatrix} \mathbf{q_0} \\ \dot{\mathbf{q}}_\mathbf{0} \end{Bmatrix} \in \mathbb{R}^{2N} \tag{5.24}$$

For distinct eigenvalues the mode shapes $\mathbf{z}_k$ form a complete set of vectors. Therefore, the solution of equation (5.23) can be expressed in terms of a linear combination of $\mathbf{z}_k$ as

$$\bar{\mathbf{z}}(s) = \sum_{k=1}^{2N} \beta_k(s)\mathbf{z}_k. \tag{5.25}$$

We only need to determine the constants $\beta_k(s)$ to obtain the complete solution.

Substituting $\mathbf{z}(s)$ from (5.25) into equation (5.23) we have

$$[s\mathbf{A} + \mathbf{B}] \sum_{k=1}^{2N} \beta_k(s)\mathbf{z}_k = \bar{\mathbf{r}}(s) + \mathbf{A}\mathbf{z_0}. \tag{5.26}$$

Premultiplying by $\mathbf{z}_j^T$ and using the orthogonality and normalization relationships (5.11)–(5.13), we have

$$\begin{aligned} &\frac{1}{\gamma_j}(s - s_j)\,\beta_j(s) = \mathbf{z}_j^T\left\{\bar{\mathbf{r}}(s) + \mathbf{A}\mathbf{z_0}\right\} \\ \text{or} \quad &\beta_j(s) = \gamma_j \frac{\mathbf{z}_j^T\bar{\mathbf{r}}(s) + \mathbf{z}_j^T\mathbf{A}\mathbf{z_0}}{s - s_j}. \end{aligned} \tag{5.27}$$

Using the expressions of $\mathbf{A}$, $\mathbf{z}_j$ and $\bar{\mathbf{r}}(s)$ from equation (5.5), (5.7) and (5.24) the term $\mathbf{z}_j^T\bar{\mathbf{r}}(s) + \mathbf{z}_j^T\mathbf{A}\mathbf{z}_0$ can be simplified and $\beta_k(s)$ can be related with mode shapes of the second-order system as

$$\beta_j(s) = \gamma_j \frac{\mathbf{u}_j^T\left\{\bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{C}\mathbf{q}_0 + s\mathbf{M}\mathbf{q}_0\right\}}{s - s_j}. \tag{5.28}$$

Since we are only interested in the displacement response, only first N rows of equation (5.25) are needed. Using the partition of $\mathbf{z}(s)$ and $\mathbf{z}_j$ we have

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{2N} \beta_j(s)\mathbf{u}_j. \tag{5.29}$$

Substituting $\beta_j(s)$ from equation (5.28) into the preceding equation one has

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{2N} \gamma_j \mathbf{u}_j \frac{\mathbf{u}_j^T\left\{\bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{C}\mathbf{q}_0 + s\mathbf{M}\mathbf{q}_0\right\}}{s - s_j} \tag{5.30}$$

$$\text{or} \quad \bar{\mathbf{q}}(s) = \mathbf{H}(s)\left\{\bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{C}\mathbf{q}_0 + s\mathbf{M}\mathbf{q}_0\right\} \tag{5.31}$$

where

$$\mathbf{H}(s) = \sum_{j=1}^{2N} \frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - s_j} \tag{5.32}$$

is the transfer function or receptance matrix. Recalling that the eigensolutions appear in complex conjugate pairs, equation (5.32) can be expanded as

$$\mathbf{H}(s) = \sum_{j=1}^{2N} \frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - s_j} = \sum_{j=1}^{N} \left[\frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - s_j} + \frac{\gamma_j^* \mathbf{u}_j^* \mathbf{u}_j^{*T}}{s - s_j^*}\right]. \tag{5.33}$$

The receptance matrix is often expressed in terms of complex natural frequencies λ_j. Substituting $s_j = i\lambda_j$ in the preceding expression we have

$$\mathbf{H}(s) = \sum_{j=1}^{N} \left[\frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - i\lambda_j} + \frac{\gamma_j^* \mathbf{u}_j^* \mathbf{u}_j^{*T}}{s + i\lambda_j^*}\right] \quad \text{and} \quad \gamma_j = \frac{1}{\mathbf{u}_j^T\left[2i\lambda_j\mathbf{M} + \mathbf{C}\right]\mathbf{u}_j}. \tag{5.34}$$

The receptance matrix $\mathbf{H}(s)$ in (5.34) reduces to its equivalent expression for the undamped case. In the undamped limit $\mathbf{C} = 0$. This results $\lambda_j = \omega_j = \lambda_j^*$ and $\mathbf{u}_j = \mathbf{x}_j = \mathbf{u}_j^*$. In view of the mass normalization relationship we also have $\gamma_j = \frac{1}{2\mathrm{i}\omega_j}$. Consider a typical term in (5.34):

$$\left[\frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - i\lambda_j} + \frac{\gamma_j^* \mathbf{u}_j^* \mathbf{u}_j^{*T}}{s + i\lambda_j^*}\right] = \left[\frac{1}{2i\omega_j}\frac{1}{\mathrm{i}\omega - \mathrm{i}\omega_j} + \frac{1}{-2i\omega_j}\frac{1}{\mathrm{i}\omega + \mathrm{i}\omega_j}\right]\mathbf{x}_j\mathbf{x}_j^T$$
$$= \frac{1}{2i^2\omega_j}\left[\frac{1}{\omega - \omega_j} - \frac{1}{\omega + \omega_j}\right]\mathbf{x}_j\mathbf{x}_j^T$$
$$= -\frac{1}{2\omega_j}\left[\frac{\omega + \omega_j - \omega + \omega_j}{(\omega - \omega_j)(\omega + \omega_j)}\right]\mathbf{x}_j\mathbf{x}_j^T = -\frac{1}{2\omega_j}\left[\frac{2\omega_j}{\omega^2 - \omega_j^2}\right]\mathbf{x}_j\mathbf{x}_j^T = \frac{\mathbf{x}_j\mathbf{x}_j^T}{\omega_j^2 - \omega^2}. \tag{5.35}$$

This was derived before for the receptance matrix of the undamped system. Therefore equation (5.31) is the most general expression of the dynamic response of damped linear dynamic systems.

Exercise: Verify that when the system is proportionally damped, equation (5.31) reduces to (4.21) as expected.

5.2.2 Time Domain Analysis

Combining equations (5.31) and (5.34) we have

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{N}\left\{\gamma_j \frac{\mathbf{u}_j^T\bar{\mathbf{f}}(s) + \mathbf{u}_j^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{u}_j^T\mathbf{C}\mathbf{q}_0 + s\mathbf{u}_j^T\mathbf{M}\mathbf{q}_0}{s - i\lambda_j}\mathbf{u}_j + \right.$$
$$\left. \gamma_j^* \frac{\mathbf{u}_j^{*T}\bar{\mathbf{f}}(s) + \mathbf{u}_j^{*T}\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{u}_j^{*T}\mathbf{C}\mathbf{q}_0 + s\mathbf{u}_j^{*T}\mathbf{M}\mathbf{q}_0}{s + i\lambda_j^*}\mathbf{u}_j^*\right\}. \tag{5.36}$$

From the table of Laplace transforms we know that

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s-a}\right] = ae^{at}, \quad t > 0. \tag{5.37}$$

Taking the inverse Laplace transform of (5.36)

$$\mathbf{q}(t) = \mathcal{L}^{-1}\left[\bar{\mathbf{q}}(s)\right] = \sum_{j=1}^{N} \gamma_j a_{1_j}(t)\mathbf{u}_j + \gamma_j^* a_{2_j}(t)\mathbf{u}_j^* \tag{5.38}$$

where

$$\begin{aligned} a_{1_j}(t) =& \mathcal{L}^{-1}\left[\frac{\mathbf{u}_j^T\bar{\mathbf{f}}(s)}{s-i\lambda_j}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-i\lambda_j}\right]\left(\mathbf{u}_j^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{u}_j^T\mathbf{C}\mathbf{q}_0\right) + \mathcal{L}^{-1}\left[\frac{s}{s-i\lambda_j}\right]\mathbf{u}_j^T\mathbf{M}\mathbf{q}_0 \\ =& \int_0^t e^{i\lambda_j(t-\tau)}\mathbf{u}_j^T\mathbf{f}(\tau)d\tau + e^{i\lambda_j t}\left(\mathbf{u}_j^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{u}_j^T\mathbf{C}\mathbf{q}_0 + i\lambda_j\mathbf{u}_j^T\mathbf{M}\mathbf{q}_0\right), \quad t>0 \end{aligned} \tag{5.39}$$

and similarly

$$\begin{aligned} a_{2_j}(t) =& \mathcal{L}^{-1}\left[\frac{\mathbf{u}_j^{*T}\bar{\mathbf{f}}(s)}{s+i\lambda_j}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+i\lambda_j}\right]\left(\mathbf{u}_j^{*T}\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{u}_j^{*T}\mathbf{C}\mathbf{q}_0\right) \\ &+ \mathcal{L}^{-1}\left[\frac{s}{s+i\lambda_j}\right]\mathbf{u}_j{*}^T\mathbf{M}\mathbf{q}_0 \\ =& \int_0^t e^{-i\lambda_j(t-\tau)}\mathbf{u}_j^{*T}\mathbf{f}(\tau)d\tau + e^{-i\lambda_j t}\left(\mathbf{u}_j^{*T}\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{u}_j^{*T}\mathbf{C}\mathbf{q}_0 - i\lambda_j\mathbf{u}_j^{*T}\mathbf{M}\mathbf{q}_0\right), \quad t>0. \end{aligned} \tag{5.40}$$

Example 3: Figure 10 shows a three DOF spring-mass system. This system is identical to the one used in Example 2, except that the damper attached with the middle block is now disconnected.

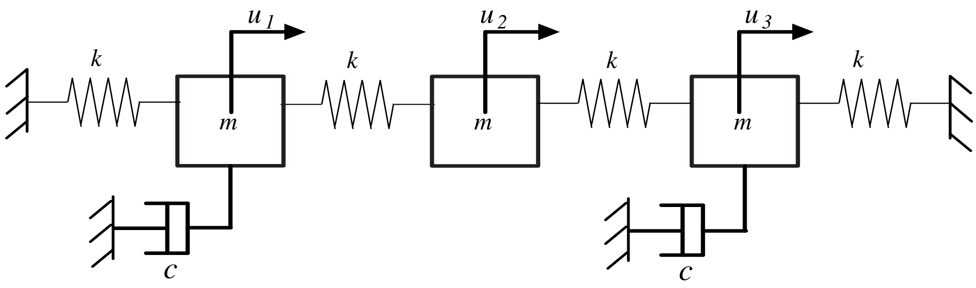

Figure 10: There DOF damped spring-mass system

1. Show that in general the system possesses complex modes.

2. Obtain approximate expressions for complex natural frequencies (using the first order perturbation method).

Solution: The mass and the stiffness matrices of the system are given before in Example 2. The damping matrix is clearly given by

$$\mathbf{C} = \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}. \tag{5.41}$$

To check if the system has complex modes we need to check if Caughey and O'Kelly's criteria, *i.e.*, $\mathbf{CM}^{-1}\mathbf{K} = \mathbf{KM}^{-1}\mathbf{C}$, is satisfied or not.

$$\mathbf{CM}^{-1}\mathbf{K} = \mathbf{C}\frac{1}{m}\mathbf{IK} = \frac{1}{m}\mathbf{CK} = \frac{ck}{m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \frac{ck}{m}\begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \tag{5.42}$$

$$\mathbf{KM}^{-1}\mathbf{C} = \mathbf{K}\frac{1}{m}\mathbf{IC} = \frac{1}{m}\mathbf{KC} = \frac{ck}{m}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{ck}{m}\begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}. \tag{5.43}$$

Thus $\mathbf{CM}^{-1}\mathbf{K} \neq \mathbf{KM}^{-1}\mathbf{C}$, that is,the system do not possess classical normal modes but has complex modes.

The modal damping matrix

$$\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X} = \frac{1}{2\sqrt{m}}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}^T \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix} \frac{1}{2\sqrt{m}}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$
$$= \frac{c}{4m}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$
$$= \frac{c}{4m}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ 0 & 0 & 0 \\ 1 & -\sqrt{2} & 1 \end{bmatrix} = \frac{c}{4m}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \frac{c}{m}\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}. \tag{5.44}$$

$\mathbf{C}'$ is not a diagonal matrix, *i.e.*, the equation of motion in the modal coordinates are coupled through the off-diagonal terms of the $\mathbf{C}'$ matrix. Approximate complex natural frequencies can be obtained from equation (5.20) as

$$\lambda_1 \approx \pm\omega_1 + iC'_{11}/2 = \pm\sqrt{\left(2-\sqrt{2}\right)\alpha} + i\frac{c}{4m}, \tag{5.45}$$

$$\lambda_2 \approx \pm\omega_2 + iC'_{22}/2 = \pm\sqrt{2\alpha} + i\frac{c}{2m} \tag{5.46}$$

$$\text{and} \quad \lambda_3 \approx \pm\omega_3 + iC'_{33}/2 = \pm\sqrt{\left(2+\sqrt{2}\right)\alpha} + i\frac{c}{4m}. \tag{5.47}$$

Exercise problem: Redo the previous example for the 3DOF system shown in Figure 11.

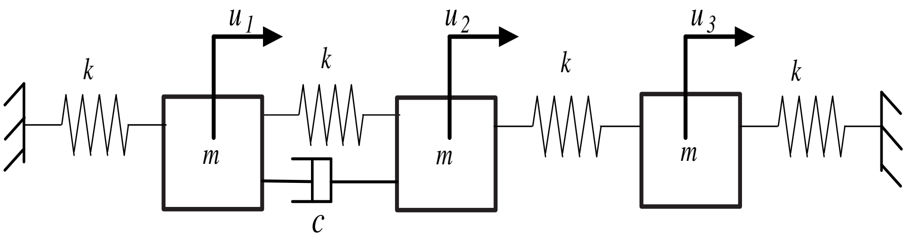

Figure 11: There DOF damped spring-mass system with dampers attached between mass 1 and 2

Hint: The damping is given by $\mathbf{C} = \begin{bmatrix} c & -c & 0 \\ -c & c & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Nomenclature

$a_j(t)$	Time dependent constants for dynamic response
$\mathbf{A}$	$2N \times 2N$ system matrix in the state-space
B_j	Constants for time domain dynamic response
$\mathbf{B}$	$2N \times 2N$ system matrix in the state-space
$\mathbf{C}$	Viscous damping matrix
$\mathbf{C}'$	Damping matrix in the modal coordinate
$\mathbf{f}(t)$	Forcing vector
$\tilde{\mathbf{f}}(t)$	Forcing vector in the modal coordinate
$\mathcal{F}$	Dissipation function
$\boldsymbol{\mathcal{G}}(t)$	Damping function in the time domain
$\mathbf{I}$	Identity matrix
i	Unit imaginary number, $i = \sqrt{-1}$
$\mathbf{K}$	Stiffness matrix
$\mathcal{L}(\bullet)$	Laplace transform of $(\bullet)$
$\mathcal{L}^{-1}(\bullet)$	Inverse Laplace transform of $(\bullet)$
$\mathbf{M}$	Mass matrix
N	Degrees-of-freedom of the system
$\mathbf{O}$	Null matrix
$\bar{\mathbf{p}}(s)$	Effective forcing vector in the Laplace domain
$\mathbf{q}(t)$	Vector of the generalized coordinates (displacements)
$\mathbf{q_0}$	Vector of initial displacements
$\dot{\mathbf{q}}_\mathbf{0}$	Vector of initial velocities

$\bar{\mathbf{q}}$	Laplace transform of $\mathbf{q}(t)$
Q_{nc_k}	Non-conservative forces
$\mathbf{r}(t)$	Forcing vector in the state-space
$\bar{\mathbf{r}}(s)$	Laplace transform of $\mathbf{r}(t)$
s	Laplace domain parameter
s_j	jth complex eigenvalue
t	Time
$\mathbf{u}_j$	Complex eigenvector in the original (N) space
$\mathbf{x}_j$	jth undamped eigenvector
$\mathbf{X}$	Matrix of the undamped eigenvectors
$\mathbf{y}(t)$	Modal coordinates
$\mathbf{z}(t)$	Response vector in the state-space
$\mathbf{z}_j$	$2N \times 1$ Complex eigenvector vector in the state-space
$\mathbf{z_0}$	Vector of initial conditions in the state-space
$\bar{\mathbf{z}}(s)$	Laplace transform of $\mathbf{z}(t)$
$\alpha_k^{(j)}$	Complex constants for jth complex mode
γ_j	j-th modal amplitude constant
$\delta(t)$	Dirac-delta function
δ_{jk}	Kroneker-delta function
ζ_j	j-th modal damping factor
$\boldsymbol{\zeta}$	Diagonal matrix containing ζ_j
θ_j	Constants for time domain dynamic response
λ_j	j-th complex natural frequency

τ	Dummy time variable
ω	frequency
ω_j	j-th undamped natural frequency
ω_{d_j}	j-th damped natural frequency
$\mathbf{\Omega}$	Diagonal matrix containing ω_j
DOF	Degrees of freedom
$\mathbb{C}$	Space of complex numbers
$\mathbb{R}$	Space of real numbers
$\mathbb{R}^{N\times N}$	Space of real $N \times N$ matrices
$\mathbb{R}^{N}$	Space of real N dimensional vectors
$\det(\bullet)$	Determinant of $(\bullet)$
diag	A diagonal matrix
$\in$	Belongs to
$\forall$	For all
$(\bullet)^T$	Matrix transpose of $(\bullet)$
$(\bullet)^{-1}$	Matrix inverse of $(\bullet)$
$(\dot{\bullet})$	Derivative of $(\bullet)$ with respect to t
$(\bullet)^*$	Complex conjugate of $(\bullet)$
$\lvert \bullet \rvert$	Absolute value of $(\bullet)$

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