Vibration of Damped Systems

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1 Introduction

Problems involving vibration occur in many areas of mechanical, civil and aerospace engineering: wave loading of offshore platforms, cabin noise in aircrafts, earthquake and wind loading of cable stayed bridges and high rise buildings, performance of machine tools – to pick only few random examples. Quite often vibration is not desirable and the interest lies in reducing it by dissipation of vibration energy or *damping*. Characterization of damping forces in a vibrating structure has long been an active area of research in structural dynamics. Since the publication of Lord Rayleigh's classic monograph 'Theory of Sound (1877)', a large body of literature can be found on damping. Although the topic of damping is an age old problem, the demands of modern engineering have led to a steady increase of interest in recent years. Studies of damping have a major role in vibration isolation in automobiles under random loading due to surface irregularities and buildings subjected to earthquake loadings. The recent developments in the fields of robotics and active structures have provided impetus towards developing procedures for dealing with general dissipative forces in the context of structural dynamics. Beside these, in the last few decades, the sophistication of modern design methods together with the development of improved composite structural materials instilled a trend towards lighter structures. At the same time, there is also a constant demand for larger structures, capable of carrying more loads at higher speeds with minimum noise and vibration level as the safety/workability and environmental criteria become more stringent. Unfortunately, these two demands are conflicting and the problem cannot be solved without proper understanding of energy dissipation or damping behaviour.

In spite of a large amount of research, understanding of damping mechanisms is quite primitive. A major reason for this is that, by contrast with inertia and stiffness forces, it is not in general clear which *state variables* are relevant to determine the damping forces. Moreover, it seems that in a realistic situation it is often the structural joints which are more responsible for the energy dissipation than the (solid) material. There have been detailed studies on the material damping and also on energy dissipation mechanisms in the joints. But here difficulty lies in representing all these tiny mechanisms in different parts of the structure in an unified manner. Even in many cases these mechanisms turn out be locally non-linear, requiring an equivalent linearization technique for a global analysis. A well known method to get rid of all these problems is to use the so called 'viscous damping'. This approach was first introduced by Rayleigh (1877) via his famous 'dissipation function', a quadratic expression for the energy dissipation rate with a symmetric matrix of coefficients, the 'damping matrix'. A further idealization, also pointed out by Rayleigh, is to assume the damping matrix to be a linear combination of the mass and stiffness matrices. Since its introduction this model has been used extensively and is now usually known as 'Rayleigh damping', 'proportional damping' or 'classical damping'. With such a damping model, the *modal analysis* procedure, originally developed for undamped systems, can be used to analyze damped systems in a very similar manner.

From an analytical point of view, models of vibrating systems are commonly divided into two broad classes – discrete, or lumped-parameter models, and continuous, or distributed-parameter models. In real life, however, systems can contain both distributed and lumped parameter models (for example, a beam with a tip mass). Distributed-parameter modelling of vibrating systems leads to *partial-differential equations* as the equations of motion. Exact solutions of such equations are possible only for a limited number of problems with simple geometry, boundary conditions, and material properties (such as constant mass density). For this reason, normally we need some kind of approximate method to solve a general problem. Such solutions are generally obtained through spatial discretization (for example, the Finite Element Method), which amounts to approximating distributed-parameter systems by lumped-parameter systems. Equations of motion of lumped-parameter systems can be shown to be expressed by a set of coupled *ordinary-differential equations*. In this lecture we will mostly deal with such lumped-parameter systems. We also restrict our attention to the linear system behavior only.

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2 Brief Review on Dynamics of Undamped Systems

The equations of motion of an undamped non-gyroscopic system with N degrees of freedom can be given by

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{2.1}$$

where $\mathbf{M} \in \mathbb{R}^{N\times N}$ is the mass matrix, $\mathbf{K} \in \mathbb{R}^{N\times N}$ is the stiffness matrix, $\mathbf{q}(t) \in \mathbb{R}^N$ is the vector of generalized coordinates and $\mathbf{f}(t) \in \mathbb{R}^N$ is the forcing vector. Equation (2.1) represents a set of coupled second-order ordinary-differential equations. The solution of this equation also requires the knowledge of the initial conditions in terms of displacements and velocities of all the coordinates. The *initial conditions* can be specified as

$$\mathbf{q}(0) = \mathbf{q_0} \in \mathbb{R}^N \quad \text{and} \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_\mathbf{0} \in \mathbb{R}^N. \tag{2.2}$$

2.1 Modal Analysis

Rayleigh (1877) has shown that undamped linear systems, equations of motion of which are given by (2.1), are capable of so-called *natural motions*. This essentially implies that all the system coordinates execute harmonic oscillation at a given frequency and form a certain displacement pattern. The oscillation frequency and displacement pattern are called *natural frequencies* and *normal modes*, respectively. The natural frequencies (ω_j) and the mode shapes ($\mathbf{x}_j$) are intrinsic characteristic of a system and can be obtained by solving the associated matrix eigenvalue problem

$$\mathbf{K}\mathbf{x}_j = \omega_j^2 \mathbf{M}\mathbf{x}_j, \quad \forall\, j = 1, \cdots, N. \tag{2.3}$$

Since the above eigenvalue problem is in terms of real symmetric non-negative definite matrices $\mathbf{M}$ and $\mathbf{K}$, the eigenvalues and consequently the eigenvectors are real, that is $\omega_j \in \mathbb{R}$ and $\mathbf{x}_j \in \mathbb{R}^N$. Premultiplying equation (2.3) by $\mathbf{x}_k^T$ we have

$$\mathbf{x}_k^T \mathbf{K}\mathbf{x}_j = \omega_j^2 \mathbf{x}_k^T \mathbf{M}\mathbf{x}_j \tag{2.4}$$

Taking transpose of the above equation and noting that $\mathbf{M}$ and $\mathbf{K}$ are symmetric matrices one has

$$\mathbf{x}_j^T \mathbf{K}\mathbf{x}_k = \omega_j^2 \mathbf{x}_j^T \mathbf{M}\mathbf{x}_k \tag{2.5}$$

Now consider the eigenvalue equation for the kth mode:

$$\mathbf{K}\mathbf{x}_k = \omega_k^2 \mathbf{M}\mathbf{x}_k \tag{2.6}$$

Premultiplying equation (2.6) by $\mathbf{x}_j^T$ we have

$$\mathbf{x}_j^T \mathbf{K}\mathbf{x}_k = \omega_k^2 \mathbf{x}_j^T \mathbf{M}\mathbf{x}_k \tag{2.7}$$

Subtracting equation (2.5) from (2.7) we have

$$\left(\omega_k^2 - \omega_j^2\right) \mathbf{x}_j^T \mathbf{M} \mathbf{x}_k = 0 \tag{2.8}$$

Since we assumed the natural frequencies are not repeated when $j \neq k$, $\omega_j \neq \omega_k$. Therefore, from equation (2.8) it follows that

$$\mathbf{x}_k^T \mathbf{M} \mathbf{x}_j = 0 \tag{2.9}$$

Using this in equation (2.5) we can also obtain

$$\mathbf{x}_k^T \mathbf{K} \mathbf{x}_j = 0 \tag{2.10}$$

If we normalize $\mathbf{x}_j$ such that $\mathbf{x}_j \mathbf{M} \mathbf{x}_j = 1$, then from equation (2.5) it follows that $\mathbf{x}_j \mathbf{K} \mathbf{x}_j = \omega_j^2$. This normalization is known as unity mass normalization, a convention often used in practice. Equations (2.9) and (2.10) are known as orthogonality relationships. These equations combined with the normalization relationships can be concisely written in terms of Kroneker delta function δ_{lj} as

$$\mathbf{x}_l^T \mathbf{M} \mathbf{x}_j = \delta_{lj} \tag{2.11}$$

$$\text{and} \quad \mathbf{x}_l^T \mathbf{K} \mathbf{x}_j = \omega_j^2 \delta_{lj}, \quad \forall\, l, j = 1, \cdots, N \tag{2.12}$$

Note that $\delta_{lj} = 1$ for $l = j$ and $\delta_{lj} = 0$ otherwise. I

This orthogonality property of the undamped modes is very powerful as it allows to transform a set of *coupled* differential equations to a set of *independent* equations. For convenience, we construct the matrices

$$\mathbf{\Omega} = \text{diag}\left[\omega_1, \omega_2, \cdots, \omega_N\right] \in \mathbb{R}^{N \times N} \tag{2.13}$$

$$\text{and} \quad \mathbf{X} = \left[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\right] \in \mathbb{R}^{N \times N} \tag{2.14}$$

where the eigenvalues are arranged such that $\omega_1 < \omega_2$, $\omega_2 < \omega_3, \cdots, \omega_k < \omega_{k+1}$. Using these matrix notations, the orthogonality relationships (2.11) and (2.12) can be rewritten as

$$\mathbf{X}^T \mathbf{M} \mathbf{X} = \mathbf{I} \tag{2.15}$$

$$\text{and} \quad \mathbf{X}^T \mathbf{K} \mathbf{X} = \mathbf{\Omega}^2 \tag{2.16}$$

where $\mathbf{I}$ is a $N \times N$ identity matrix. Use a coordinate transformation (modal transformation)

$$\mathbf{q}(t) = \mathbf{X} \mathbf{y}(t). \tag{2.17}$$

Substituting $\mathbf{q}(t)$ in equation (2.1), premultiplying by $\mathbf{X}^T$ and using the orthogonality relationships in (2.15) and (2.16), the equations of motion in the modal coordinates may be obtained as

$$\begin{aligned} \ddot{\mathbf{y}}(t) + \mathbf{\Omega}^2 \mathbf{y}(t) &= \tilde{\mathbf{f}}(t) \\ \text{or} \quad \ddot{y}_j(t) + \omega_j^2 y_j(t) &= \tilde{f}_j(t) \quad \forall\, j = 1, \cdots, N \end{aligned} \tag{2.18}$$

where $\tilde{\mathbf{f}}(t) = \mathbf{X}^T\mathbf{f}(t)$ is the forcing function in modal coordinates. Clearly, this method significantly simplifies the dynamic analysis because complex multiple degrees of freedom systems can be treated as collections of single-degree-of-freedom oscillators. This approach of analyzing linear undamped systems is known as *modal analysis*, possibly the most efficient tool for vibration analysis of complex engineering structures.

2.2 Dynamic Response

2.2.1 Frequency Domain Analysis

Taking the Laplace transform of (2.1) and considering the initial conditions in (2.2) one has

$$s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q_0} - \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) \tag{2.19}$$

$$\text{or} \quad \left[s^2\mathbf{M} + \mathbf{K}\right]\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\mathbf{M}\mathbf{q_0} = \bar{\mathbf{p}}(s) \text{ (say).} \tag{2.20}$$

Using the modal transformation

$$\bar{\mathbf{q}}(s) = \mathbf{X}\bar{\mathbf{y}}(s) \tag{2.21}$$

and premultiplying (2.20) by $\mathbf{X}^T$, we have

$$\left[s^2\mathbf{M} + \mathbf{K}\right]\mathbf{X}\bar{\mathbf{y}}(s) = \bar{\mathbf{p}}(s) \quad \text{or} \quad \left\{\mathbf{X}^T\left[s^2\mathbf{M} + \mathbf{K}\right]\mathbf{X}\right\}\bar{\mathbf{y}}(s) = \mathbf{X}^T\bar{\mathbf{p}}(s). \tag{2.22}$$

Using the orthogonality relationships in (2.15) and (2.16), this equation reduces to

$$\left[s^2\mathbf{I} + \mathbf{\Omega}^2\right]\bar{\mathbf{y}}(s) = \mathbf{X}^T\bar{\mathbf{p}}(s) \tag{2.23}$$

$$\text{or} \quad \bar{\mathbf{y}}(s) = \left[s^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T\bar{\mathbf{p}}(s) \tag{2.24}$$

$$\text{or} \quad \mathbf{X}\bar{\mathbf{y}}(s) = \mathbf{X}\left[s^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T\bar{\mathbf{p}}(s) \quad \text{(premultiplying by } \mathbf{X}) \tag{2.25}$$

$$\text{or} \quad \bar{\mathbf{q}}(s) = \mathbf{X}\left[s^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T\bar{\mathbf{p}}(s) \quad \text{(using (2.21))} \tag{2.26}$$

$$\text{or} \quad \bar{\mathbf{q}}(s) = \mathbf{X}\left[s^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T\left\{\bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\mathbf{M}\mathbf{q_0}\right\} \quad \text{(using (2.20)).} \tag{2.27}$$

Equation (2.27) is the complete solution of the undamped dynamic response using modal analysis. In structural dynamics often frequency domain analysis is used. The dynamic response in the frequency domain can be obtained by substituting $s = i\omega$ as

$$\begin{aligned}\bar{\mathbf{q}}(i\omega) &= \mathbf{X}\left[-\omega^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T\left\{\bar{\mathbf{f}}(i\omega) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + i\omega\mathbf{M}\mathbf{q_0}\right\} \\ &= \mathbf{H}(i\omega)\left\{\bar{\mathbf{f}}(i\omega) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + i\omega\mathbf{M}\mathbf{q_0}\right\}.\end{aligned} \tag{2.28}$$

The term

$$\mathbf{H}(i\omega) = \mathbf{X}\left[-\omega^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1}\mathbf{X}^T \tag{2.29}$$

is often known as the transfer function matrix or the receptance matrix. Note that $\left[-\omega^2\mathbf{I} + \mathbf{\Omega}^2\right]$ is a diagonal matrix and therefore its inverse is easy to obtain:

$$\left[-\omega^2\mathbf{I} + \mathbf{\Omega}^2\right]^{-1} = \text{diag}\left[\frac{1}{\omega_1^2 - \omega^2}, \frac{1}{\omega_2^2 - \omega^2}, \cdots, \frac{1}{\omega_N^2 - \omega^2}\right]. \tag{2.30}$$

The product $\mathbf{X}\left[-\omega^2\mathbf{I}+\boldsymbol{\Omega}^2\right]^{-1}\mathbf{X}^T$ can be expressed as

$$\mathbf{X}\left[-\omega^2\mathbf{I}+\boldsymbol{\Omega}^2\right]^{-1}\mathbf{X}^T = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N] \operatorname{diag}\left[\frac{1}{\omega_1^2-\omega^2}, \frac{1}{\omega_2^2-\omega^2}, \cdots, \frac{1}{\omega_N^2-\omega^2}\right] \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_N^T \end{bmatrix} \tag{2.31}$$

$$= [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N] \begin{bmatrix} \dfrac{\mathbf{x}_1^T}{\omega_1^2-\omega^2} \\ \dfrac{\mathbf{x}_2^T}{\omega_2^2-\omega^2} \\ \vdots \\ \dfrac{\mathbf{x}_N^T}{\omega_N^2-\omega^2} \end{bmatrix} = \left[\frac{\mathbf{x}_1\mathbf{x}_1^T}{\omega_1^2-\omega^2} + \frac{\mathbf{x}_2\mathbf{x}_2^T}{\omega_2^2-\omega^2} + \cdots + \frac{\mathbf{x}_N\mathbf{x}_N^T}{\omega_N^2-\omega^2}\right]. \tag{2.32}$$

From this we obtain the familiar expression of the receptance matrix as

$$\mathbf{H}(i\omega) = \sum_{j=1}^{N} \frac{\mathbf{x}_j\mathbf{x}_j^T}{\omega_j^2-\omega^2}. \tag{2.33}$$

Substituting $\mathbf{H}(i\omega)$ in (2.28) we have

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{N} \frac{\mathbf{x}_j\mathbf{x}_j^T\left\{\bar{\mathbf{f}}(i\omega)+\mathbf{M}\dot{\mathbf{q}}_0+i\omega\mathbf{M}\mathbf{q}_0\right\}}{\omega_j^2-\omega^2} = \sum_{j=1}^{N} \frac{\mathbf{x}_j^T\bar{\mathbf{f}}(i\omega)+\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_0+i\omega\mathbf{x}_j^T\mathbf{M}\mathbf{q}_0}{\omega_j^2-\omega^2}\mathbf{x}_j. \tag{2.34}$$

This expression shows that the dynamic response of the system is a linear combination of the mode shapes.

2.2.2 Time Domain Analysis

Rewriting equation (2.34) in the Laplace domain we have

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{N} \left\{ \frac{\mathbf{x}_j^T\bar{\mathbf{f}}(s)}{s^2+\omega_j^2} + \frac{\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_0}{s^2+\omega_j^2} + \frac{s}{s^2+\omega_j^2}\mathbf{x}_j^T\mathbf{M}\mathbf{q}_0 \right\} \mathbf{x}_j. \tag{2.35}$$

To obtain the vibration response in the time domain it is required to consider the inverse Laplace transform. Taking the inverse Laplace transform of $\bar{\mathbf{q}}(s)$ we have

$$\mathbf{q}(t) = \mathcal{L}^{-1}\left[\bar{\mathbf{q}}(s)\right] = \sum_{j=1}^{N} a_j(t)\mathbf{x}_j \tag{2.36}$$

where the time dependent constants are given by

$$a_j(t) = \mathcal{L}^{-1}\left[\frac{\mathbf{x}_j^T\bar{\mathbf{f}}(s)}{s^2+\omega_j^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2+\omega_j^2}\right]\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathcal{L}^{-1}\left[\frac{s}{s^2+\omega_j^2}\right]\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}. \tag{2.37}$$

The inverse Laplace transform of the second and third parts can be obtained from the table of Laplace transforms (see Kreyszig, 1999, for example) as

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega_j^2}\right] = \frac{\sin(\omega_j t)}{\omega_j} \tag{2.38}$$

$$\text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+\omega_j^2}\right] = \cos(\omega_j t). \tag{2.39}$$

The inverse Laplace transform of the first part can be obtained using the 'convolution theorem', which says that

$$\mathcal{L}^{-1}\left[\bar{f}(s)\bar{g}(s)\right] = \int_0^t f(\tau)g(t-\tau)d\tau. \tag{2.40}$$

Considering $\bar{g}(s) = \dfrac{1}{s^2+\omega_j^2}$, the inverse Laplace transform of the first part can be obtained as

$$\mathcal{L}^{-1}\left[\mathbf{x}_j^T\bar{\mathbf{f}}(s)\frac{1}{s^2+\omega_j^2}\right] = \int_0^t \frac{1}{\omega_j}\mathbf{x}_j^T\mathbf{f}(\tau)\sin(\omega_j(t-\tau))\,d\tau. \tag{2.41}$$

Combining (2.41), (2.38) and (2.39), from equation (2.37) we have

$$a_j(t) = \int_0^t \frac{1}{\omega_j}\mathbf{x}_j^T\mathbf{f}(\tau)\sin(\omega_j(t-\tau))\,d\tau + \frac{1}{\omega_j}\sin(\omega_j\, t)\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \cos(\omega_j\, t)\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}. \tag{2.42}$$

Collecting the terms associated with $\sin\left(\omega_{d_j}\, t\right)$ and $\cos\left(\omega_{d_j}\, t\right)$ this expression can be simplified as

$$a_j(t) = \int_0^t \frac{1}{\omega_j}\mathbf{x}_j^T\mathbf{f}(\tau)\sin(\omega_j(t-\tau))\,d\tau + B_j\cos(\omega_j t+\theta_j) \tag{2.43}$$

where

$$B_j = \sqrt{\left(\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}\right)^2 + \left(\frac{\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}}{\omega_j}\right)^2} \tag{2.44}$$

$$\text{and} \quad \tan\theta_j = -\frac{\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0}}{\omega_j\mathbf{x}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}}. \tag{2.45}$$

Observe that the second part of equation (2.43), *i.e.*, the term $B_j\cos(\omega_j t+\theta_j)$ only depends on the initial conditions and independent of the applied loading.

3 Models of Damping

Damping is the dissipation of energy from a vibrating structure. In this context, the term dissipate is used to mean the transformation of energy into the other form of energy and, therefore, a removal of energy from the vibrating system. The type of energy into which the mechanical energy is transformed is dependent on the system and the physical mechanism that cause the dissipation. For most vibrating system, a significant part of the energy is converted into heat.

The specific ways in which energy is dissipated in vibration are dependent upon the physical mechanisms active in the structure. These physical mechanisms are complicated physical process that are not totally understood. The types of damping that are present in the structure will depend on which mechanisms predominate in the given situation. Thus, any mathematical representation of the physical damping mechanisms in the equations of motion of a vibrating system will have to be a generalization and approximation of the true physical situation. Any mathematical damping model is really only a crutch which does not give a detailed explanation of the underlying physics.

For our mathematical convenience, we divide the elements that dissipate energy into three classes: (a) damping in single degree-of-freedom (SDOF) systems, (b) damping in continuous systems, and (c) damping in multiple degree-of-freedom (MDOF) systems. Elements such as dampers of a vehicle-suspension fall in the first class. Dissipation within a solid body, on the other hand, falls in the second class, demands a representation which accounts for both its intrinsic properties and its spatial distribution. Damping models for MDOF systems can be obtained by discretization of the equations of motion. Here, for the sake of generality, we will deal with damping in MDOF systems.

3.1 Viscous Damping

The most popular approach to model damping in the context of multiple degrees-of-freedom (MDOF) systems is to assume viscous damping. This approach was first introduced by Rayleigh (1877). By analogy with the potential energy and the kinetic energy, Rayleigh assumed the *dissipation function*, given by

$$\mathcal{F}(\mathbf{q}) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{jk}\dot{q}_j\dot{q}_k = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{C}\dot{\mathbf{q}}. \tag{3.1}$$

In the above expression $\mathbf{C} \in \mathbb{R}^{N\times N}$ is a non-negative definite symmetric matrix, known as the viscous damping matrix. It should be noted that not all forms of the viscous damping matrix can be handled within the scope of classical modal analysis. Based on the solution method, viscous damping matrices can be further divided into classical and non-classical damping. Further discussions on viscous damping will follow in Section 4.

3.2 Non-viscous Damping Models

It is important to avoid the widespread misconception that viscous damping is the *only* linear model of vibration damping in the context of MDOF systems. Any causal model which makes the energy dissipation functional non-negative is a possible candidate for a damping model. There have been several efforts to incorporate non-viscous damping models in MDOF systems. One popular approach is to model damping in terms of fractional derivatives of the displacements. The damping force using such models can be expressed by

$$\mathbf{F}_d = \sum_{j=1}^{l} \mathbf{g}_j D^{\nu_j}[\mathbf{q}(t)]. \tag{3.2}$$

Here $\mathbf{g}_j$ are complex constant matrices and the fractional derivative operator

$$D^{\nu_j}[\mathbf{q}(t)] = \frac{\mathrm{d}^{\nu_j}\mathbf{q}(t)}{\mathrm{d}t^{\nu_j}} = \frac{1}{\Gamma(1-\nu_j)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\mathbf{q}(t)}{(t-\tau)^{\nu_j}}\mathrm{d}\tau \tag{3.3}$$

where ν_j is a fraction and $\Gamma(\bullet)$ is the Gamma function. The familiar viscous damping appears as a special case when $\nu_j = 1$. Although this model might fit experimental data quite well, the physical justification for such models, however, is far from clear at the present time.

Possibly the most general way to model damping within the linear range is to consider non-viscous damping models which depend on the past history of motion via convolution integrals over some kernel functions. A *modified dissipation function* for such damping model can be defined as

$$\mathcal{F}(\mathbf{q}) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} \dot{q}_k \int_0^t \mathcal{G}_{jk}(t-\tau)\dot{q}_j(\tau)\mathrm{d}\tau = \frac{1}{2}\dot{\mathbf{q}}^T \int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\dot{\mathbf{q}}(\tau)\mathrm{d}\tau. \tag{3.4}$$

Here $\boldsymbol{\mathcal{G}}(t) \in \mathbb{R}^{N\times N}$ is a symmetric matrix of the damping kernel functions, $\mathcal{G}_{jk}(t)$. The kernel functions, or others closely related to them, are described under many different names in the literature of different subjects: for example, retardation functions, heredity functions, after-effect functions, relaxation functions *etc.* In the special case when $\boldsymbol{\mathcal{G}}(t-\tau) = \mathbf{C}\,\delta(t-\tau)$, where $\delta(t)$ is the Dirac-delta function, equation (3.4) reduces to the case of viscous damping as in equation (3.1). The damping model of this kind is a further generalization of the familiar viscous damping. By choosing suitable kernel functions, it can also be shown that the fractional derivative model discussed before is also a special case of this damping model. Thus, as pointed by Woodhouse (1998), this damping model is the most general damping model within the scope of a linear analysis. For further discussions on non-viscously damped system see Adhikari (2000, 2002).

Damping model of the form (3.4) is also used in the context of viscoelastic structures. The damping kernel functions are commonly defined in the frequency/Laplace domain. Conditions which $\mathbf{G}(s)$, the Laplace transform of $\boldsymbol{\mathcal{G}}(t)$, must satisfy in order to produce dissipative motion were given by Golla and Hughes (1985). Several authors have proposed several damping models and they are summarized in Table 1.

Table 1: Summary of damping functions in the Laplace domain

Damping functions	Author, Year
$G(s) = \sum_{k=1}^{n} \frac{a_k s}{s + b_k}$	Biot (1955, 1958)
$G(s) = \frac{E_1 s^\alpha - E_0 b s^\beta}{1 + b s^\beta}$ $0 < \alpha < 1, \quad 0 < \beta < 1$	Bagley and Torvik (1983)
$sG(s) = G^\infty \left[1 + \sum_k \alpha_k \frac{s^2 + 2\zeta_k \omega_k s}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}\right]$	Golla and Hughes (1985) and McTavish and Hughes (1993)
$G(s) = 1 + \sum_{k=1}^{n} \frac{\Delta_k s}{s + \beta_k}$	Lesieutre and Mingori (1990)
$G(s) = c\frac{1 - e^{-st_0}}{st_0}$	Adhikari (1998)
$G(s) = c\frac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}$	Adhikari (1998)

4 Proportionally Damped Systems

Equations of motion of a viscously damped system can be obtained from the Lagrange's equation and using the Rayleigh's dissipation function given by (3.1). The non-conservative forces can be obtained as

$$Q_{\mathrm{nc}_k} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_k}, \quad k = 1, \cdots, N \tag{4.1}$$

and consequently the equations of motion can expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t). \tag{4.2}$$

The aim is to solve this equation (together with the initial conditions) by modal analysis as described in Section 2.1. Using the transformation in (2.17), premultiplying equation (4.2) by $\mathbf{X}^T$ and using the orthogonality relationships in (2.13) and (2.14), equations of motion of a damped system in the modal coordinates may be obtained as

$$\ddot{\mathbf{y}}(t) + \mathbf{X}^T\mathbf{C}\mathbf{X}\dot{\mathbf{y}}(t) + \mathbf{\Omega}^2\mathbf{y}(t) = \tilde{\mathbf{f}}(t). \tag{4.3}$$

Clearly, unless $\mathbf{X}^T\mathbf{C}\mathbf{X}$ is a diagonal matrix, no advantage can be gained by employing modal analysis because the equations of motion will still be coupled. To solve this problem, it is common to assume *proportional damping* which we will discuss in some detail.

With proportional damping assumption, the damping matrix $\mathbf{C}$ is simultaneously diagonalizable with $\mathbf{M}$ and $\mathbf{K}$. This implies that the damping matrix in the modal coordinate

$$\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X} \tag{4.4}$$

is a diagonal matrix. The damping ratios ζ_j are defined from the diagonal elements of the modal damping matrix as

$$C'_{jj} = 2\zeta_j\omega_j \quad \forall j = 1, \cdots, N. \tag{4.5}$$

Such damping model, introduced by Rayleigh (1877), allows to analyze damped systems in very much the same manner as undamped systems since the equations of motion in the modal coordinate can be decoupled as

$$\ddot{y}_j(t) + 2\zeta_j\omega_j\dot{y}_j(t) + \omega_j^2 y_j(t) = \tilde{f}_j(t) \quad \forall\, j = 1, \cdots, N. \tag{4.6}$$

The proportional damping model expresses the damping matrix as a linear combination of the mass and stiffness matrices, that is

$$\mathbf{C} = \alpha_1\mathbf{M} + \alpha_2\mathbf{K} \tag{4.7}$$

where α_1, α_2 are real scalars. This damping model is also known as 'Rayleigh damping' or 'classical damping'. Modes of classically damped systems preserve the simplicity of the real normal modes as in the undamped case. Later, in a significant paper Caughey and O'Kelly (1965) have shown that the classical damping can exist in more general situation.

4.1 Condition for Proportional Damping

Caughey and O'Kelly (1965) have derived the condition which the system matrices must satisfy so that viscously damped linear systems possess classical normal modes. Their result can be described by the following theorem

Theorem 1 *Viscously damped system (4.2) possesses classical normal modes if and only if* $\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$.

Outline of the Proof. See the original paper by Caughey and O'Kelly (1965) for the detailed proof. Assuming $\mathbf{M}$ is not singular, premultiplying equation (4.2) by $\mathbf{M}^{-1}$ we have

$$\mathbf{I}\ddot{\mathbf{q}}(t) + [\mathbf{M}^{-1}\mathbf{C}]\dot{\mathbf{q}}(t) + [\mathbf{M}^{-1}\mathbf{K}]\mathbf{q}(t) = \mathbf{M}^{-1}\mathbf{f}(t). \tag{4.8}$$

For classical normal modes, (4.8) must be diagonalized by an orthogonal transformation. Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be diagonalized by an orthogonal transformation if and only if the commute in product, *i.e.*, $\mathbf{AB} = \mathbf{BA}$. Using this condition in (4.8) we have

$$\begin{aligned} &[\mathbf{M}^{-1}\mathbf{C}][\mathbf{M}^{-1}\mathbf{K}] = [\mathbf{M}^{-1}\mathbf{K}][\mathbf{M}^{-1}\mathbf{C}], \\ \text{or} \quad &\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C} \quad (\text{premultiplying both sides by } \mathbf{M}). \end{aligned} \tag{4.9}$$

A modified and more general version of this theorem is proved in Adhikari (2001).

Example 1: Assume that a system's mass, stiffness and damping matrices are given by

$$\mathbf{M} = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 1.0 & 2.0 & 2.0 \\ 1.0 & 2.0 & 3.0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0.5 \\ -1 & 1.2 & 0.4 \\ 0.5 & 0.4 & 1.8 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 15.25 & -9.8 & 3.4 \\ -9.8 & 6.48 & -1.84 \\ 3.4 & -1.84 & 2.22 \end{bmatrix}.$$

It may be verified that all the system matrices are positive definite. The mass-normalized undamped modal matrix is obtained as

$$\mathbf{X} = \begin{bmatrix} 0.4027 & -0.5221 & -1.2511 \\ 0.5845 & -0.4888 & 1.1914 \\ -0.1127 & 0.9036 & -0.4134 \end{bmatrix}. \tag{4.10}$$

Since Caughey and O'Kelly's condition

$$\mathbf{K}\mathbf{M}^{-1}\mathbf{C} = \mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} 125.45 & -80.92 & 28.61 \\ -80.92 & 52.272 & -18.176 \\ 28.61 & -18.176 & 7.908 \end{bmatrix}$$

is satisfied, the system possess classical normal modes and that $\mathbf{X}$ given in equation (4.10) is the modal matrix.

4.2 Generalized Proportional Damping

Obtaining a damping matrix from 'first principles' as with the mass and stiffness matrices is not possible for most systems. For this reason, assuming $\mathbf{M}$ and $\mathbf{K}$ are known, we often want to find $\mathbf{C}$ in terms of $\mathbf{M}$ and $\mathbf{K}$ such that the system still possesses classical normal modes. Of course, the earliest work along this line is the proportional damping shown in equation (4.7) by Rayleigh (1877). It can be verified that, for positive definite systems, expressing $\mathbf{C}$ in such a way will always satisfy the condition given by theorem 1. Caughey (1960) proposed that a *sufficient* condition for the existence of classical normal modes is: if $\mathbf{M}^{-1}\mathbf{C}$ can be expressed in a series involving powers of $\mathbf{M}^{-1}\mathbf{K}$. His result generalized Rayleigh's result, which turns out to be the first two terms of the series. Later, Caughey and O'Kelly (1965) proved that the series representation of damping

$$\mathbf{C} = \mathbf{M} \sum_{j=0}^{N-1} \alpha_j \left[\mathbf{M}^{-1}\mathbf{K}\right]^j \tag{4.11}$$

is the *necessary and sufficient* condition for existence of classical normal modes for systems without any repeated roots. This series is now known as the 'Caughey series'.

A further generalized and useful form of proportional damping was proposed by Adhikari (2001). It can be shown that we can express the damping matrix in the form

$$\mathbf{C} = \mathbf{M}\,\beta_1\left(\mathbf{M}^{-1}\mathbf{K}\right) + \mathbf{K}\,\beta_2\left(\mathbf{K}^{-1}\mathbf{M}\right) \tag{4.12}$$

$$\text{or}\quad \mathbf{C} = \beta_3\left(\mathbf{K}\mathbf{M}^{-1}\right)\mathbf{M} + \beta_4\left(\mathbf{M}\mathbf{K}^{-1}\right)\mathbf{K} \tag{4.13}$$

The functions $\beta_i(\bullet)$, $i = 1, \cdots, 4$ can have very general forms– they may consist of an arbitrary number of multiplications, divisions, summations, subtractions or powers of any other functions or can even be functional compositions. Thus, any conceivable form of analytic functions that are valid for scalars can be used in equations (4.12) and (4.13). In a natural way, common restrictions applicable to scalar functions are also valid, for example logarithm of a negative number is not permitted. Although the functions $\beta_i(\bullet)$, $i = 1, \cdots, 4$ are general, the expression of $\mathbf{C}$ in (4.12) or (4.13) gets restricted because of the special nature of the *arguments* in the functions. As a consequence, $\mathbf{C}$ represented in (4.12) or (4.13) does not cover the whole $\mathbb{R}^{N\times N}$, which is well known that many damped systems do not possess classical normal modes.

Rayleigh's result (4.7) can be obtained directly from equation (4.12) or (4.13) as a very special – one could almost say trivial – case by choosing each matrix function $\beta_i(\bullet)$ as real scalar times an identity matrix, that is

$$\beta_i(\bullet) = \alpha_i \mathbf{I}. \tag{4.14}$$

The damping matrix expressed in equation (4.12) or (4.13) provides a new way of interpreting the 'Rayleigh damping' or 'proportional damping' where the identity matrices (always) associated in the right or left side of $\mathbf{M}$ and $\mathbf{K}$ are replaced by arbitrary matrix functions $\beta_i(\bullet)$ with proper arguments. This kind of damping model will be called *generalized proportional damping*. We call the representation in equation (4.12) *right-functional form* and that in equation (4.13) *left-functional form*. Caughey series (4.11) is an example of right functional form. Note that if $\mathbf{M}$ or $\mathbf{K}$ is singular then the argument involving its corresponding inverse has to be removed from the functions. We will call the functions $\beta_i(\bullet)$ as *proportional damping functions* which are consistent with the definition of proportional damping constants (α_i) in Rayleighs model. From this discussion we have the following general result for damped linear systems:

Theorem 2 *A viscously damped positive definite linear system possesses classical normal modes if and only if* $\mathbf{C}$ *can be represented by*
(a) $\mathbf{C} = \mathbf{M}\,\beta_1\left(\mathbf{M}^{-1}\mathbf{K}\right) + \mathbf{K}\,\beta_2\left(\mathbf{K}^{-1}\mathbf{M}\right)$, *or*
(b) $\mathbf{C} = \beta_3\left(\mathbf{K}\mathbf{M}^{-1}\right)\mathbf{M} + \beta_4\left(\mathbf{M}\mathbf{K}^{-1}\right)\mathbf{K}$
for any $\beta_i(\bullet), i = 1, \cdots, 4$.

4.3 Dynamic Response

Dynamic response of proportionally damped systems can be obtained in a similar way to that of undamped systems.

4.3.1 Frequency Domain Analysis

Taking the Laplace transform of (4.2) and considering the initial conditions in (2.2) one has

$$s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q_0} - \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\mathbf{C}\bar{\mathbf{q}} - \mathbf{C}\mathbf{q_0} + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) \tag{4.15}$$

$$\text{or} \quad \left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{C}\mathbf{q_0} + s\mathbf{M}\mathbf{q_0}. \tag{4.16}$$

Consider the modal damping matrix

$$\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X} = 2\boldsymbol{\zeta}\boldsymbol{\Omega} \tag{4.17}$$

where

$$\boldsymbol{\zeta} = \operatorname{diag}\left[\zeta_1, \zeta_2, \cdots, \zeta_N\right] \in \mathbb{R}^{N\times N} \tag{4.18}$$

is the diagonal matrix containing the modal damping ratios. Using the mode orthogonality relationships and following the procedure similar to undamped systems, it is easy to show that

$$\bar{\mathbf{q}}(s) = \mathbf{X}\left[s^2\mathbf{I} + 2s\boldsymbol{\zeta}\boldsymbol{\Omega} + \boldsymbol{\Omega}^2\right]^{-1}\mathbf{X}^T\left\{\bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{C}\mathbf{q_0} + s\mathbf{M}\mathbf{q_0}\right\}. \tag{4.19}$$

The dynamic response in the frequency domain can be obtained by substituting $s = i\omega$. Note that $\left[s^2\mathbf{I} + 2s\boldsymbol{\zeta}\boldsymbol{\Omega} + \boldsymbol{\Omega}^2\right]$ is a diagonal matrix and therefore its inverse is easy to obtain.

Following the procedure similar to undamped systems, the transfer function matrix or the receptance matrix can be obtained as

$$\mathbf{H}(i\omega) = \mathbf{X}\left[-\omega^2\mathbf{I} + 2i\omega\boldsymbol{\zeta}\boldsymbol{\Omega} + \boldsymbol{\Omega}^2\right]^{-1}\mathbf{X}^T = \sum_{j=1}^{N} \frac{\mathbf{x}_j\mathbf{x}_j^T}{-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2}. \tag{4.20}$$

Using this, the dynamic response in the frequency domain can be conveniently represented from equation (4.19) as

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{N} \frac{\mathbf{x}_j^T\bar{\mathbf{f}}(i\omega) + \mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{x}_j^T\mathbf{C}\mathbf{q}_0 + i\omega\mathbf{x}_j^T\mathbf{M}\mathbf{q}_0}{-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2}\mathbf{x}_j. \tag{4.21}$$

Therefore, like undamped systems, the dynamic response of proportionally damped system can also be expressed as a linear combination of the undamped mode shapes.

4.3.2 Time Domain Analysis

Rewrite equation (4.21) in the Laplace domain as

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{N} \left\{ \frac{\mathbf{x}_j^T\bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} + \frac{\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{x}_j^T\mathbf{C}\mathbf{q}_0}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} + \frac{s}{s^2 + 2s\zeta_j\omega_j + \omega_j^2}\mathbf{x}_j^T\mathbf{M}\mathbf{q}_0 \right\} \mathbf{x}_j. \tag{4.22}$$

We can reorganize the denominator as

$$s^2 + 2s\zeta_j\omega_j + \omega_j^2 = (s + \zeta_j\omega_j)^2 - (\zeta_j\omega_j)^2 + \omega_j^2 = (s + \zeta_j\omega_j)^2 + \omega_{d_j}^2 \tag{4.23}$$

where

$$\omega_{d_j} = \omega_j\sqrt{1 - \zeta_j^2} \tag{4.24}$$

is known as the damped natural frequency. From the table of Laplace transforms we know that

$$\mathcal{L}^{-1}\left[\frac{1}{(s+\alpha)^2 + \beta^2}\right] = \frac{e^{-\alpha t}\sin(\beta t)}{\beta} \tag{4.25}$$

$$\text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{(s+\alpha)^2 + \beta^2}\right] = e^{-\alpha t}\cos(\beta t) - \frac{\alpha e^{-\alpha t}\sin(\beta t)}{\beta}. \tag{4.26}$$

Taking the inverse Laplace transform of (4.22) we have

$$\mathbf{q}(t) = \mathcal{L}^{-1}\left[\bar{\mathbf{q}}(s)\right] = \sum_{j=1}^{N} a_j(t)\mathbf{x}_j \tag{4.27}$$

where the time dependent constants are given by

$$
\begin{aligned}
a_j(t) =& \mathcal{L}^{-1}\left[\frac{\mathbf{x}_j^T\bar{\mathbf{f}}(s)}{(s+\zeta_j\omega_j)^2+\omega_{d_j}^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+\zeta_j\omega_j)^2+\omega_{d_j}^2}\right]\left(\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{x}_j^T\mathbf{C}\mathbf{q}_0\right) \\
&+ \mathcal{L}^{-1}\left[\frac{s}{(s+\zeta_j\omega_j)^2+\omega_{d_j}^2}\right]\mathbf{x}_j^T\mathbf{M}\mathbf{q}_0 \\
=& \int_0^t \frac{1}{\omega_{d_j}}\mathbf{x}_j^T\mathbf{f}(\tau)e^{-\zeta_j\omega_j(t-\tau)}\sin\left(\omega_{d_j}(t-\tau)\right)d\tau \\
&+ \frac{e^{-\zeta_j\omega_j t}}{\omega_{d_j}}\sin(\omega_{d_j}\,t)\left(\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{x}_j^T\mathbf{C}\mathbf{q}_0\right) \\
&+ \left\{e^{-\zeta_j\omega_j t}\cos\left(\omega_{d_j}t\right) - \frac{\zeta_j\omega_j\, e^{-\zeta_j\omega_j t}\sin\left(\omega_{d_j}\,t\right)}{\omega_{d_j}}\right\}\mathbf{x}_j^T\mathbf{M}\mathbf{q}_0
\end{aligned}
\tag{4.28}
$$

The convolution theorem is used to obtain the inverse Laplace transform of the first part. The formulae shown in (4.25) and (4.26) are used to obtain the inverse Laplace transform of second and third parts. Collecting the terms associated with $\sin\left(\omega_{d_j}\,t\right)$ and $\cos\left(\omega_{d_j}\,t\right)$ this expression can be simplified as

$$
a_j(t) = \int_0^t \frac{1}{\omega_{d_j}}\mathbf{x}_j^T\mathbf{f}(\tau)e^{-\zeta_j\omega_j(t-\tau)}\sin\left(\omega_{d_j}(t-\tau)\right)d\tau + e^{-\zeta_j\omega_j t}B_j\cos\left(\omega_{d_j}t+\theta_j\right) \tag{4.29}
$$

where

$$
B_j = \sqrt{\left(\mathbf{x}_j^T\mathbf{M}\mathbf{q}_0\right)^2 + \frac{1}{\omega_{d_j}^2}\left(\zeta_j\omega_j\mathbf{x}_j^T\mathbf{M}\mathbf{q}_0 - \mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_0 - \mathbf{x}_j^T\mathbf{C}\mathbf{q}_0\right)^2} \tag{4.30}
$$

$$
\text{and}\quad \tan\theta_j = \frac{1}{\omega_{d_j}}\left(\zeta_j\omega_j - \frac{\mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_0 + \mathbf{x}_j^T\mathbf{C}\mathbf{q}_0}{\mathbf{x}_j^T\mathbf{M}\mathbf{q}_0}\right) \tag{4.31}
$$

Exercise:

1. Verify that when damping is zero (*i.e.*, $\zeta_j = 0, \forall j$) these expressions reduce to the corresponding expressions for undamped systems obtained before.

2. Verify that equations (4.22) and (4.27) have dimensions of lengths.

3. Check that dynamic response (in the frequency and time domain) is linear with respect to the applied loading and initial conditions.

Example 2: Figure 1 shows a three DOF spring-mass system. The mass of each block is m Kg and the stiffness of each spring is k N/m. The viscous damping constant of the damper associated with each block is c Ns/m. The aim is to obtain the dynamic response for the following load cases:

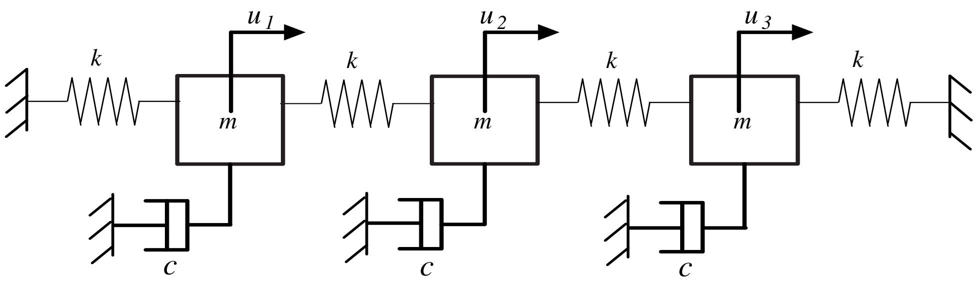

Figure 1: There DOF damped spring-mass system with dampers attached to the ground

1. When only the first mass (DOF 1) is subjected to an unit step input (see Figure 2) so that $\mathbf{f}(t) = \{f(t), 0, 0\}^T$ and $f(t) = 1 - U(t - t_0)$ with $t_0 = \frac{2\pi}{\omega_1}$ where ω_1 is the first undamped natural frequency of the system and $U(\bullet)$ is the unit step function.
2. When only the second mass (DOF 2) is subjected to unit initial displacement, *i.e.*, $\mathbf{q_0} = \{0, 1, 0\}^T$.
3. When only the second and the third masses (DOF 3) are subjected to unit initial velocities, *i.e.*, $\dot{\mathbf{q}}_\mathbf{0} = \{0, 1, 1\}^T$.
4. When all three of the above loading are acting together on the system.

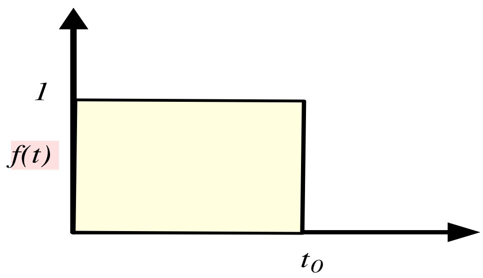

Figure 2: Unit step forcing, $t_0 = \frac{2\pi}{\omega_1}$

This problem can be solved by using the following steps

I. *Obtain the System Matrices:* The mass, stiffness and the damping matrices are given by

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}. \tag{4.32}$$

Note that the damping matrix is mass proportional, so that the system is proportionally damped.

II. *Obtain the undamped natural frequencies:* For notational convenience assume that the eigenvalues $\lambda_j = \omega_j^2$. The three DOF system has three eigenvalues and they are the roots of the following characteristic equation

$$\det\left[\mathbf{K} - \lambda\mathbf{M}\right] = 0. \tag{4.33}$$

Using the mass and the stiffness matrices from equation (4.32), this can be simplified as

$$\begin{aligned} &\det\begin{bmatrix} 2k-\lambda m & -k & 0 \\ -k & 2k-\lambda m & -k \\ 0 & -k & 2k-\lambda m \end{bmatrix} = 0 \\ \text{or} \quad & m\det\begin{bmatrix} 2\alpha-\lambda & -\alpha & 0 \\ -\alpha & 2\alpha-\lambda & -\alpha \\ 0 & -\alpha & 2\alpha-\lambda \end{bmatrix} = 0 \quad \text{where} \quad \alpha = \frac{k}{m}. \end{aligned} \tag{4.34}$$

Expanding the determinant in (4.34) we have

$$\begin{aligned} & (2\alpha-\lambda)\left\{(2\alpha-\lambda)^2-\alpha^2\right\} - \alpha\alpha\,(2\alpha-\lambda) = 0 \\ \text{or} \quad & (2\alpha-\lambda)\left\{(2\alpha-\lambda)^2-2\alpha^2\right\} = 0 \\ \text{or} \quad & (2\alpha-\lambda)\left\{(2\alpha-\lambda)^2-\left(\sqrt{2}\alpha\right)^2\right\} = 0 \\ \text{or} \quad & (2\alpha-\lambda)\left(2\alpha-\lambda-\sqrt{2}\alpha\right)\left(2\alpha-\lambda+\sqrt{2}\alpha\right) = 0 \\ \text{or} \quad & (2\alpha-\lambda)\left(\left(2-\sqrt{2}\right)\alpha-\lambda\right)\left(\left(2+\sqrt{2}\right)\alpha-\lambda\right) = 0. \end{aligned} \tag{4.35}$$

It implies that the three roots (in the increasing order) are

$$\lambda_1 = \left(2-\sqrt{2}\right)\alpha, \quad \lambda_2 = 2\alpha, \quad \text{and} \quad \lambda_3 = \left(2+\sqrt{2}\right)\alpha. \tag{4.36}$$

Since $\lambda_j = \omega_j^2$, the natural frequencies are

$$\omega_1 = \sqrt{\left(2-\sqrt{2}\right)\alpha}, \quad \omega_2 = \sqrt{2\alpha}, \quad \text{and} \quad \omega_3 = \sqrt{\left(2+\sqrt{2}\right)\alpha}. \tag{4.37}$$

III. *Obtain the undamped mode shapes:* From equation (2.3) the eigenvalue equation can be written as

$$\left[\mathbf{K} - \lambda_j\mathbf{M}\right]\mathbf{x}_j = 0. \tag{4.38}$$

For this problem, substituting $\mathbf{K}$ and $\mathbf{M}$ from equation (4.32) and dividing by m we have

$$\begin{bmatrix} 2\alpha-\lambda & -\alpha & 0 \\ -\alpha & 2\alpha-\lambda & -\alpha \\ 0 & -\alpha & 2\alpha-\lambda \end{bmatrix} \begin{Bmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \end{Bmatrix} = 0. \tag{4.39}$$

Here x_{1j}, x_{2j} and x_{3j} are the three components of jth eigenvector corresponding to the three masses. To obtain $\mathbf{x}_j$ we need to substitute λ_j from equation (4.36) in the above equation and solve for each component of $\mathbf{x}_j$ for every j.

The first eigenvector, $j = 1$:

Substituting $\lambda = \lambda_1 = \left(2-\sqrt{2}\right)\alpha$ in (4.39) we have

$$\begin{bmatrix} 2\alpha-\left(2-\sqrt{2}\right)\alpha & -\alpha & 0 \\ -\alpha & 2\alpha-\left(2-\sqrt{2}\right)\alpha & -\alpha \\ 0 & -\alpha & 2\alpha-\left(2-\sqrt{2}\right)\alpha \end{bmatrix} \begin{Bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{Bmatrix} = 0 \tag{4.40}$$

$$\text{or} \quad \begin{bmatrix} \sqrt{2}\alpha & -\alpha & 0 \\ -\alpha & \sqrt{2}\alpha & -\alpha \\ 0 & -\alpha & \sqrt{2}\alpha \end{bmatrix} \begin{Bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{Bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{bmatrix} \begin{Bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{Bmatrix} = 0. \tag{4.41}$$

This can be separated into three equations as

$$\sqrt{2}x_{11} - x_{21} = 0, \quad -x_{11} + \sqrt{2}x_{21} - x_{31} = 0 \quad \text{and} \quad -x_{21} + \sqrt{2}x_{31} = 0. \tag{4.42}$$

These three equations cannot be solved uniquely but once we fix one element, the other two elements can be expressed in terms of it. This implies that the ratios between the modal amplitudes are unique. Solving the system of linear equations (4.42) we have $x_{21} = \sqrt{2}x_{11}$ and $x_{21} = \sqrt{2}x_{31}$, that is $x_{11} = x_{31} = \gamma_1$ (say). Therefore the first eigenvector is given by

$$\mathbf{x}_1 = \begin{Bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{Bmatrix} = \gamma_1 \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}. \tag{4.43}$$

The constant γ_1 can be obtained from the mass normalization condition in (2.11), that is

$$\mathbf{x}_1^T \mathbf{M} \mathbf{x}_1 = 1 \quad \text{or} \quad \gamma_1 \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \gamma_1 \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} = 1 \tag{4.44}$$

$$\text{or} \quad \gamma_1^2 m \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} = 1 \tag{4.45}$$

$$\text{or} \quad \gamma_1^2 m \left(1 + \sqrt{2}\sqrt{2} + 1\right) = 1 \quad \text{that is} \quad \gamma_1 = \frac{1}{2\sqrt{m}}. \tag{4.46}$$

Thus the mass normalized first eigenvector is given by

$$\mathbf{x}_1 = \frac{1}{2\sqrt{m}} \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}. \tag{4.47}$$

The second eigenvector, $j = 2$:

Substituting $\lambda = \lambda_2 = 2\alpha$ in (4.39) we have

$$\begin{bmatrix} 2\alpha - 2\alpha & -\alpha & 0 \\ -\alpha & 2\alpha - 2\alpha & -\alpha \\ 0 & -\alpha & 2\alpha - 2\alpha \end{bmatrix} \begin{Bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{Bmatrix} = 0 \tag{4.48}$$

$$\text{or} \quad \begin{bmatrix} 0 & -\alpha & 0 \\ -\alpha & 0 & -\alpha \\ 0 & -\alpha & 0 \end{bmatrix} \begin{Bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{Bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{Bmatrix} = 0. \tag{4.49}$$

This implies that $x_{22} = 0$ and $x_{12} = -x_{32} = \gamma_2$ (say). Therefore the second eigenvector is given by

$$\mathbf{x}_2 = \begin{Bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{Bmatrix} = \gamma_2 \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}. \tag{4.50}$$

Using the mass normalization condition

$$\mathbf{x}_2^T \mathbf{M} \mathbf{x}_2 = 1 \quad \text{or} \quad \gamma_2 \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \gamma_2 \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = 1 \tag{4.51}$$

$$\text{or} \quad \gamma_2^2 m \left(1 + 1\right) = 1 \quad \text{that is} \quad \gamma_2 = \frac{1}{\sqrt{2m}} = \frac{\sqrt{2}}{2\sqrt{m}}. \tag{4.52}$$

Thus, the mass normalized second eigenvector is given by

$$\mathbf{x}_2 = \frac{1}{2\sqrt{m}} \begin{Bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{Bmatrix}. \tag{4.53}$$

The third eigenvector, $j = 3$:

Substituting $\lambda = \lambda_3 = \left(2 + \sqrt{2}\right) \alpha$ in (4.39) we have

$$\begin{bmatrix} 2\alpha - \left(2+\sqrt{2}\right)\alpha & -\alpha & 0 \\ -\alpha & 2\alpha - \left(2+\sqrt{2}\right)\alpha & -\alpha \\ 0 & -\alpha & 2\alpha - \left(2+\sqrt{2}\right)\alpha \end{bmatrix} \begin{Bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{Bmatrix} = 0 \tag{4.54}$$

$$\text{or} \quad \begin{bmatrix} -\sqrt{2}\alpha & -\alpha & 0 \\ -\alpha & -\sqrt{2}\alpha & -\alpha \\ 0 & -\alpha & -\sqrt{2}\alpha \end{bmatrix} \begin{Bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{Bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \begin{Bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{Bmatrix} = 0. \tag{4.55}$$

This implies that

$$\begin{aligned} \sqrt{2}x_{13} + x_{23} = 0 \Rightarrow x_{23} = -\sqrt{2}x_{13} \\ x_{13} + \sqrt{2}x_{23} + x_{33} = 0 \\ \text{and} \quad x_{23} + \sqrt{2}x_{33} = 0 \Rightarrow x_{23} = -\sqrt{2}x_{33} \end{aligned} \tag{4.56}$$

that is $x_{13} = x_{33} = \gamma_3$ (say). Therefore the third eigenvector is given by

$$\mathbf{x}_3 = \begin{Bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{Bmatrix} = \gamma_3 \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}. \tag{4.57}$$

Using the mass normalization condition

$$\mathbf{x}_3^T \mathbf{M} \mathbf{x}_3 = 1 \quad \text{or} \quad \gamma_3 \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \gamma_3 \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix} = 1 \tag{4.58}$$

$$\text{or} \quad \gamma_3^2 m \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix} = 1 \tag{4.59}$$

$$\text{or} \quad \gamma_3^2 m \left(1 + \sqrt{2}\sqrt{2} + 1\right) = 1, \quad \text{that is,} \quad \gamma_3 = \frac{1}{2\sqrt{m}}. \tag{4.60}$$

Thus the mass normalized third eigenvector is given by

$$\mathbf{x}_3 = \frac{1}{2\sqrt{m}} \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}. \tag{4.61}$$

Combining the three eigenvectors, the mass normalized undamped modal matrix is now given by

$$\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = \frac{1}{2\sqrt{m}} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}. \tag{4.62}$$

Here the modal matrix turns out to be symmetric. But in general this is not the case.

IV. *Obtain the modal damping ratios:* The damping matrix in the modal coordinate can be obtained from (4.4) as

$$\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X} = \frac{1}{2\sqrt{m}} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}^T \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \frac{1}{2\sqrt{m}} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} = \frac{c}{m}\mathbf{I}. \tag{4.63}$$

Therefore

$$2\zeta_j\omega_j = \frac{c}{m} \quad \text{or} \quad \zeta_j = \frac{c}{2m\omega_j}. \tag{4.64}$$

Since ω_j becomes bigger for higher modes, modal damping gets smaller, *i.e.*, higher modes are less damped.

V. *Response due to applied loading:* The applied loading $\mathbf{f}(t) = \{f(t), 0, 0\}^T$ where $f(t) = 1 - U(t - t_0)$ with $t_0 = \frac{2\pi}{\omega_1}$. In the Laplace domain

$$\bar{f}(s) = \mathcal{L}\left[1 - U(t - t_0)\right] = 1 - \frac{e^{-st_0}}{s}. \tag{4.65}$$

Thus, the term $\mathbf{x}_j^T\bar{\mathbf{f}}(s)$ can be obtained as

$$\mathbf{x}_j^T\bar{\mathbf{f}}(s) = \begin{Bmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \end{Bmatrix}^T \begin{Bmatrix} \bar{f}(s) \\ 0 \\ 0 \end{Bmatrix} = x_{1j}\left(1 - \frac{e^{-st_0}}{s}\right) \quad \forall j. \tag{4.66}$$

Since the initial conditions are zero, the dynamic response in the Laplace domain can be obtained from equation (4.22) as

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{3} \left\{ \frac{x_{1j}\left(1 - \frac{e^{-st_0}}{s}\right)}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} \right\} \mathbf{x}_j = \sum_{j=1}^{3} \left\{ \frac{x_{1j}\left(s - e^{-st_0}\right)}{s\left(s^2 + 2s\zeta_j\omega_j + \omega_j^2\right)} \right\} \mathbf{x}_j. \tag{4.67}$$

In the frequency domain, the response is given by

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{3} \left\{ \frac{x_{1j}\left(i\omega - e^{-i\omega t_0}\right)}{i\omega\left(-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2\right)} \right\} \mathbf{x}_j. \tag{4.68}$$

For the numerical calculations we assume $m = 1$, $k = 1$ and $c = 0.2$. Using these values, from (4.68), the absolute value of the frequency domain response of the three masses are plotted in Figure 3. The three peaks in the diagram correspond to the three natural frequencies of the system. The time domain response can be obtained by evaluating the

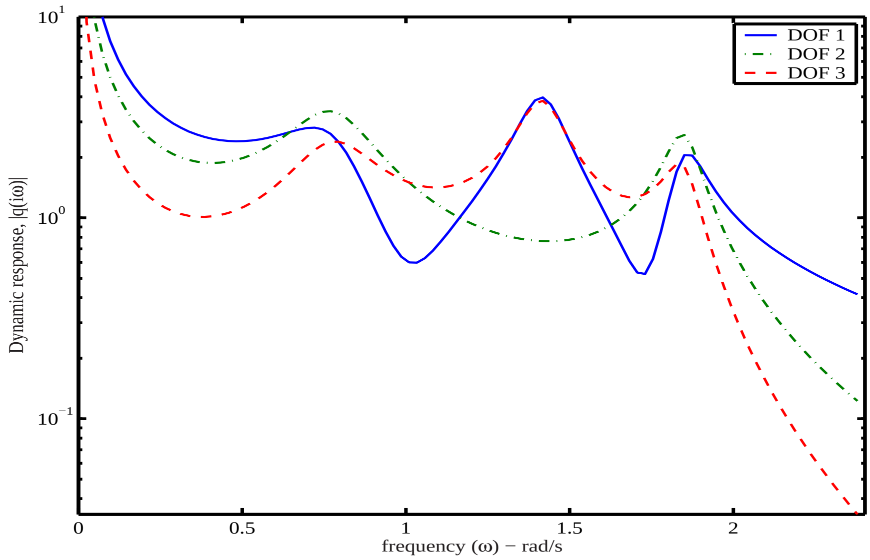

Figure 3: Absolute value of the frequency domain response of the three masses due to applied step loading at first DOF

convolution integral in (4.29) and substituting $a_j(t)$ in equation (4.27). In practice, usually numerical integration methods are used to evaluate this integral. For this problem a closed-from expression can be obtained. We have

$$\mathbf{x}_j^T \mathbf{f}(\tau) = x_{1j} f(\tau). \tag{4.69}$$

From Figure 2 it can be noted that

$$f(\tau) = \begin{cases} 1 & \text{if} \quad \tau < t_0, \\ 0 & \text{if} \quad \tau > t_0. \end{cases} \tag{4.70}$$

Because of this, the limit of the integral in (4.29) can be changed as

$$\begin{aligned} a_j(t) &= \int_0^t \frac{1}{\omega_{d_j}} \mathbf{x}_j^T \mathbf{f}(\tau) e^{-\zeta_j\omega_j(t-\tau)} \sin\left(\omega_{d_j}(t-\tau)\right) d\tau = \int_0^{t_0} \frac{1}{\omega_{d_j}} x_{1j} e^{-\zeta_j\omega_j(t-\tau)} \sin\left(\omega_{d_j}(t-\tau)\right) d\tau \\ &= \frac{x_{1j}}{\omega_{d_j}} \int_0^{t_0} e^{-\zeta_j\omega_j(t-\tau)} \sin\left(\omega_{d_j}(t-\tau)\right) d\tau. \end{aligned} \tag{4.71}$$

By making a substitution $\tau' = t - \tau$, this integral can be evaluated as

$$a_j(t) = \frac{x_{1j}}{\omega_{d_j}} \frac{e^{-\zeta_j \omega_j t}}{\omega_j^2} \left\{ \alpha_j \sin\left(\omega_{d_j} t\right) + \beta_j \cos\left(\omega_{d_j} t\right) \right\} \tag{4.72}$$

$$\text{where} \quad \alpha_j = \left\{ \omega_{d_j} \sin\left(\omega_{d_j} t_0\right) + \zeta_j \omega_j \cos\left(\omega_{d_j} t_0\right) \right\} e^{\zeta_j \omega_j t_0} - \zeta_j \omega_j \tag{4.73}$$

$$\text{and} \quad \beta_j = \left\{ \omega_{d_j} \cos\left(\omega_{d_j} t_0\right) - \zeta_j \omega_j \sin\left(\omega_{d_j} t_0\right) \right\} e^{\zeta_j \omega_j t_0} - \omega_{d_j}. \tag{4.74}$$

The time domain responses of the three masses obtained using equation (4.27) are shown in Figure 4. From the diagram observe that, because the forcing is applied to the first

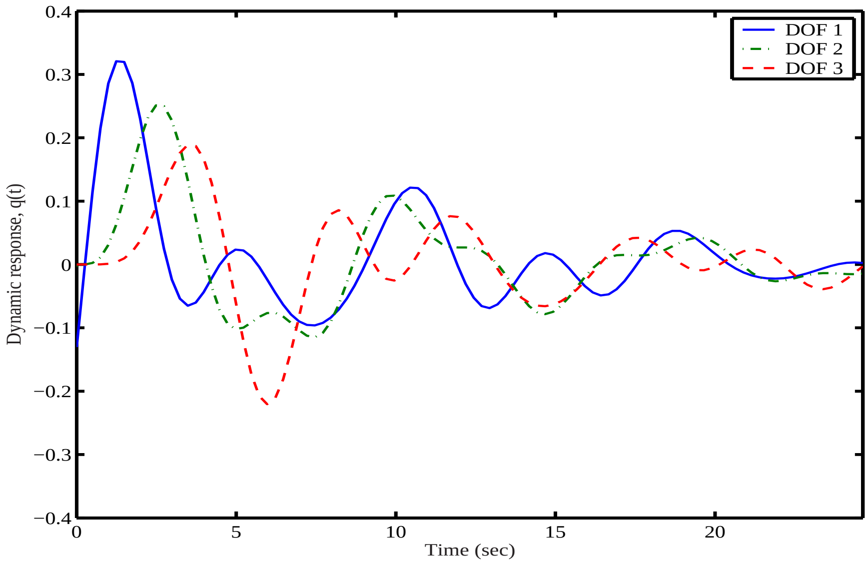

Figure 4: Time domain response of the three masses due to applied step loading at first DOF

mass (DOF 1), it moves earlier and comparatively more than the other two masses.

VI. *Response due to initial displacement:* When $\mathbf{q_0} = \{0, 1, 0\}^T$ we have

$$\mathbf{x}_j^T \mathbf{C} \mathbf{q_0} = \begin{Bmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \end{Bmatrix}^T \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = x_{2j} c \quad \forall j. \tag{4.75}$$

Similarly

$$\mathbf{x}_j^T \mathbf{M} \mathbf{q_0} = x_{2j} m \quad \forall j. \tag{4.76}$$

The dynamic response in the Laplace domain can be obtained from equation (4.22) as

$$\begin{aligned} \bar{\mathbf{q}}(s) &= \sum_{j=1}^{3} \left\{ \frac{x_{2j} c}{s^2 + 2s\zeta_j \omega_j + \omega_j^2} + \frac{x_{2j} m s}{s^2 + 2s\zeta_j \omega_j + \omega_j^2} \right\} \mathbf{x}_j \\ &= \sum_{j=1}^{3} x_{2j} \left\{ \frac{c + ms}{s^2 + 2s\zeta_j \omega_j + \omega_j^2} \right\} \mathbf{x}_j. \end{aligned} \tag{4.77}$$

From equation (4.64) note that $c = 2\zeta_j\omega_j m$. Substituting this in the above equation we have

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{3} x_{2j} m \left\{ \frac{2\zeta_j\omega_j + s}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} \right\} \mathbf{x}_j. \tag{4.78}$$

In the frequency domain the response is given by

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{3} x_{2j} m \left\{ \frac{2\zeta_j\omega_j + i\omega}{-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2} \right\} \mathbf{x}_j. \tag{4.79}$$

Figure 5 shows the responses of the three masses. Note that the second peak is 'missing' and the responses of the first and the third masses are exactly the same. This is due to the fact that in the second mode of vibration the middle mass remains stationary while the two other masses move equal amount in the opposite directions (see the second mode in equation (4.53)). The time domain response can be obtained by directly taking the

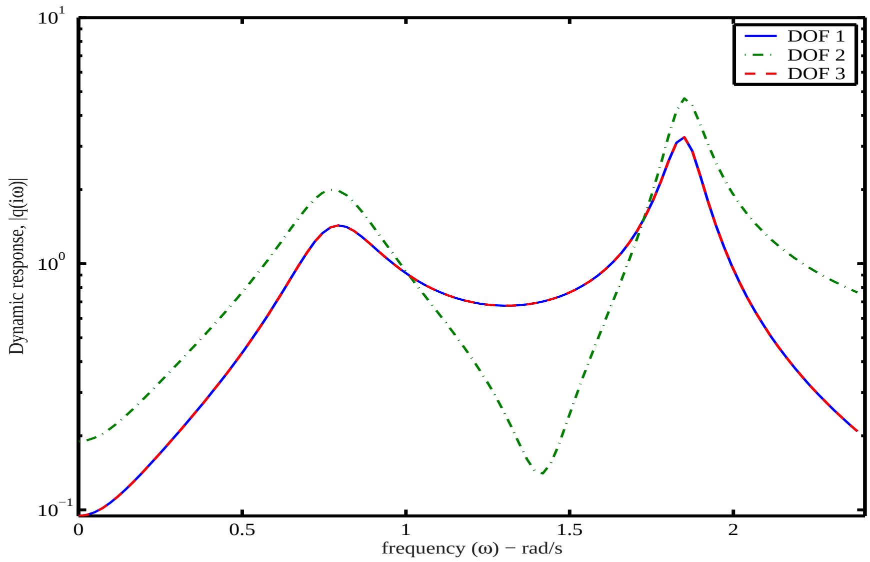

Figure 5: Absolute value of the frequency domain response of the three masses due to unit initial displacement at second DOF

inverse Laplace transform of (4.78) as

$$\mathbf{q}(t) = \mathcal{L}^{-1}\left[\bar{\mathbf{q}}(s)\right] = \sum_{j=1}^{3} x_{2j} m \mathcal{L}^{-1}\left[\frac{2\zeta_j\omega_j + s}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} \right] \mathbf{x}_j = \sum_{j=1}^{3} x_{2j} m e^{-\zeta_j\omega_j t} \cos\left(\omega_{d_j} t\right) \mathbf{x}_j. \tag{4.80}$$

This expression is plotted in Figure 6. Observe that initial displacement of the second mass is unity, which verify that the initial condition has been applied correctly. Because of the symmetry of the system the displacements of the two other masses are identical.

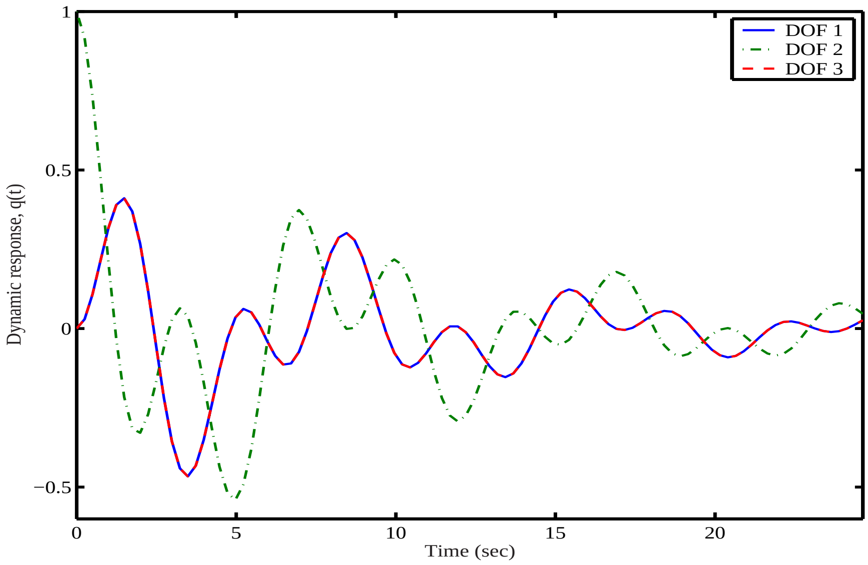

Figure 6: Time domain response of the three masses due to unit initial displacement at second DOF

VII. *Response due to initial velocity:* When $\dot{\mathbf{q}}_\mathbf{0} = \{0, 1, 1\}^T$ we have

$$\mathbf{x}_j^T \mathbf{M} \dot{\mathbf{q}}_\mathbf{0} = \begin{Bmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = (x_{2j} + x_{3j})\, m \quad \forall j. \tag{4.81}$$

The dynamic response in the Laplace domain can be obtained from equation (4.22) as

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{3} \left\{ \frac{(x_{2j} + x_{3j})\, m}{s^2 + 2s\zeta_j\omega_j + \omega_j^2} \right\} \mathbf{x}_j. \tag{4.82}$$

The time domain response can be obtained using the inverse Laplace transform as

$$\mathbf{q}(t) = \sum_{j=1}^{3} (x_{2j} + x_{3j})\, \frac{m}{\omega_{d_j}} e^{-\zeta_j\omega_j t} \sin\left(\omega_{d_j} t\right) \mathbf{x}_j. \tag{4.83}$$

The responses of the three masses in frequency domain and in the time domain are respectively shown in Figures 7 and 8. In this case all the modes of the system can be observed. Because the initial conditions of the second and the third masses are the same, their initial displacements are close to each other. However, as the time passes the displacements of these two masses start differing from each other.

VIII. *Combined Response:* Exercise.

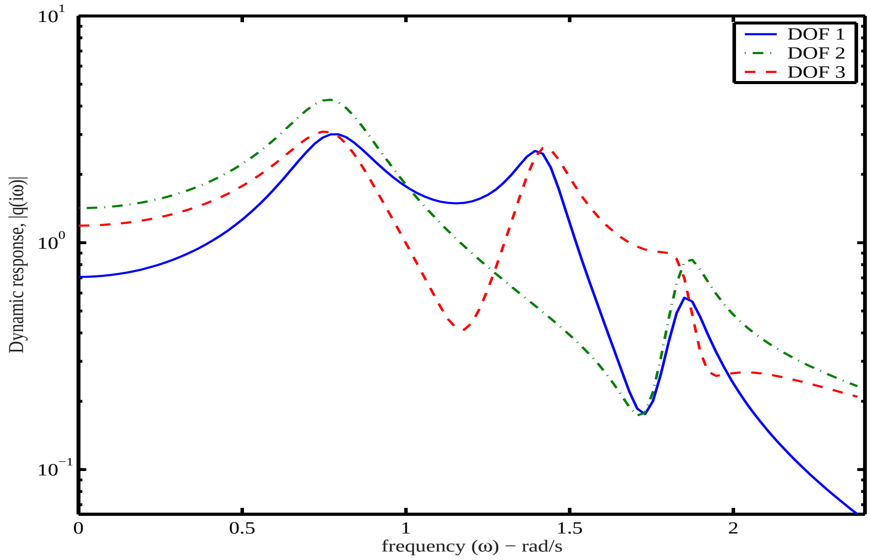

Figure 7: Absolute value of the frequency domain response of the three masses due to unit initial velocity at the second and third DOF

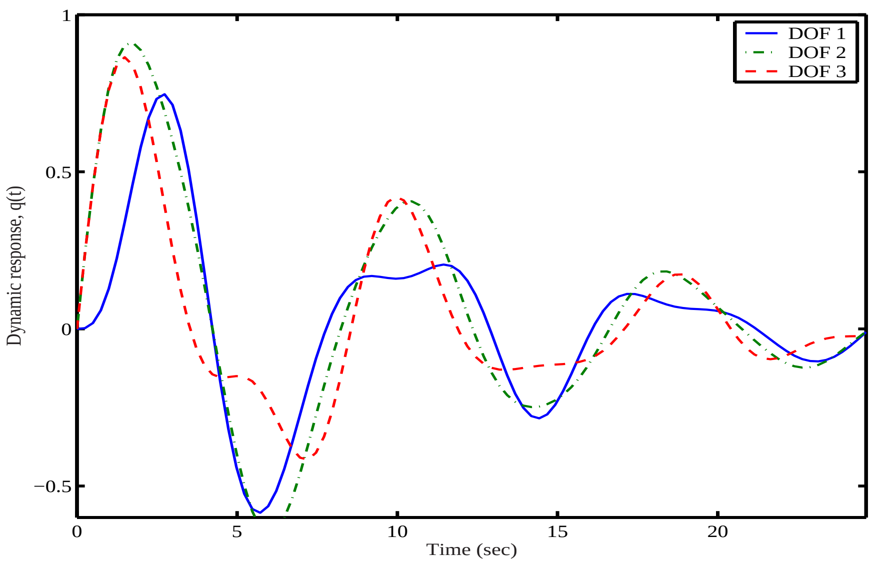

Figure 8: Time domain response of the three masses due to unit initial velocity at the second and third DOF

Exercise problem: Redo the previous example (a) for undamped system, and (b) for the 3DOF system shown in Figure 9.

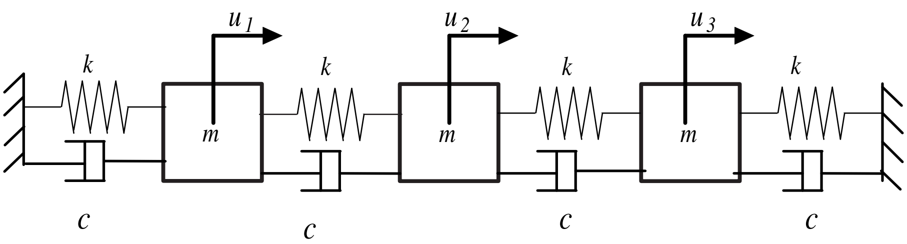

Figure 9: There DOF damped spring-mass system with dampers attached to each other

Hint: The damping is stiffness proportional. Use the associated Matlab program.

5 Non-proportionally Damped Systems

Modes of proportionally damped systems preserve the simplicity of the real normal modes as in the undamped case. Unfortunately there is no physical reason why a general system should behave like this. In fact practical experience in modal testing shows that most real-life structures do not do so, as they possess complex modes instead of real normal modes. This implies that in general linear systems are non-classically damped. When the system is non-classically damped, some or all of the N differential equations in (4.3) are coupled through the $\mathbf{X}^T\mathbf{C}\mathbf{X}$ term and cannot be reduced to N second-order uncoupled equation. This coupling brings several complication in the system dynamics – the eigenvalues and the eigenvectors no longer remain real and also the eigenvectors do not satisfy the classical orthogonality relationships as given by equations (2.11) and (2.12).

5.1 Free Vibration and Complex Modes

The complex eigenvalue problem associated with equation (4.2) can be represented by

$$s_j^2\mathbf{M}\mathbf{u}_j + s_j\mathbf{C}\mathbf{u}_j + \mathbf{K}\mathbf{u}_j = \mathbf{0} \tag{5.1}$$

where $s_j \in \mathbb{C}$ is the jth eigenvalue and $\mathbf{u}_j \in \mathbb{C}^N$ is the jth eigenvector. The eigenvalues, s_j, are the roots of the characteristic polynomial

$$\det\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right] = 0. \tag{5.2}$$

The order of the polynomial is $2N$ and if the roots are complex they appear in complex conjugate pairs. The methods for solving this kind of complex problem follow mainly two routes, the state-space method and the methods in configuration space or 'N-space'. A brief discussion of these two approaches is taken up in the following subsections.

5.1.1 The State-Space Method:

The state-space method is based on transforming the N second-order coupled equations into a set of $2N$ first-order coupled equations by augmenting the displacement response vectors with the velocities of the corresponding coordinates. We can write equation (4.2) together with a trivial equation $\mathbf{M}\dot{\mathbf{q}}(t) - \mathbf{M}\dot{\mathbf{q}}(t) = 0$ in a matrix form as

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}(t) \\ \ddot{\mathbf{q}}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & -\mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{Bmatrix} \tag{5.3}$$

$$\text{or} \quad \mathbf{A}\,\dot{\mathbf{z}}(t) + \mathbf{B}\,\mathbf{z}(t) = \mathbf{r}(t) \tag{5.4}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \in \mathbb{R}^{2N\times 2N}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & -\mathbf{M} \end{bmatrix} \in \mathbb{R}^{2N\times 2N},$$
$$\mathbf{z}(t) = \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix} \in \mathbb{R}^{2N}, \quad \text{and} \quad \mathbf{r}(t) = \begin{Bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{Bmatrix} \in \mathbb{R}^{2N}. \tag{5.5}$$

In the above equation $\mathbf{O}$ is the $N \times N$ null matrix. This form of equations of motion is also known as the 'Duncan form'.

The eigenvalue problem associated with equation (5.4) can be expressed as

$$s_j \mathbf{A} \mathbf{z}_j + \mathbf{B} \mathbf{z}_j = \mathbf{0} \tag{5.6}$$

where $s_j \in \mathbb{C}$ is the j-th eigenvalue and $\mathbf{z}_j \in \mathbb{C}^{2N}$ is the j-th eigenvector. This eigenvalue problem is similar to the undamped eigenvalue problem (2.3) except (a) the dimension of the matrices are $2N$ as opposed to N, and (b) the matrices are not positive definite. Because of (a) the computational cost to obtain the eigensolutions of (5.6) is much higher compared to the undamped eigensolutions and due to (b) the eigensolutions in general become complex valued. From a phenomenological point of view, this implies that the modes are not synchronous, *i.e.*, there is a 'phase lag' so that different degrees of freedom do not simultaneously reach to their corresponding 'peaks' and 'troughs'. Thus, both from computational and conceptual point of view, complex modes significantly complicates the problem and in practice they are often avoided. From the expression of $\mathbf{z}(t)$ in equation (5.5), the state-space complex eigenvectors $\mathbf{z}_j$ can be related to the jth eigenvector of the second-order system as

$$\mathbf{z}_j = \begin{Bmatrix} \mathbf{u}_j \\ s_j \mathbf{u}_j \end{Bmatrix}. \tag{5.7}$$

Since $\mathbf{A}$ and $\mathbf{B}$ are real matrices, taking complex conjugate ($(\bullet)^*$ denotes complex conjugation) of the eigenvalue equation (5.6) it is trivial to see that

$$s_j^* \mathbf{A} \mathbf{z}_j^* + \mathbf{B} \mathbf{z}_j^* = \mathbf{0}. \tag{5.8}$$

This implies that the eigensolutions must appear in complex conjugate pairs. For convenience arrange the eigenvalues and the eigenvectors so that

$$s_{j+N} = s_j^* \tag{5.9}$$
$$\mathbf{z}_{j+N} = \mathbf{z}_j^*, \quad j = 1, 2, \cdots, N \tag{5.10}$$

Like real normal modes, complex modes in the state-space also satisfy orthogonal relationships over the $\mathbf{A}$ and $\mathbf{B}$ matrices. For distinct eigenvalues it is easy to show that

$$\mathbf{z}_j^T \mathbf{A} \mathbf{z}_k = 0 \quad \text{and} \quad \mathbf{z}_j^T \mathbf{B} \mathbf{z}_k = 0; \quad \forall j \neq k. \tag{5.11}$$

Premultiplying equation (5.6) by $\mathbf{y}_j^T$ one obtains

$$\mathbf{z}_j^T \mathbf{B} \mathbf{z}_j = -s_j \mathbf{z}_j^T \mathbf{A} \mathbf{z}_j. \tag{5.12}$$

The eigenvectors may be normalized so that

$$\mathbf{z}_j^T \mathbf{A} \mathbf{z}_j = \frac{1}{\gamma_j} \tag{5.13}$$

where $\gamma_j \in \mathbb{C}$ is the normalization constant. In view of the expressions of $\mathbf{z}_j$ in equation (5.7) the above relationship can be expressed in terms of the eigensolutions of the second-order system as

$$\mathbf{u}_j^T \left[2s_j \mathbf{M} + \mathbf{C}\right] \mathbf{u}_j = \frac{1}{\gamma_j}. \tag{5.14}$$

There are several ways in which the normalization constants can be selected. The one that is most consistent with traditional modal analysis practice, is to choose $\gamma_j = 1/2s_j$. Observe that this degenerates to the familiar mass normalization relationship $\mathbf{u}_j^T \mathbf{M} \mathbf{u}_j = 1$ when the damping is zero.

5.1.2 Approximate Methods in the Configuration Space

It has been pointed out that the state-space approach towards the solution of equation of motion in the context of linear structural dynamics is not only computationally expensive but also fails to provide the physical insight which modal analysis in the configuration space or N-space offers. In this section, assuming that the damping is light, a simple *first-order perturbation method* is used to obtain complex modes and frequencies in terms of undamped modes and frequencies.

The undamped modes form a *complete* set of vectors so that each complex mode $\mathbf{u}_j$ can be expressed as a linear combination of $\mathbf{x}_j$. Assume that

$$\mathbf{u}_j = \sum_{k=1}^{N} \alpha_k^{(j)} \mathbf{x}_k \tag{5.15}$$

where $\alpha_k^{(j)}$ are complex constants which we want to determine. Since damping is assumed light, $\alpha_k^{(j)} \ll 1, \forall j \neq k$ and $\alpha_j^{(j)} = 1, \forall j$. Suppose the complex natural frequencies are denoted by λ_j, which are related to complex eigenvalues s_j through

$$s_j = i\lambda_j. \tag{5.16}$$

Substituting s_j and $\mathbf{u}_j$ in the eigenvalue equation (5.1) we have

$$\left[-\lambda_j^2 \mathbf{M} + i\lambda_j \mathbf{C} + \mathbf{K}\right] \sum_{k=1}^{N} \alpha_k^{(j)} \mathbf{x}_k = \mathbf{0}. \tag{5.17}$$

Premultiplying by $\mathbf{x}_j^T$ and using the orthogonality conditions (2.11) and (2.12), we have

$$-\lambda_j^2 + i\lambda_j \sum_{k=1}^{N} \alpha_k^{(j)} C'_{jk} + \omega_j^2 = 0 \tag{5.18}$$

where $C'_{jk} = \mathbf{x}_j^T \mathbf{C} \mathbf{x}_k$, is the jkth element of the modal damping matrix $\mathbf{C}'$. Due to small damping assumption we can neglect the product $\alpha_k^{(j)} C'_{jk}, \forall j \neq k$ since they are small compared to $\alpha_j^{(j)} C'_{jj}$. Thus equation (5.18) can be approximated as

$$-\lambda_j^2 + i\lambda_j \alpha_j^{(j)} C'_{jj} + \omega_j^2 \approx 0. \tag{5.19}$$

By solving the quadratic equation we have

$$\lambda_j \approx \pm\omega_j + iC'_{jj}/2. \tag{5.20}$$

This is the expression of approximate complex natural frequencies. Premultiplying equation (5.17) by $\mathbf{x}_k^T$ using the orthogonality conditions (2.11) and (2.12) and light damping assumption, it can be shown that

$$\alpha_k^{(j)} \approx \frac{i\omega_j C'_{kj}}{\omega_j^2 - \omega_k^2}, \quad k \neq j. \tag{5.21}$$

Substituting this in (5.15), approximate complex modes can be given by

$$\mathbf{u}_j \approx \mathbf{x}_j + \sum_{k \neq j}^{N} \frac{i\omega_j C'_{kj} \mathbf{x}_k}{\omega_j^2 - \omega_k^2}. \tag{5.22}$$

This expression shows that (a) the imaginary parts of complex modes are approximately orthogonal to the real parts, and (b) the 'complexity' of the modes will be more if ω_j and ω_k are close, *i.e.*, modes will be significantly complex when the natural frequencies of a system are closely spaced. It should be recalled that the first-order perturbation expressions are only valid when damping is small. For moderate to large damping values more general expression derived by Adhikari (1999) should be used.

5.2 Dynamic Response

Once the complex mode shapes and natural frequencies are obtained (either from the state-space method or from the approximate method), the dynamic response can be obtained using the orthogonality properties of the complex eigenvectors in the state-space. We will derive the expressions for the general dynamic response in the frequency and time domain.

5.2.1 Frequency Domain Analysis

Taking the Laplace transform of equation (5.4) we have

$$s\mathbf{A}\bar{\mathbf{z}}(s) - \mathbf{A}\mathbf{z_0} + \mathbf{B}\bar{\mathbf{z}}(s) = \bar{\mathbf{r}}(s) \tag{5.23}$$

where $\bar{\mathbf{z}}(s)$ is the Laplace transform of $\mathbf{z}(t)$, $\mathbf{z_0}$ is the vector of initial conditions in the state-space and $\bar{\mathbf{r}}(s)$ is the Laplace transform of $\mathbf{r}(t)$. From the expressions of $\mathbf{z}(t)$ and $\mathbf{r}(t)$ in equation (5.5) it is obvious that

$$\bar{\mathbf{z}}(s) = \left\{ \begin{array}{c} \bar{\mathbf{q}}(s) \\ s\bar{\mathbf{z}}(s) \end{array} \right\} \in \mathbb{C}^{2N}, \quad \bar{\mathbf{r}}(s) = \left\{ \begin{array}{c} \bar{\mathbf{f}}(s) \\ \mathbf{0} \end{array} \right\} \in \mathbb{C}^{2N} \quad \text{and} \quad \mathbf{z_0} = \left\{ \begin{array}{c} \mathbf{q_0} \\ \dot{\mathbf{q}}_\mathbf{0} \end{array} \right\} \in \mathbb{R}^{2N} \tag{5.24}$$

For distinct eigenvalues the mode shapes $\mathbf{z}_k$ form a complete set of vectors. Therefore, the solution of equation (5.23) can be expressed in terms of a linear combination of $\mathbf{z}_k$ as

$$\bar{\mathbf{z}}(s) = \sum_{k=1}^{2N} \beta_k(s)\mathbf{z}_k. \tag{5.25}$$

We only need to determine the constants $\beta_k(s)$ to obtain the complete solution.

Substituting $\mathbf{z}(s)$ from (5.25) into equation (5.23) we have

$$[s\mathbf{A} + \mathbf{B}] \sum_{k=1}^{2N} \beta_k(s)\mathbf{z}_k = \bar{\mathbf{r}}(s) + \mathbf{A}\mathbf{z_0}. \tag{5.26}$$

Premultiplying by $\mathbf{z}_j^T$ and using the orthogonality and normalization relationships (5.11)–(5.13), we have

$$\begin{aligned} \frac{1}{\gamma_j}(s - s_j)\,\beta_j(s) &= \mathbf{z}_j^T \left\{ \bar{\mathbf{r}}(s) + \mathbf{A}\mathbf{z_0} \right\} \\ \text{or} \quad \beta_j(s) &= \gamma_j \frac{\mathbf{z}_j^T \bar{\mathbf{r}}(s) + \mathbf{z}_j^T \mathbf{A}\mathbf{z_0}}{s - s_j}. \end{aligned} \tag{5.27}$$

Using the expressions of $\mathbf{A}$, $\mathbf{z}_j$ and $\bar{\mathbf{r}}(s)$ from equation (5.5), (5.7) and (5.24) the term $\mathbf{z}_j^T \bar{\mathbf{r}}(s) + \mathbf{z}_j^T \mathbf{A}\mathbf{z_0}$ can be simplified and $\beta_k(s)$ can be related with mode shapes of the second-order system as

$$\beta_j(s) = \gamma_j \frac{\mathbf{u}_j^T \left\{ \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{C}\mathbf{q_0} + s\mathbf{M}\mathbf{q_0} \right\}}{s - s_j}. \tag{5.28}$$

Since we are only interested in the displacement response, we only need to determine the first N rows of equation (5.25). Using the partition of $\mathbf{z}(s)$ and $\mathbf{z}_j$ we have

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{2N} \beta_j(s)\mathbf{u}_j. \tag{5.29}$$

Substituting $\beta_j(s)$ from equation (5.28) into the preceding equation one has

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{2N} \gamma_j \mathbf{u}_j \frac{\mathbf{u}_j^T \left\{ \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{C}\mathbf{q_0} + s\mathbf{M}\mathbf{q_0} \right\}}{s - s_j} \tag{5.30}$$

$$\text{or} \quad \bar{\mathbf{q}}(s) = \mathbf{H}(s) \left\{ \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{C}\mathbf{q_0} + s\mathbf{M}\mathbf{q_0} \right\} \tag{5.31}$$

where

$$\mathbf{H}(s) = \sum_{j=1}^{2N} \frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - s_j} \tag{5.32}$$

is the transfer function or receptance matrix. Recalling that the eigensolutions appear in complex conjugate pairs, using (5.9) equation (5.32) can be expanded as

$$\mathbf{H}(s) = \sum_{j=1}^{2N} \frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - s_j} = \sum_{j=1}^{N} \left[\frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - s_j} + \frac{\gamma_j^* \mathbf{u}_j^* \mathbf{u}_j^{*^T}}{s - s_j^*} \right]. \tag{5.33}$$

The receptance matrix is often expressed in terms of complex natural frequencies λ_j. Substituting $s_j = i\lambda_j$ in the preceding expression we have

$$\mathbf{H}(s) = \sum_{j=1}^{N} \left[\frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - i\lambda_j} + \frac{\gamma_j^* \mathbf{u}_j^* \mathbf{u}_j^{*^T}}{s + i\lambda_j^*} \right] \quad \text{and} \quad \gamma_j = \frac{1}{\mathbf{u}_j^T \left[2i\lambda_j \mathbf{M} + \mathbf{C}\right] \mathbf{u}_j}. \tag{5.34}$$

It can be shown that the receptance matrix $\mathbf{H}(s)$ in (5.34) reduces to its equivalent expression for the undamped case in (2.33). In the undamped limit $\mathbf{C} = 0$. This results $\lambda_j = \omega_j = \lambda_j^*$ and $\mathbf{u}_j = \mathbf{x}_j = \mathbf{u}_j^*$. In view of the mass normalization relationship we also have $\gamma_j = \frac{1}{2\mathrm{i}\omega_j}$. Consider a typical term in (5.34):

$$\begin{aligned}
&\left[\frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{s - i\lambda_j} + \frac{\gamma_j^* \mathbf{u}_j^* \mathbf{u}_j^{*^T}}{s + i\lambda_j^*} \right] = \left[\frac{1}{2i\omega_j} \frac{1}{\mathrm{i}\omega - \mathrm{i}\omega_j} + \frac{1}{-2i\omega_j} \frac{1}{\mathrm{i}\omega + \mathrm{i}\omega_j} \right] \mathbf{x}_j \mathbf{x}_j^T \\
&= \frac{1}{2i^2\omega_j} \left[\frac{1}{\omega - \omega_j} - \frac{1}{\omega + \omega_j} \right] \mathbf{x}_j \mathbf{x}_j^T \\
&= -\frac{1}{2\omega_j} \left[\frac{\omega + \omega_j - \omega + \omega_j}{(\omega - \omega_j)(\omega + \omega_j)} \right] \mathbf{x}_j \mathbf{x}_j^T = -\frac{1}{2\omega_j} \left[\frac{2\omega_j}{\omega^2 - \omega_j^2} \right] \mathbf{x}_j \mathbf{x}_j^T = \frac{\mathbf{x}_j \mathbf{x}_j^T}{\omega_j^2 - \omega^2}.
\end{aligned} \tag{5.35}$$

Note that this term was derived before for the receptance matrix of the undamped system in equation (2.33). Therefore equation (5.31) is the most general expression of the dynamic response of damped linear dynamic systems.

Exercise: Verify that when the system is proportionally damped, equation (5.31) reduces to (4.21) as expected.

5.2.2 Time Domain Analysis

Combining equations (5.31) and (5.34) we have

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{N} \left\{ \gamma_j \frac{\mathbf{u}_j^T \bar{\mathbf{f}}(s) + \mathbf{u}_j^T \mathbf{M} \dot{\mathbf{q}}_0 + \mathbf{u}_j^T \mathbf{C} \mathbf{q}_0 + s\mathbf{u}_j^T \mathbf{M} \mathbf{q}_0}{s - i\lambda_j} \mathbf{u}_j + \gamma_j^* \frac{\mathbf{u}_j^{*^T} \bar{\mathbf{f}}(s) + \mathbf{u}_j^{*^T} \mathbf{M} \dot{\mathbf{q}}_0 + \mathbf{u}_j^{*^T} \mathbf{C} \mathbf{q}_0 + s\mathbf{u}_j^{*^T} \mathbf{M} \mathbf{q}_0}{s + i\lambda_j^*} \mathbf{u}_j^* \right\}. \tag{5.36}$$

From the table of Laplace transforms we know that

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s-a}\right] = ae^{at}, \quad t > 0. \tag{5.37}$$

Taking the inverse Laplace transform of (5.36), dynamic response in the time domain can be obtained as

$$\mathbf{q}(t) = \mathcal{L}^{-1}\left[\bar{\mathbf{q}}(s)\right] = \sum_{j=1}^{N} \gamma_j a_{1_j}(t)\mathbf{u}_j + \gamma_j^* a_{2_j}(t)\mathbf{u}_j^* \tag{5.38}$$

where

$$\begin{aligned} a_{1_j}(t) =& \mathcal{L}^{-1}\left[\frac{\mathbf{u}_j^T\bar{\mathbf{f}}(s)}{s-i\lambda_j}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-i\lambda_j}\right]\left(\mathbf{u}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{u}_j^T\mathbf{C}\mathbf{q}_\mathbf{0}\right) + \mathcal{L}^{-1}\left[\frac{s}{s-i\lambda_j}\right]\mathbf{u}_j^T\mathbf{M}\mathbf{q}_\mathbf{0} \\ =& \int_0^t e^{i\lambda_j(t-\tau)}\mathbf{u}_j^T\mathbf{f}(\tau)d\tau + e^{i\lambda_j t}\left(\mathbf{u}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{u}_j^T\mathbf{C}\mathbf{q}_\mathbf{0} + i\lambda_j\mathbf{u}_j^T\mathbf{M}\mathbf{q}_\mathbf{0}\right), \quad \text{for} \quad t > 0 \end{aligned} \tag{5.39}$$

and similarly

$$\begin{aligned} a_{2_j}(t) =& \mathcal{L}^{-1}\left[\frac{\mathbf{u}_j^{*T}\bar{\mathbf{f}}(s)}{s+i\lambda_j}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+i\lambda_j}\right]\left(\mathbf{u}_j^{*T}\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{u}_j^{*T}\mathbf{C}\mathbf{q}_\mathbf{0}\right) + \mathcal{L}^{-1}\left[\frac{s}{s+i\lambda_j}\right]\mathbf{u}_j{*}^T\mathbf{M}\mathbf{q}_\mathbf{0} \\ =& \int_0^t e^{-i\lambda_j(t-\tau)}\mathbf{u}_j^{*T}\mathbf{f}(\tau)d\tau + e^{-i\lambda_j t}\left(\mathbf{u}_j^{*T}\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{u}_j^{*T}\mathbf{C}\mathbf{q}_\mathbf{0} - i\lambda_j\mathbf{u}_j^{*T}\mathbf{M}\mathbf{q}_\mathbf{0}\right), \quad \text{for} \quad t > 0. \end{aligned} \tag{5.40}$$

Example 3: Figure 10 shows a three DOF spring-mass system. This system is identical to the one used in Example 2, except that the damper attached with the middle block is now disconnected.

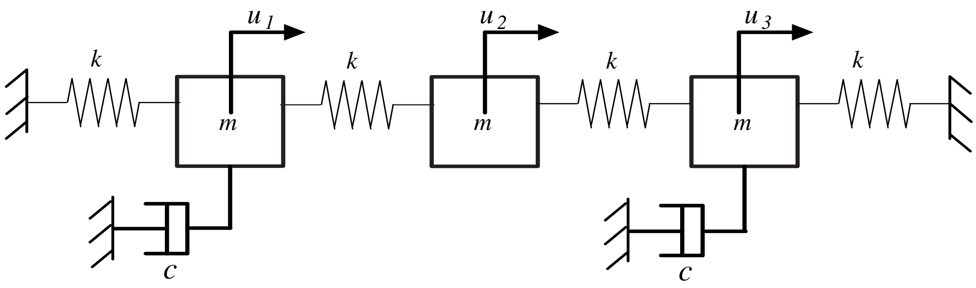

Figure 10: There DOF damped spring-mass system

1. Show that in general the system possesses complex modes.

2. Obtain approximate expressions for complex natural frequencies (using the first order perturbation method).

Solution: The mass and the stiffness matrices of the system are same as in Example 2 (given in equation (4.32)). The damping matrix is clearly given by

$$\mathbf{C} = \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}. \tag{5.41}$$

From this (and also from observation) it clear that the damping matrix is neither proportional to the mass matrix nor it is proportional to the stiffness matrix. Therefore, it is *likely* that the system will not have classical normal mode. It order to be sure we need to check if Caughey and O'Kelly's criteria, *i.e.*, $\mathbf{CM}^{-1}\mathbf{K} = \mathbf{KM}^{-1}\mathbf{C}$, is satisfied or not. Since $\mathbf{M} = m\mathbf{I}$ a diagonal matrix, $\mathbf{M}^{-1} = \frac{1}{m}\mathbf{I}$. Recall that for any matrix $\mathbf{A}$, $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$. Using the system matrices we have

$$\mathbf{CM}^{-1}\mathbf{K} = \mathbf{C}\frac{1}{m}\mathbf{IK} = \frac{1}{m}\mathbf{CK} = \frac{ck}{m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \frac{ck}{m}\begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \tag{5.42}$$

$$\text{and} \quad \mathbf{KM}^{-1}\mathbf{C} = \mathbf{K}\frac{1}{m}\mathbf{IC} = \frac{1}{m}\mathbf{KC} = \frac{ck}{m}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{ck}{m}\begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}. \tag{5.43}$$

It is clear that $\mathbf{C}\mathbf{M}^{-1}\mathbf{K} \neq \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$, that is, Caughey and O'Kelly's condition is not satisfied by the system matrices. This confirms that the system do not possess classical normal modes but has complex modes.

Exact complex modes of the system can be obtained using the state-space approach outlined before. For a three DOF system we need to solve an eigenvalue problem of the order six. This calculation is very difficult to do without computer. Here we will obtain approximate natural frequencies using the first-order perturbation method described before. Using the undamped modal matrix in equation (4.62), the damping matrix in the modal coordinated can be obtained as

$$\begin{aligned}
\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X} &= \frac{1}{2\sqrt{m}}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}^T \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix} \frac{1}{2\sqrt{m}} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \\
&= \frac{c}{4m}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \\
&= \frac{c}{4m}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ 0 & 0 & 0 \\ 1 & -\sqrt{2} & 1 \end{bmatrix} = \frac{c}{4m}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \frac{c}{m}\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}.
\end{aligned} \tag{5.44}$$

Notice that unlike Example 2, $\mathbf{C}'$ is not a diagonal matrix, *i.e.*, the equation of motion in the modal coordinates are coupled through the off-diagonal terms of the $\mathbf{C}'$ matrix. Approximate complex natural frequencies can be obtained from equation (5.20) as

$$\lambda_1 \approx \pm\omega_1 + iC'_{11}/2 = \pm\sqrt{\left(2-\sqrt{2}\right)\alpha} + i\frac{c}{4m}, \tag{5.45}$$

$$\lambda_2 \approx \pm\omega_2 + iC'_{22}/2 = \pm\sqrt{2\alpha} + i\frac{c}{2m} \tag{5.46}$$

$$\text{and} \quad \lambda_3 \approx \pm\omega_3 + iC'_{33}/2 = \pm\sqrt{\left(2+\sqrt{2}\right)\alpha} + i\frac{c}{4m}. \tag{5.47}$$

In the above equations, undamped eigenvalues obtained in Example 2 (equation (4.37)) have been used. The second complex mode is most heavily damped (as the imaginary part is twice compared to the other two modes). This is because in the second mode the middle mass is stationary (look at the second mode shape in (4.50)) while the two 'damped' masses move maximum distance away from it. This causes maximum 'stretch' to both the dampers and results maximum damping in this mode.

Exercise problem: Redo the previous example for the 3DOF system shown in Figure 11.

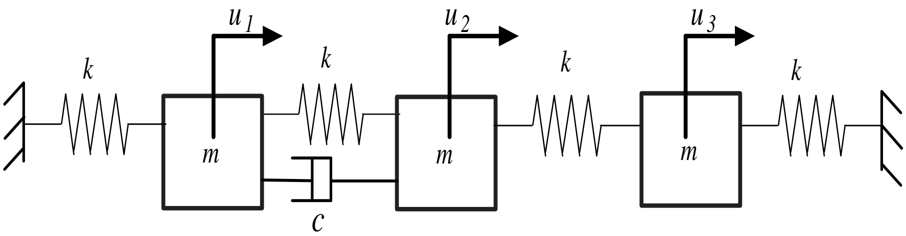

Figure 11: There DOF damped spring-mass system with dampers attached between mass 1 and 2

Hint: The damping is given by $\mathbf{C} = \begin{bmatrix} c & -c & 0 \\ -c & c & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Nomenclature

$a_j(t)$	Time dependent constants for dynamic response
$\mathbf{A}$	$2N \times 2N$ system matrix in the state-space
B_j	Constants for time domain dynamic response
$\mathbf{B}$	$2N \times 2N$ system matrix in the state-space
$\mathbf{C}$	Viscous damping matrix
$\mathbf{C}'$	Damping matrix in the modal coordinate
$\mathbf{f}(t)$	Forcing vector
$\tilde{\mathbf{f}}(t)$	Forcing vector in the modal coordinate
$\mathcal{F}$	Dissipation function
$\boldsymbol{\mathcal{G}}(t)$	Damping function in the time domain
$\mathbf{I}$	Identity matrix
i	Unit imaginary number, $i = \sqrt{-1}$
$\mathbf{K}$	Stiffness matrix
$\mathcal{L}(\bullet)$	Laplace transform of $(\bullet)$
$\mathcal{L}^{-1}(\bullet)$	Inverse Laplace transform of $(\bullet)$
$\mathbf{M}$	Mass matrix
N	Degrees-of-freedom of the system
$\mathbf{O}$	Null matrix
$\bar{\mathbf{p}}(s)$	Effective forcing vector in the Laplace domain
$\mathbf{q}(t)$	Vector of the generalized coordinates (displacements)
$\mathbf{q_0}$	Vector of initial displacements
$\dot{\mathbf{q}}_\mathbf{0}$	Vector of initial velocities
$\bar{\mathbf{q}}$	Laplace transform of $\mathbf{q}(t)$
Q_{nc_k}	Non-conservative forces
$\mathbf{r}(t)$	Forcing vector in the state-space
$\bar{\mathbf{r}}(s)$	Laplace transform of $\mathbf{r}(t)$
s	Laplace domain parameter
s_j	jth complex eigenvalue
t	Time
$\mathbf{u}_j$	Complex eigenvector in the original (N) space
$\mathbf{x}_j$	jth undamped eigenvector
$\mathbf{X}$	Matrix of the undamped eigenvectors
$\mathbf{y}(t)$	Modal coordinates
$\mathbf{z}(t)$	Response vector in the state-space
$\mathbf{z}_j$	$2N \times 1$ Complex eigenvector vector in the state-space
$\mathbf{z_0}$	Vector of initial conditions in the state-space

$\bar{\mathbf{z}}(s)$	Laplace transform of $\mathbf{z}(t)$
$\alpha_k^{(j)}$	Complex constants for jth complex mode
γ_j	j-th modal amplitude constant
$\delta(t)$	Dirac-delta function
δ_{jk}	Kroneker-delta function
ζ_j	j-th modal damping factor
$\boldsymbol{\zeta}$	Diagonal matrix containing ζ_j
θ_j	Constants for time domain dynamic response
λ_j	j-th complex natural frequency
τ	Dummy time variable
ω	frequency
ω_j	j-th undamped natural frequency
ω_{d_j}	j-th damped natural frequency
$\boldsymbol{\Omega}$	Diagonal matrix containing ω_j
DOF	Degrees of freedom
$\mathbb{C}$	Space of complex numbers
$\mathbb{R}$	Space of real numbers
$\mathbb{R}^{N\times N}$	Space of real $N \times N$ matrices
$\mathbb{R}^N$	Space of real N dimensional vectors
$\det(\bullet)$	Determinant of $(\bullet)$
diag	A diagonal matrix
$\in$	Belongs to
$\forall$	For all
$(\bullet)^T$	Matrix transpose of $(\bullet)$
$(\bullet)^{-1}$	Matrix inverse of $(\bullet)$
$(\dot{\bullet})$	Derivative of $(\bullet)$ with respect to t
$(\bullet)^*$	Complex conjugate of $(\bullet)$
$\lvert \bullet \rvert$	Absolute value of $(\bullet)$

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