DOMAINS OF DIRICHLET FORMS AND EFFECTIVE RESISTANCE ESTIMATES ON P.C.F. FRACTALS

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Abstract. In this paper we consider *post-critically finite* self-similar fractals with *regular harmonic structures*. We first obtain *effective resistance* estimates in terms of the Euclidean metric, which particularly imply the embedding theorem for the *domains* of the *Dirichlet forms* associated with the harmonic structures. We then characterize the domains of the Dirichlet forms.

1. Introduction

Let $(K, \{F_i\}_{i=1}^M)$ be a post-critically finite (p.c.f.) self-similar fractal in $\mathbb{R}^n (n \geq 1)$ with a regular harmonic structure $(H, \mathbf{r})$, and let $(\mathcal{E}, \mathcal{D})$ be the Dirichlet form associated with $(H, \mathbf{r})$. Let R be the effective resistance determined by the form $(\mathcal{E}, \mathcal{D})$. In this paper, we are concerned with the following problems:

(1) What is the relationship between R and the Euclidean metric?
(2) How to characterize the domain $\mathcal{D}$ of $\mathcal{E}$?

These two problems are important in studying the dynamical aspects of fractals, such as PDE's, Brownian motions, heat kernels, and function spaces on fractals.

Recall that the first problem above is obvious, if K is a bounded open interval of $\mathbb{R}$ and $\mathcal{E}$ is the classical energy form with respect to the Lebesgue measure

$$\mathcal{E}(f, g) = \frac{1}{2} \int_K \nabla f \cdot \nabla g dx. \tag{1.1}$$

As a matter of fact, there exists some $c > 0$ such that, for all $x, y \in K \subset \mathbb{R}$,

$$c^{-1}|x - y| \leq R(x, y) \leq c\,|x - y|. \tag{1.2}$$

The second inequality in (1.2) follows by using the definition of R, see (2.9) below, and the Sobolev embedding theorem:

$$|f(x) - f(y)| \leq c^{1/2}|x - y|^{1/2}\mathcal{E}(f, f)^{1/2}, \tag{1.3}$$

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see for example [1, Formula (9), p.98]. The first inequality in (1.2) also follows; this is because for any $x_0 < y_0$ in K, letting

$$f_0(x) = \begin{cases} 0, & \text{if } x \le x_0, \\ \frac{x-x_0}{y_0-x_0}, & \text{if } x_0 \le x \le y_0, \\ 1, & \text{if } x \ge y_0, \end{cases}$$

we obtain that

$$\mathcal{E}(f_0, f_0) = \frac{1}{2}\int_K |f_0'(x)|^2 dx = \frac{1}{2}|y_0 - x_0|^{-1},$$

and then use the definition (2.8) below (cf. [18, Sect 1.6]). Note that for the higher-dimensional case, there does not exist such an elegant estimate for R as in (1.2).

The second problem above is also clear for the classical case: if K is an open domain of $\mathbb{R}^n (n \ge 1)$ and $\mathcal{E}$ is as in (1.1), then the *domain* $\mathcal{D}$ of $\mathcal{E}$ is just $W^{1,2}(K)$, the usual *Sobolev space* on K.

However, for the fractal case, the above two problems are non-trivial. Recall that for the problem (1), if K is a *nested* fractal, there exists a *geodesic metric* d on K, and Barlow [2, Lemma 8.17] obtained the relationship between R and d as follows

$$R(x,y) \sim d(x,y)^{\theta},$$

where $\theta = \log\rho/\log\gamma$, and ρ, γ are the *resistance* and *shortest path* scaling factors, respectively. If K is a Sirepinski gasket in $\mathbb{R}^2$, Strichartz [18, Sect 1.6] obtained a relationship between R and the Euclidean metric

$$R(x,y) \sim |x-y|^{d_w - d_f}, \tag{1.4}$$

where $d_f = \frac{\ln 3}{\ln 2}$ and $d_w = \frac{\ln 5}{\ln 2}$ are, respectively, the *Hausdorff* and *walk dimensions* of the Sierpinski gasket. Under certain mild conditions, we shall obtain in this paper the relationship between R and the Euclidean metric for p.c.f. fractals with *regular* harmonic structures, see Section 3. (If the harmonic structure is not regular, then $R(x,y)$ may be infinite for some points x and y, but $|x-y| < \infty$ for any $x, y \in K$ since K is bounded. So R can not be controlled from above by the Euclidean metric. Therefore, the estimate (1.4) fails.) In particular, we show that (1.4) holds with different exponents for a certain class of nested fractals. (One may use the heat kernel estimates in [12] to derive (1.4) for some nested fractals. This is another story.)

As for the problem (2), the first result was obtained by Jonsson [9] for the *Sierpinski gasket* K in $\mathbb{R}^n$. It was shown that the domain $\mathcal{D}$ of the energy form $\mathcal{E}$ is equivalent to a *Sobolev-type* space on K, that is,

$$\mathcal{D} \simeq W^{\beta/2,2}(\mu) := \{f \in L^2(K,\mu) : W_{\beta/2,2}(f) < \infty\}, \tag{1.5}$$

where μ is the $\alpha := \frac{\log(n+1)}{\log 2}$-dimensional Hausdorff measure on K, and $\beta = \frac{\log(n+3)}{\log 2}$ is the walk dimension, and

$$W_{\beta/2,2}(f) = \sup_{0<r<1} r^{-(\alpha+\beta)} \int_K \int_{B(x,r)} |f(y) - f(x)|^2 d\mu(y) d\mu(x). \tag{1.6}$$

Here $B(x,r) = \{y \in K : |y-x| < r\}$ is a ball of radius r and center x in K under the Euclidean metric. (Note that Jonsson used a different notion $\mathrm{Lip}(\beta/2, 2, \infty)(K, \mu)$ to denote the space $W^{\beta/2,2}(\mu)$.) Pietruska-Pałuba generalized Jonsson's result to a certain class of nested fractals in $\mathbb{R}^n$, see [16].

On the other hand, one can characterize the domain $\mathcal{D}$ of the energy $\mathcal{E}$ with the help of *heat kernel* estimates. Assume that the heat kernel (or the transition density) $p(t,x,y)$ exists on K, and satisfies

$$t^{-\alpha/\beta}\Phi_1\left(t^{-1/\beta}|x-y|\right) \le p(t,x,y) \le t^{-\alpha/\beta}\Phi_2\left(t^{-1/\beta}|x-y|\right) \tag{1.7}$$

for all $x, y \in K$ and $0 < t < 1$, where $\alpha, \beta > 0$ and $\Phi_i \ge 0$ is continuous and decreasing on $[0,\infty)$ for $i = 1, 2$. Under certain mild assumptions on Φ_1 and Φ_2, one can obtain that the domain $\mathcal{D}$ of the Dirichlet form $(\mathcal{E}, \mathcal{D})$ associated with the heat kernel $p(t,x,y)$ is equivalent to $W^{\beta/2,2}(\mu)$, where μ is the α-measure on K (that is, $\mu(B(x,r)) \sim r^\alpha$), see [17] for the Euclidean case and [4] for metric spaces. However, it is much complicated to obtain heat kernel estimates like (1.7), see [7, 13] for p.c.f fractals with regular harmonic structures.

In Section 4, we characterize the domain $\mathcal{D}$ of the energy $\mathcal{E}$ on p.c.f. fractals with regular harmonic structures. We avoid using the heat kernel estimates, and present a direct proof. We do follow the technique in [9], but there are some new twists in our proof. The effective metric R and the self-similar measure μ with the *standard* weights will be used in defining the functions spaces $W^{\beta/2,2}(\mu)$. We mention in passing here that a closely related problem was studies in [11], where the domain of the Laplacian on p.c.f self-similar sets was characterized.

Notation. The constants in this paper sometimes change from line to line while they are all denoted by a single letter c. The integers M, M_i and constants c_i are fixed for $i \ge 0$. For two non-negative functions f, g, by $f \sim g$ we mean that there is some $c > 0$ such that $c^{-1} f \le g \le c\, f$.

2. Preliminaries

2.1. **p.c.f. fractals and Dirichlet forms.**

We first recall the concept of p.c.f. fractals introduced by Kigami [10].

Let $M \ge 2$ be an integer, and set $S = \{1, 2, \cdots, M\}$. Let $W_* = \bigcup_{m\ge 0} S^m$ be the collection of all *finite* words. Let (X, d) be a complete metric space, and let $\{F_i\}_{i=1}^M$ be a family of strict contractions on (X, d). Then there exists a unique non-empty compact subset K of X such that

$$K = \bigcup_{i=1}^{M} F_i(K), \tag{2.1}$$

see [8] or [3]. For any word $w = i_1 i_2 \cdots i_m \in S^m$, any sequence of positive numbers $\{p_i\}_{i=1}^M$, and any function $f : K \to \mathbb{R}$, denote by $|w| = m$ the length of w, and set

$$F_w = F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_m}, \quad K_w = F_w(K),$$
$$p_w = p_{i_1} p_{i_2} \cdots p_{i_m}, \quad f_w = f \circ F_w.$$

For the empty word w, set $p_w = 1$ and $F_w = id$. Denote by $B(x_0, r) = \{y \in K : |y - x_0| < r\}$ for $x_0 \in K$ and $r > 0$.

Define a continuous surjection $\pi : S^{\mathbb{N}} \to K$ by

$$\{\pi(w)\} = \bigcap_{m \geq 1} F_{i_1 \cdots i_m}(K)$$

for any infinite word $w = i_1 i_2 \cdots i_m \cdots \in S^{\mathbb{N}}$. Let

$$\mathcal{C} = \bigcup_{i \neq j} (K_i \cap K_j),$$
$$\Gamma = \pi^{-1}(\mathcal{C}), \quad \mathcal{P} = \bigcup_{n \geq 1} \sigma^n(\Gamma),$$

where $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}$ is the shift map defined by

$$\sigma(i_1 i_2 i_3 \cdots) = i_2 i_3 \cdots .$$

If $\mathcal{P}$ is finite, the triple $(K, S, \{F_i\}_{i \in S})$ is termed a *post-critically finite* self-similar set, see [10, Definition 1.3.13, p.23]. Let

$$V_0 = \pi(\mathcal{P}), \quad V_m = \bigcup_{w \in S^m} F_w(V_0) \quad (m \geq 1), \quad V_* = \bigcup_{m \geq 0} V_m. \tag{2.2}$$

If $(K, \{F_i\}_{i=1}^M)$ is a p.c.f. fractal, then $V_m \subset V_{m+1}$ $(m \geq 0)$. From now on we assume that $(K, \{F_i\}_{i=1}^M)$, or simply K, is a p.c.f. fractal.

We now recall how to construct a Dirichlet form on a p.c.f. fractal K. Let V_0 be as in (2.2), and $\ell(V_0) = \{f | f : V_0 \to \mathbb{R}\}$ be the collection of all real functions on V_0. Let H be a *Laplace* matrix, or simply a Laplace, on V_0, that is H satisfies that, for any $f, g \in \ell(V_0)$,

- $H_{pq} = H_{qp} \geq 0$ for any $p \neq q \in V_0$, where H_{pq} are the entries of H;
- $(f, Hg) := \sum_{p \in V_0} f(p) \Big(\sum_{q \in V_0} H_{pq} g(q) \Big) \leq 0$;
- $Hf = 0$ if and only if f is constant.

Given a Laplace H on V_0, and a family of positive numbers $\mathbf{r} = \{r_i\}_{i=1}^M$, we define an *energy form* $\mathcal{E}_m$ on V_m for $m \geq 0$ by

$$\begin{aligned} &\mathcal{E}_0(f,g) = -(f, Hg), \\ &\mathcal{E}_m(f,g) = \sum_{w \in S^m} r_w^{-1} \mathcal{E}_0(f_w, g_w) \quad (m \geq 1), \end{aligned} \tag{2.3}$$

for $f, g : V_m \to \mathbb{R}$. In the sequel, we write $\mathcal{E}_m(f) := \mathcal{E}_m(f, f)$ for simplicity. If there exists a pair $(H, \mathbf{r})$ such that the following *variational problem*

$$\min\{\mathcal{E}_1(g) : g|_{V_0} = f\} = \mathcal{E}_0(f) \tag{2.4}$$

is solvable for any $f \in \ell(V_0)$, then we say that K possesses a *harmonic structure.* If in addition $r_i < 1$ for all $1 \leq i \leq M$, then the harmonic structure is said to be *regular*, see [10, Definition. 3.1.2, p.69].

Form now on we assume that K possesses a regular harmonic structure $\left(H, \{r_i\}_{i=1}^M\right)$.

Note that (2.4) implies that the sequence $\{\mathcal{E}_m(f)\}_{m \geq 0}$ is non-decreasing in m for any $f : V_* \to \mathbb{R}$. Let

$$\mathcal{E}(f) := \lim_{m \to \infty} \mathcal{E}_m(f), \tag{2.5}$$

$$\mathcal{D} = \{f \in C(K) : \mathcal{E}(f) < \infty\}, \tag{2.6}$$

where $C(K)$ is the space of all continuous functions on K. It is known that $(\mathcal{E}, \mathcal{D})$ defined as in (2.5) and (2.6) is a *local, regular, irreducible* Dirichlet form on $L^2(K, \mu)$ for any Borel measure μ which charges every set of the form K_w for $w \in S^m$, see [2, Theorem 7.14, p.99] or [10, Theorem 3.4.6, p.92]. Clearly $\mathcal{E}$ is *self-similar*: for any $f \in \mathcal{D}$, we have that $f \circ F_i \in \mathcal{D}$ for each i, and

$$\mathcal{E}(f) = \sum_{i \in S} r_i^{-1} \mathcal{E}(f \circ F_i). \tag{2.7}$$

We call $\{r_i\}_{i=1}^M$ the *weights* of the energy $\mathcal{E}$.

In order to characterize the domain $\mathcal{D}$ of the form $(\mathcal{E}, \mathcal{D})$, we need the *effective resistance* R on K. Let $R : K \times K \to [0, \infty]$ be defined by $R(x, x) = 0$ for $x \in K$, and

$$R(x,y)^{-1} = \inf\{\mathcal{E}(f) : f(x) = 0, f(y) = 1\} \tag{2.8}$$

for any $x \neq y \in K$. Note that (2.8) is equivalent to

$$R(x,y) = \sup\left\{\frac{|f(x) - f(y)|^2}{\mathcal{E}(f)} \,:\, \mathcal{E}(f) > 0\right\}. \tag{2.9}$$

for $x \neq y \in K$. It turns out that R is a *metric* on K, and the topology induced by R is equal to the original topology on K, see [10, Theorem 3.3.4, p.85] or [2, Proposition 7.18, p.101].

2.2. **Partition.** The partition will be useful in our analysis (the idea of partition on p.c.f. goes back to Hambly [6]). Let $\mathbf{a} = \{a_i\}_{i=1}^M$ be a family of numbers with $0 < a_i < 1$ for each i. For $0 < \lambda < 1$, define

$$\Lambda_{\mathbf{a}}(\lambda) = \{w = i_1 i_2 \cdots i_m : a_w \le \lambda < a_{i_1} a_{i_2} \cdots a_{i_{m-1}}\} \tag{2.10}$$

with the convention that $a_\emptyset = 1$. We call $\Lambda_{\mathbf{a}}(\lambda)$ the *partition* with respect to $\mathbf{a}$ and λ. Note that the set $\Lambda_{\mathbf{a}}(\lambda)$ is *finite*; this is easily seen since $0 < \lambda < 1$ and $0 < a_i < 1$ for each i. For simplicity, denote by

$$\tau \sim w \quad \text{for } \tau, w \in \Lambda_{\mathbf{r}}(\lambda)$$

if $K_\tau \cap K_w \ne \emptyset$. For $x, y \in K$, we write $x \sim y$, if $x, y \in K_w$ for some $w \in \Lambda_{\mathbf{a}}(\lambda)$. Clearly, by (2.1) and (2.7), we see that, for any partition $\Lambda_{\mathbf{a}}(\lambda)$,

$$K = \bigcup_{w \in \Lambda_{\mathbf{a}}(\lambda)} K_w,$$

$$\mathcal{E}(f) = \sum_{w \in \Lambda_{\mathbf{a}}(\lambda)} r_w^{-1} \mathcal{E}(f_w) \quad (f \in \mathcal{D}). \tag{2.11}$$

For $f : V_* \to \mathbb{R}$ and $0 < \lambda < 1$, define

$$\mathcal{E}_\lambda(f) := \sum_{w \in \Lambda_{\mathbf{r}}(\lambda)} r_w^{-1} \mathcal{E}_0(f_w). \tag{2.12}$$

Proposition 2.1. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal with a regular harmonic structure $\big(H, \{r_i\}_{i=1}^M\big)$. Then $\{\mathcal{E}_\lambda(f)\}$ is increasing as $\lambda \searrow 0$ for any f, and*

$$\lim_{\lambda \to 0} \mathcal{E}_\lambda(f) = \mathcal{E}(f), \quad f \in \mathcal{D}. \tag{2.13}$$

Proof. Let $0 < \lambda_1 < \lambda_2 < 1$. Then we have two partitions $\Lambda_{\mathbf{r}}(\lambda_1)$ and $\Lambda_{\mathbf{r}}(\lambda_2)$. And $\Lambda_{\mathbf{r}}(\lambda_2)$ is a "father" of $\Lambda_{\mathbf{r}}(\lambda_1)$, that is, any word $w \in \Lambda_{\mathbf{r}}(\lambda_1)$ can be written as $w = \tau w'$ with $\tau \in \Lambda_{\mathbf{r}}(\lambda_2)$; $w' \in W_*$ being possibly an empty word. Indeed, let

$$w = i_1 i_2 \cdots i_m \in \Lambda_{\mathbf{r}}(\lambda_1) \backslash \Lambda_{\mathbf{r}}(\lambda_2) \quad (m \ge 1).$$

Then we have $\lambda_2 \ge r_{i_1} \cdots r_{i_{m-1}}$; otherwise we would see that

$$r_{i_1} \cdots r_{i_m} \le \lambda_1 < \lambda_2 < r_{i_1} \cdots r_{i_{m-1}},$$

and so $w = i_1 i_2 \cdots i_m \in \Lambda_{\mathbf{r}}(\lambda_2)$ by the definition, a contradiction. Let $1 \le k \le m-1$ be an integer such that

$$r_{i_1} \cdots r_{i_k} \le \lambda_2 < r_{i_1} \cdots r_{i_{k-1}}.$$

This implies $\tau := i_1 i_2 \cdots i_k \in \Lambda_{\mathbf{r}}(\lambda_2)$. Setting $w' := i_{k+1} \cdots i_m$, we see that $w = \tau w'$ with $\tau \in \Lambda_{\mathbf{r}}(\lambda_2)$. This shows that $\Lambda_{\mathbf{r}}(\lambda_2)$ is a father of $\Lambda_{\mathbf{r}}(\lambda_1)$. Therefore, for $f \in \mathcal{D}$,

$$\mathcal{E}_{\lambda_1}(f) = \sum_{w \in \Lambda_{\mathbf{r}}(\lambda_1)} r_w^{-1} \mathcal{E}_0(f_w) \ge \sum_{\tau \in \Lambda_{\mathbf{r}}(\lambda_2)} r_\tau^{-1} \mathcal{E}_0(f_\tau) = \mathcal{E}_{\lambda_2}(f),$$

proving that $\{\mathcal{E}_\lambda(f)\}$ is decreasing in λ for any f. Here, the above inequality follows from both the harmonic structure and the post-critical finiteness of K.

Finally, for $0 < \lambda < 1$, letting

$$m_1 = m_1(\mathbf{r}, \lambda) = \min\{|w| : w \in \Lambda_{\mathbf{r}}(\lambda)\},$$
$$m_2 = m_2(\mathbf{r}, \lambda) = \max\{|w| : w \in \Lambda_{\mathbf{r}}(\lambda)\},$$

we see that S^{m_1} is a father of $\Lambda_{\mathbf{r}}(\lambda)$, and $\Lambda_{\mathbf{r}}(\lambda)$ a father of S^{m_2}. Hence,

$$\mathcal{E}_{m_1}(f) \le \mathcal{E}_\lambda(f) \le \mathcal{E}_{m_2}(f) \quad (f \in \mathcal{D}).$$

Thus

$$\mathcal{E}(f) = \lim_{m_1\to\infty} \mathcal{E}_{m_1}(f) \le \lim_{\lambda\to 0} \mathcal{E}_\lambda(f) \le \lim_{m_2\to\infty} \mathcal{E}_{m_2}(f) = \mathcal{E}(f).$$

This finishes the proof. □

3. Effective resistance estimates

In this section we will give two-sided estimates of the effective resistance R in terms of the Euclidean metric. Two exponents appearing in the two-sided estimates of R are calculated for some nested fractals and some non-nested fractals.

Theorem 3.1. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal in $\mathbb{R}^n$ with a regular harmonic structure $\big(H, \{r_i\}_{i=1}^M\big)$. Assume that $s_i < 1$ is the contraction ratio of F_i, that is*

$$|F_i(x) - F_i(y)| \le s_i|x - y| \quad \text{for } x, y \in \mathbb{R}^n.$$

Then there exists some $c > 0$ such that, for all $x, y \in K$,

$$c^{-1}|x-y|^{\alpha_1} \le R(x,y) \tag{3.1}$$

where $\alpha_1 = \max_{1\le i\le M}\left\{\frac{\ln r_i}{\ln s_i}\right\}$.

Proof. Let $x_0 \ne y_0 \in K$. Without loss of generality, assume that

$$R(x_0, y_0) < (2c_1)^{-1}$$

where $c_1 > 0$ will be determined below. Set

$$\lambda = 2c_1 R(x_0, y_0) < 1.$$

Then $\Lambda_{\mathbf{r}}(\lambda)$ is a partition. There exist two words $w_1, w_2 \in \Lambda_{\mathbf{r}}(\lambda)$ such that $x_0 \in K_{w_1}$ and $y_0 \in K_{w_2}$. We claim that

$$K_{w_1} \cap K_{w_2} \ne \emptyset.$$

Otherwise there would exist a $\Lambda_{\boldsymbol{r}}(\lambda)$-*harmonic* function f satisfying

$$f|_{V_{w_1}} = 1 \quad \text{and} \quad f|_{V_\lambda \setminus V_{w_1}} = 0, \tag{3.2}$$

where $V_w := F_w(V_0)$ for $w \in W_*$, and $V_\lambda = \cup_{w\in\Lambda_{\mathbf{r}}(\lambda)} F_w(V_0)$. (We say that a function f on K is $\Lambda_{\boldsymbol{r}}(\lambda)$-*harmonic* if

$$\mathcal{E}(f, \varphi) = 0$$

for any $\varphi \in \mathcal{D}$ with $\varphi|_{V_\lambda} = 0$.)

Note that $f(x_0) = 1$ and $f(y_0) = 0$. Since f is $\Lambda_{\mathbf{r}}(\lambda)$-harmonic, we have that

$$\begin{aligned}\mathcal{E}(f) = \mathcal{E}_\lambda(f) &= \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} r_w^{-1}\mathcal{E}_0(f_w)\\ &= \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} r_w^{-1}\Big(\frac{1}{2}\sum_{p,q\in V_0} H_{pq}(f(F_w(p)) - f(F_w(q)))^2\Big). \end{aligned}\tag{3.3}$$

Using (3.2), we see that the right-hand side of (3.3) is actually equal to the summation of the terms

$$r_w^{-1}H_{pq}\left(f(F_w(p)) - f(F_w(q))\right)^2 = r_w^{-1}H_{pq},$$

with $w(\neq w_1)$ being taken over all the words in $\Lambda_{\mathbf{r}}(\lambda)$ with $w \sim w_1$, and p and q being taken over V_0 such that $F_w(p) \in V_{w_1}$ (so that $f(F_w(p)) = 1$ and $f(F_w(q)) = 0$); all the other terms are equal to zero. Hence, noting that $r_w \geq \lambda\, r_{min}$, the right-hand side of (3.3) is bounded by

$$H_{max}M_0(M_0-1)r_w^{-1} \leq H_{max}\big(r_{min}\big)^{-1}M_0(M_0-1)\lambda^{-1} := c_1\lambda^{-1},$$

where $H_{max} = \max_{p\neq q\in V_0}\{H_{pq}\}, M_0 = \#(V_0)$ and $r_{min} = \min\{r_i\}$. Therefore, by (2.8), it follows that

$$R(x_0,y_0)^{-1} \leq c_1\lambda^{-1},$$

and so

$$R(x_0,y_0) \geq c_1^{-1}\lambda = 2R(x_0,y_0),$$

yielding a contradiction. So the claim holds.

Now let $z_0 \in K_{w_1}\cap K_{w_2}$. Since $x_0, z_0 \in K_{w_1}$, we see that, writing $x_0 = F_{w_1}(x_0')$ and $z_0 = F_{w_1}(z_0')$ for some $x_0', z_0' \in K$,

$$\begin{aligned}|x_0 - z_0| = |F_{w_1}(x_0') - F_{w_1}(z_0')| &\leq s_{w_1}\mathrm{diam}(K)\\ &\leq (r_{w_1})^{1/\alpha_1}\,\mathrm{diam}(K) \leq \lambda^{1/\alpha_1}\mathrm{diam}(K).\end{aligned}$$

Similarly, noting that $y_0, z_0 \in K_{w_2}$, we see that

$$|y_0 - z_0| \leq \lambda^{1/\alpha_1}\mathrm{diam}(K).$$

Therefore,

$$|x_0 - y_0| \leq |x_0 - z_0| + |z_0 - y_0| \leq 2\lambda^{1/\alpha_1}\mathrm{diam}(K) = c\,R(x_0,y_0)^{1/\alpha_1},$$

giving that $R(x_0,y_0) \geq c^{-1}|x_0-y_0|^{\alpha_1}$. □

In order to bound R from above, we need the following *separation property*:

(C1): There exist a family of numbers $\mathbf{b} = \{b_i\}_{i=1}^M$ with $0 < b_i < 1$ for every i, and a constant $c_2 > 0$ such that, for any $0 < \lambda < 1$,

$$\mathrm{dist}(K_w, K_\tau) \geq c_2\lambda,$$

if $K_w \cap K_\tau = \emptyset$ for $w, \tau \in \Lambda_{\mathbf{b}}(\lambda)$.

We remark here that c_2 is independent of λ, but may depend on $\{b_i\}_{i=1}^M$. Condition (C1) says that any two disjoint components obtained from any partition $\Lambda_{\mathbf{b}}(\lambda)$ with $0 < \lambda < 1$ are apart away by distance $c_2\lambda$.

Theorem 3.2. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal in $\mathbb{R}^n$ $(n \geq 1)$ with a regular harmonic structure $\big(H, \{r_i\}_{1=1}^M\big)$. Assume that condition (C1) holds for some $\boldsymbol{b} = \{b_i\}_{i=1}^M$. Then*

$$R(x,y) \leq c\,|x-y|^{\alpha_2} \tag{3.4}$$

for all $x, y \in K$, where $\alpha_2 = \min_{1\leq i\leq M}\left\{\frac{\ln r_i}{\ln b_i}\right\}$ and $c > 0$.

Proof. First note that, for all $x, y \in K$,

$$R(x,y) \leq c < \infty,$$

since the harmonic structure is regular, see [10, Theorem 3.3.4, p. 85]. This implies that

$$|f(x) - f(y)|^2 \leq R(x,y)\,\mathcal{E}(f) \leq c\,\mathcal{E}(f)$$

for any $f \in \mathcal{D}$. In particular, for $x, y \in K_w$ $(w \in W_*)$, writing $x = F_w(x')$ and $y = F_w(y')$ for some $x', y' \in K$, we have that

$$|f(x) - f(y)|^2 = |f_w(x') - f_w(y')|^2 \leq c\,\mathcal{E}(f_w). \tag{3.5}$$

Now let $x_0 \neq y_0 \in K$. Without loss of generality, we assume that

$$|x_0 - y_0| < \frac{c_2}{2},$$

where c_2 is the same as in condition (C1). Let

$$\lambda = \frac{2}{c_2}|x_0 - y_0| < 1.$$

There are two words $w_1, w_2 \in \Lambda_{\mathbf{b}}(\lambda)$ such that $x_0 \in K_{w_1}$ and $y_0 \in K_{w_2}$. Then $K_{w_1} \cap K_{w_2} \neq \emptyset$; otherwise, we would have from condition (C1) that

$$|x_0 - y_0| \geq \operatorname{dist}(K_{w_1}, K_{w_2}) \geq c_2\lambda = 2|x_0 - y_0|,$$

a contradiction. Let $z_0 \in K_{w_1} \cap K_{w_2}$. For $f \in \mathcal{D}$, using the fact that $x_0, z_0 \in K_{w_1}$, we see from (3.5) and (2.11) that

$$\begin{aligned} |f(x_0) - f(z_0)|^2 &\leq c\,\mathcal{E}(f_{w_1}) = c\,r_{w_1}\,(r_{w_1})^{-1}\,\mathcal{E}(f_{w_1}) \\ &\leq c\,r_{w_1}\,\mathcal{E}(f) \leq c\,(b_w)^{\alpha_2}\,\mathcal{E}(f) \leq c\,\lambda^{\alpha_2}\,\mathcal{E}(f). \end{aligned} \tag{3.6}$$

Similarly, since $z_0, y_0 \in K_{w_2}$, we have that

$$|f(z_0) - f(y_0)|^2 \leq c\,\lambda^{\alpha_2}\mathcal{E}(f).$$

Therefore,

$$\begin{aligned}|f(x_0)-f(y_0)|^2 &\le 2\left(|f(x_0)-f(z_0)|^2+|f(z_0)-f(y_0)|^2\right)\\ &\le c\,\lambda^{\alpha_2}\,\mathcal{E}(f)=c\,|x_0-y_0|^{\alpha_2}\,\mathcal{E}(f),\end{aligned}$$

which gives that

$$R(x_0,y_0)\le c|x_0-y_0|^{\alpha_2}.$$

Thus (3.4) follows. □

Condition (C1) may be replaced by the following *connectivity property*:

(C2): There exist $\widehat{\mathbf{b}}=\left\{\widehat{b}_i\right\}_{i=1}^M$ with $0<\widehat{b}_i<1$ for each i, and a (small) constant $c_3>0$ and an integer M_1 such that, for any $0<\lambda<1$ and for any $x_0\in K$, any point $y\in B(x_0,c_3\lambda)$ can be connected to x_0 by a sequence of points $\{x_k\}_{k=0}^{n_0}$ in K with $1\le n_0\le M_1$, where $x_{n_0}=y$ and $x_{k-1}\sim x_k$ for $1\le k\le n_0$.

For a partition $\Lambda_{\widehat{\mathbf{b}}}(\lambda)$ with $0<\lambda<1$, the condition (C2) means that any point y in any ball $B\left(x_0,c_3\lambda\right)$ can be connected to its center x_0 by at most M_1 components obtained from the partition $\Lambda_{\widehat{\mathbf{b}}}(\lambda)$.

Theorem 3.3. *Let $(K,\{F_i\}_{i=1}^M)$ be a p.c.f. fractal in $\mathbb{R}^n\,(n\ge 1)$ with a regular harmonic structure $\left(H,\{r_i\}_{1=1}^M\right)$. Assume that condition (C2) holds for some $\widehat{\boldsymbol{b}}=\left\{\widehat{b}_i\right\}_{i=1}^M$. Then, for all $x,y\in K$,*

$$R(x,y)\le c\,|x-y|^{\alpha_3}, \tag{3.7}$$

where $\alpha_3=\min_{1\le i\le M}\left\{\frac{\ln r_i}{\ln\widehat{b}_i}\right\}$ and $c>0$.

Proof. Let $x_0\neq y_0\in K$. Without loss of generality, we assume that $|x_0-y_0|<\frac{c_3}{2}$ where c_3 is the same as in (C2). Set $\lambda:=2c_3^{-1}|x_0-y_0|$. Let $\Lambda_{\widehat{\mathbf{b}}}(\lambda)$ be the partition with respect to $\widehat{b}$ and λ. Note that $y_0\in B(x_0,c_3\lambda)$. Then, by condition (C2), there exists a sequence of points $\{x_k\}_{k=0}^{n_0}$ with $1\le n_0\le M_1$ satisfying that $x_{n_0}=y$, and

$$x_{k-1},x_k\in K_{w_k}\quad\text{for some } w_k\in\Lambda_{\widehat{\mathbf{b}}}(\lambda)\quad(k=1,\cdots,n_0).$$

For $f\in\mathcal{D}$, as in (3.6), we have that

$$|f(x_k)-f(x_{k-1})|^2\le c\,r_{w_k}\mathcal{E}(f)\le c\left(\widehat{b}_{w_k}\right)^{\alpha_3}\mathcal{E}(f)\le c\lambda^{\alpha_3}\mathcal{E}(f),\quad k=1,\cdots,n_0.$$

Therefore,

$$\begin{aligned}|f(x_0)-f(y_0)|^2 &=\left(\sum_{k=1}^{n_0}(f(x_k)-f(x_{k-1}))\right)^2\le n_0\sum_{k=1}^{n_0}(f(x_k)-f(x_{k-1}))^2\\ &\le c\,M_1^2\lambda^{\alpha_3}\mathcal{E}(f)=c\,|x_0-y_0|^{\alpha_3}\mathcal{E}(f),\end{aligned}$$

which implies that

$$R(x_0, y_0) \le c|x_0 - y_0|^{\alpha_3}.$$

Thus (3.7) follows. □

We remark that Theorem 3.2 or 3.3 implies the *Morrey-Sobolev embedding* of the function space $\mathcal{D}$:

$$|f(x) - f(y)| \le c\,|x - y|^{\beta}\sqrt{\mathcal{E}(f)} \tag{3.8}$$

for all $x, y \in K$ and all $f \in \mathcal{D}$, for some $c, \beta > 0$.

We now give some examples of p.c.f. fractals where condition (C1) or (C2) holds so that Theorem 3.2 or 3.3 is true.

- *Nested fractals.* The nested fractal $\left(K, \{F_i\}_{i=1}^M\right)$ was introduced by Lindstrøm [14]. It belongs to the class of p.c.f. fractals in $\mathbb{R}^n$ with the same contraction ratio $0 < \rho < 1$, that is

$$|F_i(x) - F_i(y)| = \rho|x - y| \quad (x, y \in \mathbb{R}^n).$$

It is known that K possesses a regular harmonic structure $\left(H, \{r_i\}_{i=1}^M\right)$ with $r_i := r < 1$ for $1 \le i \le M$, see for example [15]. And condition (C2) holds for $\widehat{b_i} = \rho$ for a certain class of nested fractals, see [12, Lemma 5.4]. Thus, by Theorems 3.1 and 3.3, we see that there exists some $c > 0$ such that, for all $x, y \in K$,

$$c^{-1}|x - y|^{\frac{\ln r}{\ln \rho}} \le R(x, y) \le c\,|x - y|^{\frac{\ln r}{\ln \rho}}.$$

As a typical representative of nested fractals, the Sierpinski gasket K in $\mathbb{R}^n$ admits an effective resistance R satisfying

$$c^{-1}|x - y|^{\frac{\ln\left(\frac{n+3}{n+1}\right)}{\ln 2}} \le R(x, y) \le c\,|x - y|^{\frac{\ln\left(\frac{n+3}{n+1}\right)}{\ln 2}},$$

by taking $r_i = \frac{n+1}{n+3}$ in constructing the Dirichlet form. Note that the exponent

$$\frac{\ln\left(\frac{n+3}{n+1}\right)}{\ln 2} = \frac{\ln(n+3)}{\ln 2} - \frac{\ln(n+1)}{\ln 2} := d_w - d_f,$$

is the difference between the walk dimension d_w and Hausdorff dimension d_f of the Sierpinski gasket, see also [18] for $n = 2$.

- *Vicsek sets.* Let

$$p_1 = (0, 0)\,,\ p_2 = (1, 0)\,,\ p_3 = (1, 1)\,,\ p_4 = (0, 1)\,,\ p_5 = \left(\frac{1}{2}, \frac{1}{2}\right)$$

be the four corners and center of the unit square in the plane. Define

$$F_i(x) = \frac{1}{4}(x - p_i) + p_i\ (1 \le i \le 4), \quad F_5 = \frac{1}{2}(x - p_5) + p_5 \quad (x \in \mathbb{R}^2).$$

The *Vicsek set* K is determined by $K = \bigcup_{i=1}^5 F_i(K)$. It is a p.c.f. fractal but not a nested fractal, and the boundary is $V_0 = \{p_1, p_2, p_3, p_4\}$.

Let $\mathbf{b} = \{b_i\}_{i=1}^5$ where $b_i = \frac{1}{4}$ each i. Then the Vicsek set satisfies the condition (C1). Indeed, for $0 < \lambda < 1$, let $m \geq 1$ be an integer such that $4^{-m} \leq \lambda < 4^{-(m-1)}$. Then $\Lambda_{\mathbf{b}}(\lambda) = S^m$, and

$$\mathrm{dist}(K_w, K_\tau) \geq \frac{1}{2}\, 4^{-(m-1)} > \frac{1}{2}\,\lambda$$

if $K_w \cap K_\tau = \emptyset$ for $w, \tau \in S^m$. It is not hard to construct a regular harmonic structure $(H, \{r_i\}_{i=1}^5)$ on the Vicsek set. In fact, let

$$H = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix},$$

and $r_1 = r_2 = r_3 = r_4 = \frac{1}{2}(1-r)$ and $r_5 = r$ with $0 < r < 1$. One can verify that $(H, \{r_i\}_{i=1}^5)$ is a regular harmonic structure on K for any $0 < r < 1$. Thus we see from Theorems 3.1 and 3.2 that, for all $x, y \in K$,

$$c^{-1}|x-y|^{\alpha_1} \leq R(x,y) \leq c\,|x-y|^{\alpha_2}, \tag{3.9}$$

where $\alpha_1 = \max\left\{-\frac{\ln(\frac{1}{2}(1-r))}{\ln 4}, -\frac{\ln r}{\ln 2}\right\}$ and $\alpha_2 = \min\left\{-\frac{\ln(\frac{1}{2}(1-r))}{\ln 4}, -\frac{\ln r}{\ln 4}\right\}$.

Note that the effective resistance R here can not be controlled by any powered Euclidean metric, that is, the following

$$R(x,y) \sim |x-y|^\theta \quad (\forall\, x, y \in K) \tag{3.10}$$

fails for any $\theta > 0$. In fact, let $m \geq 1$ be any integer and set $w_1 = 11\cdots 1 \in S^m$. Choose a family of points $\{(x_m, y_m)\}_{m\geq 1}$ in K, where

$$x_m = (0,0) = F_{w_1}(p_1) \quad \text{and} \quad y_m = \big(4^{-m}, 0\big) = F_{w_1}(p_2).$$

Clearly $(x_m, y_m) \in F_{w_1}(V_0)$ with $|x_m - y_m| = 4^{-m}$. Let f be the S^m-*harmonic* function on K satisfying $f(x_m) = 1$ and $f|_{V_m \setminus \{x_m\}} = 0$. Then we have

$$\mathcal{E}(f) = \mathcal{E}_m(f) = 3\left(2^{-1}(1-r)\right)^{-m},$$

which gives that

$$\begin{aligned} R(x_m, y_m)^{-1} &= \inf\{\mathcal{E}(u) : \ u \in \mathcal{D} \text{ and } u(x_m) = 1, u(y_m) = 0\} \\ &\leq \mathcal{E}(f) = 3\left(2^{-1}(1-r)\right)^{-m} \\ &= 3 \cdot 4^{m\theta_1} = 3|x_m - y_m|^{-\theta_1}, \end{aligned}$$

where $\theta_1 = -\frac{\ln(\frac{1}{2}(1-r))}{\ln 4}$. Therefore, $R(x_m, y_m) \geq 3^{-1}|x_m - y_m|^{\theta_1}$. On the other hand, for any $u \in \mathcal{D}$, we see that

$$\begin{aligned} \mathcal{E}(u) \geq \mathcal{E}_m(u) &\geq r_{w_1}^{-1}\left(u(x_m) - u(y_m)\right)^2 = \left(2^{-1}(1-r)\right)^{-m}\left(u(x_m) - u(y_m)\right)^2 \\ &= |x_m - y_m|^{-\theta_1}\left(u(x_m) - u(y_m)\right)^2, \end{aligned}$$

which implies that $R(x_m, y_m) \le |x_m - y_m|^{\theta_1}$ by using (2.9). Therefore,

$$3^{-1}|x_m - y_m|^{\theta_1} \le R(x_m, y_m) \le |x_m - y_m|^{\theta_1}. \tag{3.11}$$

Similarly, let $w_2 = 55\cdots 5 \in S^m$, and take $x'_m = F_{w_2}(p_1)$ and $y'_m = F_{w_2}(p_2)$. Clearly $|x'_m - y'_m| = 2^{-m}$. By the same calculation as above, we can obtain that

$$\frac{1-r}{3(1+r)} \cdot |x'_m - y'_m|^{\theta_2} \le R\left(x'_m, y'_m\right) \le |x'_m - y'_m|^{\theta_2}, \tag{3.12}$$

where $\theta_2 = -\frac{\ln r}{\ln 2}$. If $r \ne \frac{1}{2}$, we see that $\theta_1 \ne \theta_2$. It follows from (3.11) and (3.12) that (3.10) can not hold for any $\theta > 0$, provided that $r \ne \frac{1}{2}$.

4. Domains of Dirichlet forms

Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal with a regular harmonic structure $\left(H, \{r_i\}_{i=1}^M\right)$, and let $(\mathcal{E}, \mathcal{D})$ be the associated Dirichlet form defined as in (2.5) and (2.6). In this section, we give a characterization of the domain $\mathcal{D}$.

In order to characterize $\mathcal{D}$, we need to introduce a measure μ. We choose μ to be the normalized *self-similar* measure with the *standard* weights $\{p_i\}_{i=1}^M$, that is

$$\mu = \sum_{i=1}^M p_i \cdot \mu \circ F_i^{-1} \tag{4.1}$$

where $p_i = r_i^\alpha$, with α given by

$$\sum_{i=1}^M r_i^\alpha = 1. \tag{4.2}$$

For any $w \ne \tau \in W_*$, we have that

$$\mu(K_w) = (r_w)^\alpha \quad \text{and} \quad \mu(K_w \cap K_\tau) = 0. \tag{4.3}$$

Observe that there exists two constants $0 < c_4 \le c_5$ independent of x and λ such that for any $x \in K$ and $0 < \lambda < 1$,

$$B_R(x, c_4\lambda) \subset N_\lambda(x) \subset B_R(x, c_5\lambda), \tag{4.4}$$

where $B_R(x, \lambda) = \{y \in K : R(y, x) < \lambda\}$ is a ball under the metric R, and

$$N_\lambda(x) = \bigcup \{K_w : x \in K_w \text{ and } w \in \Lambda_{\mathbf{r}}(\lambda)\},$$

is the union of all components K_w ($w \in \Lambda_{\mathbf{r}}(\lambda)$) to which x belongs. Indeed, the first inclusion in (4.4) follows since, for $x \in K$ and $y \notin N_\lambda(x)$, one can find a function f such that $f(x) = 1$ and $f(y) = 0$, and $\mathcal{E}(f) \le (c_4\lambda)^{-1}$ for some $c_4 > 0$, see the proof of Theorem 3.1. So $R(x, y) \ge c_4\lambda$ by using (2.8). Therefore $B_R(x, c_4\lambda) \subset N_\lambda(x)$. The second inclusion follows from ([2], Prop. 8.9, p.110).

Theorem 4.1. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal with a regular harmonic structure $\big(H, \{r_i\}_{i=1}^M\big)$, and let $(\mathcal{E}, \mathcal{D})$ be the associated Dirichlet form defined as in (2.5) and (2.6). Let μ be a self-similar measure with standard weights. Then there exists some $c > 0$ such that*

$$c^{-1} W_\alpha(f) \le \mathcal{E}(f) \le c\, W_\alpha(f) \tag{4.5}$$

for all $f \in C(K)$, where

$$W_\alpha(f) := \sup_{0<\lambda<1} \lambda^{-(2\alpha+1)} \int_K \int_{B_R(x, c_4\lambda)} |f(x) - f(y)|^2 d\mu(y) d\mu(x), \tag{4.6}$$

and the constants α and c_4 are the same as in (4.2) and in (4.4), respectively. In particular, we have that $\mathcal{D} = \{f \in C(K) : W_\alpha(f) < \infty\}$.

We decompose Theorem 4.1 into Lemmas 4.3 and 4.4 below. In order to prove Lemma 4.3, we need the following proposition.

Proposition 4.2. *Let $\big(K, \{F_i\}_{i=1}^M\big)$, $(\mathcal{E}, \mathcal{D})$ and μ be as in Theorem 4.1. Then, for $0 < \lambda < 1$ and $f \in C(K)$,*

$$\sum_{w \in \Lambda_r(\lambda)} \int_{K_w} |f(x) - f_w(x_0)|^2 d\mu(x) \le c\, \lambda^{\alpha+1} \mathcal{E}(f). \tag{4.7}$$

where x_0 is any point in V_0, and c is independent of λ, f and x_0.

Proof. The proof is motivated by [5]. Without loss of generality, assume that $f \in \mathcal{D}$ and $x_0 \in V_0$. Since $\Lambda_{\mathbf{r}}(\lambda)$ is a partition, we see that the set

$$\big\{ w\tau : w \in \Lambda_{\mathbf{r}}(\lambda) \text{ and } \tau \in S^k \big\}$$

is also a partition for any $k \ge 1$. Therefore, for μ-almost all $x \in K$, there is exactly one $\tau \in S^k$ such that $x \in K_{w\tau}$. We define $f_k(x) := f_{w\tau}(x_0)$ if $x \in K_{w\tau}$. Obviously the function f_k is defined μ-almost everywhere on K, and is constant on each component of the form $K_{w\tau}$ where $w \in \Lambda_{\mathbf{r}}(\lambda)$ and $\tau \in S^k$. Since f is continuous, we see that $f_k(x) \to f(x)$ for μ-almost all $x \in K$ as $k \to \infty$. In order to derive (4.7), it is enough to show that

$$\sum_{w \in \Lambda_{\mathbf{r}}(\lambda)} \int_{K_w} |f_k(x) - f_w(x_0)|^2 d\mu(x) \le c\, \lambda^{\alpha+1} \mathcal{E}(f). \tag{4.8}$$

In fact, if (4.8) holds, letting $k \to \infty$ in (4.8) and using the dominated convergence theorem, we then obtain (4.7).

Fix $w \in \Lambda_{\mathbf{r}}(\lambda)$ and $\tau := i_1 i_2 \cdots i_k$ for $k \ge 1$ temporally. Let

$$x_l = F_{w i_1 i_2 \cdots i_l}(x_0), \quad 1 \le l \le k.$$

Note that

$$
\begin{aligned}
(f(x_k)-f(x_0))^2 &= \Big(\sum_{l=0}^{k-1} a_l^{-1/2}\cdot a_l^{1/2}(f(x_{l+1})-f(x_l))\Big)^2 \\
&\le \Big(\sum_{l=0}^{k-1} a_l^{-1}\Big)\Big(\sum_{l=0}^{k-1} a_l(f(x_{l+1})-f(x_l))^2\Big) \\
&\le c\sum_{l=0}^{k-1} a_l\,(f(x_{l+1})-f(x_l))^2, \qquad (4.9)
\end{aligned}
$$

where $\{a_l\}_{l=0}^{\infty}$ is a sequence of positive numbers satisfying $\sum_{l=0}^{\infty} a_l^{-1}<\infty$, which will be specified later on. Observing that

$$
\begin{aligned}
(f(x_{l+1})-f(x_l))^2 &= \big(f_{wi_1\cdots i_l}(F_{i_{l+1}}(x_0))-f_{wi_1\cdots i_l}(x_0)\big)^2 \\
&\le c\,\mathcal{E}_1(f_{wi_1\cdots i_l})\le c\,\mathcal{E}(f_{wi_1\cdots i_l}),
\end{aligned}
$$

we see from (4.9) that

$$
(f(x_k)-f(x_0))^2\le c\sum_{l=0}^{k-1} a_l\,\mathcal{E}(f_{wi_1\cdots i_l}).
$$

Therefore, using the fact that $\mu(K_{w\tau})=(r_w)^{\alpha}\,\mu(K_\tau)\le\lambda^{\alpha}\mu(K_\tau)$,

$$
\begin{aligned}
\int_{K_{w\tau}}(f_k(x)-f_w(x_0))^2d\mu(x) &= \mu(K_{w\tau})\,(f(x_k)-f(x_0))^2 \\
&\le c\,\lambda^{\alpha}\mu(K_{i_1i_2\cdots i_k})\sum_{l=0}^{k-1} a_l\mathcal{E}(f_{wi_1\cdots i_l})
\end{aligned}
$$

for any $w\in\Lambda_{\mathbf{r}}(\lambda)$ and any $\tau:=i_1i_2\cdots i_k\in S^k\,(k\ge 1)$. Hence,

$$
\begin{aligned}
\int_{K_w}(f_k(x)-f_w(x_0))^2d\mu(x) &= \sum_{\tau\in S^k}\int_{K_{w\tau}}(f_k(x)-f_w(x_0))^2d\mu(x) \\
&\le c\,\lambda^{\alpha}\sum_{i_1,\cdots,i_k}\mu(K_{i_1i_2\cdots i_k})\sum_{l=0}^{k-1} a_l\mathcal{E}(f_{wi_1\cdots i_l}) \\
&\le c\,\lambda^{\alpha}\sum_{l=0}^{k-1} a_l\sum_{i_1,\cdots,i_l}\mathcal{E}(f_{wi_1\cdots i_l}). \qquad (4.10)
\end{aligned}
$$

In the last inequality above, we have exchanged the order of the summations, and then used the fact that $\sum_{i_{l+1},\cdots,i_k\in S}\mu(K_{i_{l+1}\cdots i_k})=1$ and $\mu(K_{i_1\cdots i_l})\le 1\,(l\ge 1)$. On the other hand, we have that, for any $l \geq 0$,

$$\begin{aligned}\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}\sum_{i_1,\cdots,i_l}\mathcal{E}(f_{wi_1\cdots i_l}) &= \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}\sum_{i_1,\cdots,i_l}(r_{wi_1\cdots i_l})\,(r_{wi_1\cdots i_l})^{-1}\,\mathcal{E}(f_{wi_1\cdots i_l}) \\ &\leq \lambda\,(r_{max})^l\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}\sum_{i_1,\cdots,i_l}(r_{wi_1\cdots i_l})^{-1}\,\mathcal{E}(f_{wi_1\cdots i_l}) \\ &= \lambda\,(r_{max})^l\,\mathcal{E}(f), \qquad (4.11)\end{aligned}$$

since $r_{wi_1\cdots i_l} = r_w r_{i_1\cdots i_l} \leq \lambda\,(r_{max})^l$, where $r_{max} := \max_i\{r_i\} < 1$. Therefore, we obtain from (4.10) and (4.11) that

$$\begin{aligned}\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}\int_{K_w}(f_k(x)-f_w(x_0))^2d\mu(x) &\leq c\,\lambda^\alpha\sum_{l=0}^{k-1}a_l\Big(\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}\sum_{i_1,\cdots,i_l}\mathcal{E}(f_{wi_1\cdots i_l})\Big) \\ &\leq c\,\lambda^{\alpha+1}\mathcal{E}(f)\sum_{l=0}^{\infty}a_l\,(r_{max})^l \leq c\,\lambda^{\alpha+1}\mathcal{E}(f),\end{aligned}$$

where we have chosen $a_l := (r_{max})^{-l/2}$ that satisfies $\sum_{l=0}^{\infty}a_l^{-1} < \infty$. Thus (4.8) follows. This finishes the proof. □

Lemma 4.3. *Let $(K,\{F_i\}_{i=1}^M)$, $(\mathcal{E},\mathcal{D})$ and μ be as in Theorem 4.1. Then there exists some $c > 0$ such that, for all $f \in C(K)$,*

$$W_\alpha(f) \leq c\,\mathcal{E}(f) \qquad (4.12)$$

where $W_\alpha(f)$ is defined as in (4.6).

Proof. Assume that $f \in \mathcal{D}$; otherwise (4.12) automatically holds. Let $0 < \lambda < 1$. Note that $\mu(K_w \cap K_\tau) = 0$ for any distinct $w,\tau \in \Lambda_{\mathbf{r}}(\lambda)$. Set

$$I_\lambda(f) := \int_K\int_{B_R(x,c_4\lambda)}|f(x)-f(y)|^2d\mu(y)d\mu(x).$$

Then we have that, using (4.4),

$$\begin{aligned}I_\lambda(f) &= \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}\int_{K_w}\int_{B_R(x,c_4\lambda)}|f(x)-f(y)|^2d\mu(y)d\mu(x) \\ &\leq \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}\sum_{\tau\sim w}\int_{K_w}\int_{K_\tau}|f(x)-f(y)|^2d\mu(y)d\mu(x).\end{aligned}$$

For $x \in K_w$, $y \in K_\tau$, let $z_0 \in K_w \cap K_\tau = F_w(V_0) \cap F_\tau(V_0)$ (if $w = \tau$, we simply take any point $z_0 \in F_w(V_0)$ and run the same proof as below. So we only consider the case $w \neq \tau$). Using the elementary inequality

$$|f(x)-f(y)|^2 \leq 2(|f(x)-f(z_0)|^2 + |f(z_0)-f(y)|^2),$$

and the fact that $\sharp\{\tau : \tau \sim w\} \leq M_2$ for an integer M_2 independent of w and λ (cf. [10, Lemma 4.2.3, p.139]), we obtain that

$$I_\lambda(f) \leq c\,\lambda^\alpha \sum_{\substack{w\in\Lambda_{\mathbf{r}}(\lambda)\\ z_0\in F_w(V_0)}} \int_{K_w} |f(x)-f(z_0)|^2 d\mu(x).$$

Let $z_0 = F_w(x_0)$ for some $x_0 \in V_0$. By (4.7) and the fact that $\sharp V_0 < \infty$, it immediately follows that

$$I_\lambda(f) \leq c\,\lambda^{2\alpha+1}\mathcal{E}(f),$$

proving (4.12). □

Lemma 4.4. *Let $(K, \{F_i\}_{i=1}^M)$, $(\mathcal{E}, \mathcal{D})$ and μ be as in Theorem 4.1. Then*

$$\mathcal{E}(f) \leq c\,W_\alpha(f) \tag{4.13}$$

for all $f \in C(K)$, where $c > 0$.

Proof. Let $0 < \lambda < c_4/c_5 \leq 1$. Let $f \in C(K)$. Without loss of generality, we assume that $W_\alpha(f) < \infty$. We have that

$$\mathcal{E}_\lambda(f) = \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} r_w^{-1}\mathcal{E}_0(f_w) \leq c \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} r_w^{-1}\Big(\sum_{p,q\in F_w(V_0)} \big(f(p)-f(q)\big)^2\Big). \tag{4.14}$$

Noting that, for any $x_0 \in K_w$,

$$|f(p)-f(q)|^2 \leq 2\big((f(p)-f(x_0))^2 + (f(x_0)-f(q))^2\big),$$

we see that

$$|f(p)-f(q)|^2 \leq \frac{2}{\mu(K_w)}\Big(\int_{K_w}\big(f(p)-f(x_0)\big)^2 d\mu(x_0) + \int_{K_w}\big(f(x_0)-f(q)\big)^2 d\mu(x_0)\Big).$$

Hence,

$$\sum_{p,q\in F_w(V_0)} \big(f(p)-f(q)\big)^2 \leq M_3 \sum_{p\in F_w(V_0)} \frac{1}{\mu(K_w)}\int_{K_w}\big(f(p)-f(x_0)\big)^2 d\mu(x_0), \tag{4.15}$$

where $M_3 = 4\#(F_w(V_0)) = 4\#(V_0)$. Let $w \in \Lambda_{\mathbf{r}}(\lambda)$ and $p \in F_w(V_0)$ be fixed. We now estimate the integral

$$\int_{K_w}\big(f(p)-f(x_0)\big)^2 d\mu(x_0).$$

Let $0 < a < 1$ be any fixed number (for example, $a = \frac{1}{2}$). For each integer $l \geq 0$, let $\Lambda(\lambda a^l) := \Lambda_{\mathbf{r}}(\lambda a^l)$ be a partition. We choose a sequence of subsets of K_w:

$$K_w,\ K_{w\tau_1},\ K_{w\tau_2},\ \cdots,\ K_{w\tau_l},\ \cdots$$

such that $w\tau_l \in \Lambda(\lambda a^l)$ and $p \in K_{w\tau_l}$ for each $l \geq 0$. Note that, for any $l > i \geq 0$,

$$K_{w\tau_l} \subset K_{w\tau_i};$$

this is because $\lambda a^l < \lambda a^i$, and so the partition $\Lambda(\lambda a^i)$ is a father of $\Lambda(\lambda a^l)$. For simplicity, we denote by

$$K'_0 = K_w \quad \text{and} \quad K'_l = K_{w\tau_l} \quad (l \geq 1).$$

Note that, for any $x_l \in K'_l \, (l \geq 0)$,

$$\begin{aligned}(f(p) - f(x_0))^2 &= \Big(\big(f(p) - f(x_k)\big) + \sum_{l=0}^{k-1} a_l^{-1/2} \cdot a_l^{1/2}\big(f(x_{l+1}) - f(x_l)\big)\Big)^2 \\ &\leq 2(f(p) - f(x_k))^2 + 2\Big(\sum_{l=0}^{\infty} a_l^{-1}\Big)\Big(\sum_{l=0}^{k-1} a_l\big(f(x_{l+1}) - f(x_l)\big)^2\Big),\end{aligned}$$

where $\{a_l\}_{l=0}^{\infty}$ is a sequence of positive numbers satisfying

$$\sum_{l=0}^{\infty} a_l^{-1} < \infty,$$

which will be determined below. Integrating the above inequality with respect to each $x_l \in K'_l$ for $0 \leq l \leq k$, and then dividing by $\mu(K'_0) \cdots \mu(K'_k)$, we obtain that

$$\begin{aligned}\frac{1}{\mu(K_w)} \int_{K_w} (f(p) - f(x_0))^2 d\mu(x_0) &\leq \frac{2}{\mu(K'_k)} \int_{K'_k} (f(p) - f(x_k))^2 d\mu(x_k) \\ &+ c \sum_{l=0}^{k-1} \frac{a_l}{\mu(K'_{l+1})\mu(K'_l)} \int_{K'_l} \int_{K'_{l+1}} \big(f(x_{l+1}) - f(x_l)\big)^2 d\mu(x_{l+1}) d\mu(x_l). \end{aligned} \tag{4.16}$$

Note that the first term on the right-hand side of (4.16) tends to zero as $k \to \infty$, since $\{K'_k\}$ shrinks to p as $k \to \infty$ and f is continuous. In order to estimate the second term, we denote by

$$A_{w,k}(f) := \sum_{l=0}^{k-1} \frac{a_l}{\mu(K'_{l+1})\mu(K'_l)} \int_{K'_l} \int_{K'_{l+1}} \big(f(x_{l+1}) - f(x_l)\big)^2 d\mu(x_{l+1}) d\mu(x_l). \tag{4.17}$$

By (4.4), we have that $K'_l \subset N_{\lambda a^l}(x_l) \subset B_R(x_l, c_5 \lambda a^l)$ for any $x_l \in K'_l$ and $l \geq 0$. Using the fact that $K'_{l+1} \subset K'_l \subset K_w$, we obtain that

$$\begin{aligned}&\int_{K'_l} \int_{K'_{l+1}} \big(f(x_{l+1}) - f(x_l)\big)^2 d\mu(x_{l+1}) d\mu(x_l) \\ &\qquad \leq \int_{K_w} \int_{B_R(x_l, c_5 \lambda a^l)} \big(f(x_{l+1}) - f(x_l)\big)^2 d\mu(x_{l+1}) d\mu(x_l).\end{aligned}$$

Note that, using (4.3) and the fact that $w\tau_l \in \Lambda_{\mathbf{r}}(\lambda a^l)$,

$$\mu\left(K'_l\right) = (r_{w\tau_l})^{\alpha} \sim \left(\lambda a^l\right)^{\alpha} \quad \text{for any } l \geq 0.$$

Therefore, it follows from (4.17) that

$$A_{w,k}(f) \le c \sum_{l=0}^{k-1} a_l (\lambda a^l)^{-2\alpha} \int_{K_w} \int_{B_R(x, c_5\lambda a^l)} \big(f(y)-f(x)\big)^2 d\mu(y) d\mu(x). \tag{4.18}$$

Therefore, combining (4.14), (4.15) and (4.16), we obtain that, for any $k \ge 0$,

$$\mathcal{E}_\lambda(f) \le c \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} r_w^{-1} A_{w,k}(f) + \sum_{\substack{w\in\Lambda_{\mathbf{r}}(\lambda)\\ p\in F_w(V_0)}} r_w^{-1} \frac{c}{\mu(K'_k)} \int_{K'_k} (f(p)-f(z))^2 d\mu(z). \tag{4.19}$$

On the other hand, noting that $r_w \sim \lambda$ for $w \in \Lambda_{\mathbf{r}}(\lambda)$, it follows from (4.18) that

$$\begin{aligned}
\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} r_w^{-1} A_{w,k}(f) &\le c\,\lambda^{-1} \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} A_{w,k}(f)\\
&\le c\,\lambda^{-1} \sum_{l=0}^{k-1} a_l (\lambda a^l)^{-2\alpha} \int_K \int_{B_R(x,c_5\lambda a^l)} \big(f(y)-f(x)\big)^2 d\mu(y)d\mu(x)\\
&= c \sum_{l=0}^{k-1} a_l\, a^l \Big\{ (c_5\lambda a^l)^{-(2\alpha+1)} \int_K \int_{B_R(x,c_5\lambda a^l)} \big(f(y)-f(x)\big)^2 d\mu(y)d\mu(x) \Big\}\\
&\le c\, W_\alpha(f) \sum_{l=0}^{k-1} a_l\, a^l \le c\, W_\alpha(f). \qquad (4.20)
\end{aligned}$$

Here we have chosen $a_l := a^{-l/2}$ that satisfies $\sum_{l\ge 0} a_l^{-1} < \infty$. Therefore, by (4.19) and (4.20),

$$\mathcal{E}_\lambda(f) \le c\, W_\alpha(f) + \sum_{\substack{w\in\Lambda_{\mathbf{r}}(\lambda)\\ p\in F_w(V_0)}} r_w^{-1} \frac{c}{\mu(K'_k)} \int_{K'_k} (f(p)-f(z))^2 d\mu(z) \quad (k \ge 0),$$

where c is independent of k and λ. Letting $k \to \infty$, we see that

$$\mathcal{E}_\lambda(f) \le c\, W_\alpha(f).$$

This gives (4.13) by letting $\lambda \to 0$ and using (2.13). □

Finally we remark that Theorem 4.1 directly follows from Lemmas 4.3 and 4.4.

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