Notes on Fuzzy Set Ordination

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1 Introduction

The membership of an element in a fuzzy set A is denoted by $\mu_A(x)$ and the fuzzy set A is defined as:

$$A = \{(x, \mu_A(x)), x \in X\} \tag{1}$$

where X denotes the universe. Kaufmann [1] proposes the difference between two fuzzy sets A and B as

$$A - B = A \cap \bar{B} \tag{2}$$

The fuzzy compliment is defined per Zadeh [2] as

$$\forall x \in X \quad \mu_{\bar{A}}(x) = 1 - \mu_A(x) \tag{3}$$

Intersection has three definitions:

$$\forall x \in X \quad \mu_{A\cap B}(x) = min(\mu_A(x), \mu_B(x)) \quad A \cap B[2] \tag{4}$$

$$\forall x \in X \quad \mu_{A\cap B}(x) = \mu_A(x).\mu_B(x) \quad A.B \tag{5}$$

$$\forall x \in X \quad \mu_{A\cap B}(x) = max(0, \mu_A(x) + \mu_B(x) - 1) \quad A \Cap B[3] \tag{6}$$

The three difference operators are $A \cap \bar{B}$, $A.\bar{B}$, and $A \Cap \bar{B}$, respectively. In addition, Zadeh [4] introduced bounded difference, defined as

$$\forall x \in X \quad \mu_{A|-|B}(x) = max(0, \mu_A(x) - \mu_B(x)). \tag{7}$$

and can be shown that $\mu_{A\Cap B}(x) = \mu_{A|-|B}(x)$.
The anticommutative difference is defined as

$$\forall x \in X \quad \mu_{A|\bar{B}}(x) = \frac{[1 + (\mu_{\bar{B}}(x)^2 - \mu_{\bar{A}}(x)^2)]}{2} \tag{8}$$

and is termed anticommutative difference because

$$\forall x \in X \quad \mu_{A|\bar{B}}(x) = 1 - \mu_{B|\bar{A}}(x) \tag{9}$$

so that $A|\bar{B} = \overline{B|\bar{A}}$. As both $\mu_A(x)$ and $\mu_B(x)$ range over $[0, 1]$, so likewise do $\mu_{\bar{A}}(x)$ and $\mu_{\bar{B}}(x)$. Therefore, $\mu_{\bar{B}}(x)^2 - \mu_{\bar{A}}(x)^2$ ranges over $[-1, +1]$ and to map this expression back to

$[0, 1]$, we add one and divide by two. When A and B partition X, the following relationship is true:

$$\forall x \in X \quad \mu_{A|\bar{B}}(x) = \mu_{A\cap\bar{B}}(x) = \mu_A(x) \tag{10}$$

For boolean operators, $A \cap \bar{B}$, $A.\bar{B}$, and $A| - |B$, recover the difference when applied to crisp sets, however the anticummutative difference $A|\bar{B}$ fails to recover classical difference and does not qualify as a binary operator for crisp sets. On the other hand, anticommutative difference is an interactive operator, i.e., a change in either $\mu_A(x)$ or $\mu_B(x)$ necessarily changes $\mu_{A|\bar{B}}(x)$, where as fuzzy difference based on the *min* definition of intersection, and bounded difference are non-interactive. $A.\bar{B}$ is interactive, but with crisp sets reduces to a Boolean operator and interaction is masked. Zadeh's [4] bounded difference $(A|-|B)$ is the most conservative operator. $A| - |B = \emptyset$ unless $\exists x \in X : \mu_A(x) - \mu_B(x) > 0$. Dubois and Prade [5] interpret $A| - |B$ as a set of elements which belong more to A than to B. The next most conservative operator is $A.\bar{B}$

$$\forall x \in X \quad \mu_{A|-|B}(x) \leqslant \mu_{A.\bar{B}}(x) \tag{11}$$

Also, $\mu_{A.\bar{B}}(x) = \mu_{A|-|B}(x)$ only when $\mu_A(x)$ or $\mu_B(x) = 0$ or 1. In the order of decreasing conservativeness, the next operator is $A \cap \bar{B}$:

$$\forall x \in X \quad \mu_{A.\bar{B}}(x) \leqslant \mu_{A\cap\bar{B}}(x) \tag{12}$$

Finally, the last operator is anticommutative difference:

$$\forall x \in X \quad \mu_{A\cap\bar{B}}(x) \leqslant \mu_{A|\bar{B}}(x) \tag{13}$$

And so we have the following relations of the membership functions:

$$A| - |B \quad \subset \quad A.\bar{B} \quad \subset \quad A \cap \bar{B} \quad \subset \quad A|\bar{B} \tag{14}$$

1.1 Ordination based on Bray-Curtis similarity [7]

The coefficient of similarity by Bray and Curtis [6] is defined as

$$\mathbf{C} = \frac{2\mathbf{w}}{\mathbf{a} + \mathbf{b}} \tag{15}$$

where $\mathbf{a}$ is the sum of quantitative values (abundance) of all species in one sample, $\mathbf{b}$ is the sum of quantitative values in another sample, and $\mathbf{w}$ is the sum of quantitative values the two samples have in common for each species. For example, given three species, abundance count for Sample 1 is $(10, 20, 30)$, Sample 2 is $(24, 23, 15)$. To get $\mathbf{w}$, the sum of the lowest value for each species between the two samples is $10 + 20 + 15 = 55$. Therefore, $\mathbf{C} = \frac{2\times55}{60+62} = 0.9016393$; i.e., Sample 1 and 2 have a similarity of 90%. This index ranges from zero, if the two samples have no species in common, to 1.00 if they are by chance identical.

The next step is to construct a matrix showing coefficients of similarity for each of the samples with all other samples. The coefficients are totaled for each sample, and the sample with the lowest sum could be considered the sample most different from all the others. It is used as one end of the first or x-axis of the ordination. The other end sample of this axis is the sample having the least in common with the first. Since the ordination attempts to arrange the samples according to their relative dissimilarity, inverse values of the coefficients of similarity are used (by subtracting the coefficient from 1.00). Expressed in these units, the distance between samples is called *dissimilarity* values.

The length of the axis of the ordination is equal to the dissimilarity between the two reference samples (Fig. 1). Each of the other samples is located by drawing arcs representing the

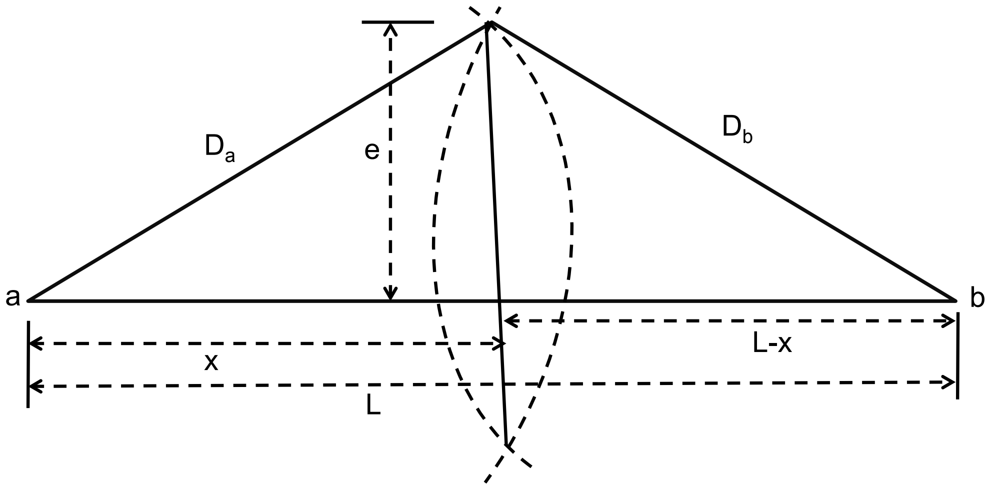

Figure 1: Location of sample along an axis of ordination, by projection of the point of arc intersection on x-axis where **a** and **b** are the reference samples, **L** is the dissimilarity value between the reference samples, $\mathbf{D}_a$ and $\mathbf{D}_b$ are dissimilarity values of a given sample from the two reference samples, and **x** is the location of that sample along the axis.

dissimilarity from the two ends. These arcs intersect each other above and below a line drawn between the two reference samples. Two triangles are given with sides $\mathbf{exD}_a$ and $\mathbf{e(L\text{-}x)D}_b$ with hypotenuse as known dissimilarities from the reference samples. The triangles have the following equations according to the Pythogorean theorem:

$$\mathbf{e}^2 + \mathbf{x}^2 = \mathbf{D}_a^2 \tag{16}$$

$$\mathbf{e}^2 + (\mathbf{L} - \mathbf{x})^2 = \mathbf{D}_b^2 \tag{17}$$

Subtracting one equation from the other, to eliminate $\mathbf{e}^2$, and solving for **x** gives

$$\mathbf{x} = \frac{\mathbf{L}^2 + \mathbf{D}_a^2 - \mathbf{D}_b^2}{2\mathbf{L}} \tag{18}$$

The above calculation can be simplified since **L** is constant for all samples along a given axis. When all samples are located along the x-axis, there are samples placed close together which in reality are quite dissimilar. Therefore a second or y-axis is constructed to separate these. The first reference sample on the y-axis is selected on the basis of the highest **e** value along the x-axis and this **e** value is calculated from

$$\mathbf{e}^2 = \mathbf{D}_a^2 - \mathbf{x}^2 \tag{19}$$

The other end sample is the most dissimilar one to the first end within a distance from the latter, along the x-axis, of less than 10% of the total length of the x-axis. In this way, the second axis approximates a perpendicular relationship to the first. The samples can then be plotted on a two-dimensional graph. In [7], distances between samples on the ordination were calculated as $\mathbf{d}_x^2 + \mathbf{d}_y^2$ and the correlation between these distances and the respective coefficients of similarity for a random sample of 50 intersample distances was remarkably high ($\mathbf{r} = -.922$), indicating that the method yields a close approximation of samples to one another based on the coefficient of similarity.

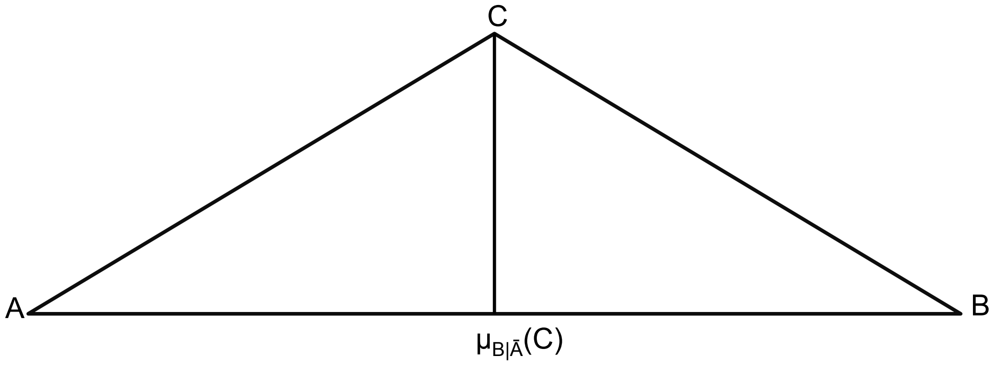

Figure 2: Graphical derivation of the anticommutative difference operator.

1.2 Anticommutative difference operator as an ordination [8]

The formula 18 is essentially the anticommutative difference operator with a slight difference. In fuzzy set notation, Beal's formula would be

$$\frac{(\mu_{AC}(x))^2 + (\mu_{BC}(x))^2 - (\mu_{AB}(x))^2}{2(\mu_{AC}(x))} \tag{20}$$

where $\mu_{AC}(x)$ is the dissimilarity of point **A** to point **C**, and similarly for the other symbols (Fig. 2). This technique is for a Euclidean space as the transitivity of the relation is known. The maximum value of $\mu_{AC}(x)$ is one, and so when the transitivity of the relation is not known, $\mu_{AC}(x)$ must be assumed to be one or the operator may fail as a binary operator. For fuzzy sets, the value of $\mu_{AC}(x)$ must always be one to insure the operator quality as a binary operator on the interval $[0, 1]$, and to ensure anticommutativity.

2 One-dimensional FSO

A one-dimensional FSO is defined through a series of operations on intermediate sets. Given a vector of environmental data X, the first fuzzy set (A) is simply the relativized value of the vector X.

$$\forall x \in X \quad \mu_A(x) = \frac{x - min(x)}{max(x) - min(x)} \tag{21}$$

The second fuzzy set (B) is the complement of the first

$$\forall x \in X \quad \mu_B(x) = 1 - \mu_A(x) \tag{22}$$

The third fuzzy set (C) is the set of plots similar to plots with higher value of X

$$\mu_C(x) = \frac{\sum_{y \neq x} [S_{xy}(\mu_A(y))]}{\sum_{y \neq x}(\mu_A(y))} \tag{23}$$

where S_{xy} is the similarity of samples x and y. Fuzzy set D is the set of samples similar to samples with low value of X

$$\mu_D(x) = \frac{\sum_{y \neq x} [S_{xy}(\mu_B(y))]}{\sum_{y \neq x}(\mu_B(y))} \tag{24}$$

Fuzzy set E, the fuzzy set actually plotted in the FSO, is calculated as the anticommutative difference of C and D

$$\mu_E(x) = \frac{[1 + \mu_{\bar{D}}(x)^2 - \mu_{\bar{C}}(x)^2]}{2} \tag{25}$$

where $\mu_{\bar{C}}(x) = 1 - \mu_C(x)$.

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