

Uncertainty Quantification Using Random Matrix Theory

S ADHIKARI AND D TARTAKOVSKY (UCSD)

School of Engineering, University of Wales Swansea, Swansea, U.K.

Email: S.Adhikari@swansea.ac.uk

URL: <http://engweb.swan.ac.uk/~adhikaris>

Outline

- Motivation
- Current methods for response UQ
- Matrix variate probability density functions
- Inverse of a general real symmetric random matrix
- Response moments
- Numerical example
- Conclusions

Background

- In many stochastic mechanics problems we need to solve a system of linear stochastic equations:

$$\mathbf{K}\mathbf{u} = \mathbf{f}. \quad (1)$$

$\mathbf{K} \in \mathbb{R}^{n \times n}$ is a $n \times n$ real non-negative definite random matrix, $\mathbf{f} \in \mathbb{R}^n$ is a n -dimensional real deterministic input vector and $\mathbf{u} \in \mathbb{R}^n$ is a n -dimensional real uncertain output vector which we want to determine.

- This typically arise due to the discretisation of stochastic partial differential equations (eg. in the stochastic finite element method)

Background

- In the context of linear structural mechanics, \mathbf{K} is known as the stiffness matrix, \mathbf{f} is the forcing vector and \mathbf{u} is the vector of structural displacements.
- Often, the objective is to determine the probability density function (pdf) and consequently the cumulative distribution function (cdf) of \mathbf{u} . This will allow one to calculate the reliability of the system.
- It is generally difficult to obtain the probably density function (pdf) of the response. As a consequence, engineers often intend to obtain only the fist few moments (typically the fist two) of the response quantity.

Objectives

- We propose an exact analytical method for the inverse of a real symmetric (in general non-Gaussian) random matrix of arbitrary dimension.
- The method is based on random matrix theory and utilizes the Jacobian of the underlying nonlinear matrix transformation.
- Exact expressions for the mean and covariance of the response vector is obtained in closed-form.

Current Approaches

The random matrix can be represented as

$$\mathbf{K} = \mathbf{K}^0 + \Delta\mathbf{K} \quad (2)$$

$\mathbf{K}^0 \in \mathbb{R}^{n \times n}$ is the deterministic part and the random part:

$$\Delta\mathbf{K} = \sum_{j=1}^m \xi_j \mathbf{K}_j^I + \sum_{j=1}^m \sum_{l=1}^m \xi_j \xi_l \mathbf{K}_{jl}^{II} + \dots \quad (3)$$

m is the number of random variables, $\mathbf{K}_j^I, \mathbf{K}_{jl}^{II} \in \mathbb{R}^{n \times n}$, $\forall j, l$ are deterministic matrices and $\xi_j, \forall j$ are real random variables.

Perturbation based approach

Represent the response as

$$\mathbf{u} = \mathbf{u}^0 + \xi_j \mathbf{u}_j^I + \sum_{j=1}^m \sum_{l=1}^m \xi_j \xi_l \mathbf{u}_{jl}^{II} + \dots \quad (4)$$

where

$$\mathbf{u}^0 = \mathbf{K}^{0^{-1}} \mathbf{f} \quad (5)$$

$$\mathbf{u}_j^I = -\mathbf{K}^{0^{-1}} \mathbf{K}_j^I \mathbf{u}^0, \quad \forall j \quad (6)$$

$$\text{and } \mathbf{u}_{jl}^{II} = -\mathbf{K}^{0^{-1}} [\mathbf{K}_{jl}^{II} \mathbf{u}^0 + \mathbf{K}_j^I \mathbf{u}_l^I + \mathbf{K}_l^I \mathbf{u}_j^I], \quad \forall j, l. \quad (7)$$

Neumann expansion

Provided $\left\| \mathbf{K}^{0^{-1}} \Delta \mathbf{K} \right\|_{\text{F}} < 1$,

$$\begin{aligned} \mathbf{K}^{-1} &= \left[\mathbf{K}_0 (\mathbf{I}_n + \mathbf{K}^{0^{-1}} \Delta \mathbf{K}) \right]^{-1} \\ &= \mathbf{K}^{0^{-1}} - \mathbf{K}^{0^{-1}} \Delta \mathbf{K} \mathbf{K}^{0^{-1}} + \mathbf{K}^{0^{-1}} \Delta \mathbf{K} \mathbf{K}^{0^{-1}} \Delta \mathbf{K} \mathbf{K}^{0^{-1}} - \dots \end{aligned}$$

Therefore,

$$\mathbf{u} = \mathbf{K}^{-1} \mathbf{f} = \mathbf{u}^0 - \mathbf{T} \mathbf{u}_0 + \mathbf{T}^2 \mathbf{u}_0 + \dots \quad (8)$$

where $\mathbf{T} = \mathbf{K}^{0^{-1}} \Delta \mathbf{K} \in \mathbb{R}^{n \times n}$ is a random matrix.

Projection methods

Here one ‘projects’ the solution vector onto a complete stochastic basis. Depending on how the basis is selected, several methods are proposed.

Using the classical Polynomial Chaos (PC) projection scheme

$$\mathbf{u} = \sum_{j=0}^{P-1} \mathbf{u}_j \Psi_j(\boldsymbol{\xi}) \quad (9)$$

where $\mathbf{u}_j \in \mathbb{R}^n$, $\forall j$ are unknown vectors and $\Psi_j(\boldsymbol{\xi})$ are multidimensional Hermite polynomials in ξ_r .

A partial summary

Methods	Sub-methods
1. Perturbation based methods	First and second order perturbation ^{1,2} , Neumann expansion ^{3,4} , improved perturbation method ⁵ .
2. Projection methods	Polynomial chaos expansion ⁶ , random eigenfunction expansion ⁴ , stochastic reduced basis method ⁷⁻⁹ , Wiener–Askey chaos expansion ¹⁰⁻¹² , domain decomposition method ^{13,14} .
3. Monte carlo simulation and other methods	Simulation methods ^{15,16} , Analytical methods in references ¹⁷⁻²¹ , Exact solutions for beams ^{22,23} .

The Central Idea of the Proposed Work

- We aim to obtain moments of \mathbf{u} (or part of \mathbf{u}) from a system of linear stochastic equations

$$\mathbf{K}\mathbf{u} = \mathbf{f} \quad (10)$$

considering the fact that we ‘know’ the probability density function of the matrix \mathbf{K} .

- For different physical problems (e.g., flow through stochastic porous media, circuit theory, static structural mechanics etc.) the matrix \mathbf{K} (i.e., its pdf) can be obtained in a manner specific to the discipline.
- The ‘final problem’ to be solved is however, the same and

is the central topic of this paper.

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If \mathbf{A} is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \rightarrow \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\Sigma \otimes \Psi$, where $\Sigma \in \mathbb{R}_n^+$ and $\Psi \in \mathbb{R}_p^+$ provided the pdf of \mathbf{X} is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} |\Sigma|^{-p/2} |\Psi|^{-n/2} \operatorname{etr} \left\{ -\frac{1}{2} \Sigma^{-1} (\mathbf{X} - \mathbf{M}) \Psi^{-1} (\mathbf{X} - \mathbf{M})^T \right\} \quad (11)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \Sigma \otimes \Psi)$.

Symmetric Gaussian matrix

If $\mathbf{Y} \in \mathbb{R}^{n \times n}$ is a symmetric Gaussian random matrix then its pdf is given by

$$p_{\mathbf{Y}}(\mathbf{Y}) = (2\pi)^{-n(n+1)/4} \left| \mathbf{B}_n^T (\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}) \mathbf{B}_n \right|^{-1/2} \text{etr} \left\{ -\frac{1}{2} \boldsymbol{\Sigma}^{-1} (\mathbf{Y} - \mathbf{M}) \boldsymbol{\Psi}^{-1} (\mathbf{Y} - \mathbf{M})^T \right\}. \quad (12)$$

This is denoted as $\mathbf{Y} = \mathbf{Y}^T \sim SN_{n,n}(\mathbf{M}, \mathbf{B}_n^T (\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}) \mathbf{B}_n)$. The elements of the translation matrix $\mathbf{B}_n \in \mathbb{R}^{n^2 \times n(n+1)/2}$ are:

$$(B_n)_{ij,gh} = \frac{1}{2} (\sigma_{ig}\sigma_{jh} + \sigma_{ih}\sigma_{jg}), \quad i \leq n, j \leq n, g \leq h \leq n, \quad (13)$$

Matrix variate Gamma distribution

A $n \times n$ symmetric positive definite matrix random \mathbf{W} is said to have a matrix variate gamma distribution with parameters a and $\Psi \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{ \Gamma_n(a) |\Psi|^{-a} \right\}^{-1} |\mathbf{W}|^{a - \frac{1}{2}(n+1)} \text{etr} \{-\Psi \mathbf{W}\}; \Re(a) > \frac{1}{2}(n-1)$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \Psi)$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^n \Gamma \left[a - \frac{1}{2}(k-1) \right]; \text{ for } \Re(a) > (n-1)/2$$

Wishart matrix

A $n \times n$ symmetric positive definite random matrix \mathbf{S} is said to have a Wishart distribution with parameters $p \geq n$ and $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \Gamma_n \left(\frac{1}{2}p \right) |\boldsymbol{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \text{etr} \left\{ -\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{S} \right\} \quad (14)$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma})$.

Note: $G_n(a, \boldsymbol{\Psi}) = W_n(2a, \boldsymbol{\Psi}^{-1}/2)$, so that Gamma and Wishart are equivalent distributions.

Some books

- Muirhead, *Aspects of Multivariate Statistical Theory*, John Wiley and Sons, 1982.
- Mehta, *Random Matrices*, Academic Press, 1991.
- Gupta and Nagar, *Chapman & Hall/CRC*, 2000.

Inverse of a scalar

$$ku = f \quad (15)$$

where $k, u, f \in \mathbb{R}$. Suppose the pdf k is $p_k(k)$ and we are interested in deriving the pdf of

$$h = k^{-1}. \quad (16)$$

The Jacobian of the above transformation

$$J = \left| \frac{\partial h}{\partial k} \right| = |-k^{-2}| = |k|^{-2}. \quad (17)$$

Using the Jacobian, the pdf of h can be obtained as

$$p_h(h)(dh) = p_k(k)(dk) \quad (18)$$

$$\text{or } p_h(h) = \frac{1}{\left| \frac{\partial h}{\partial k} \right|} p_k(k) \quad (19)$$

$$\text{or } p_h(h) = \frac{1}{J(k = h^{-1})} p_k(k = h^{-1}) = |h|^{-2} p_k(h^{-1}). \quad (20)$$

The case of $n \times n$ matrices

- Suppose the matrix variate probability density function of the non-singular matrix \mathbf{K} is given by $p_{\mathbf{K}}(\mathbf{K}) : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$. Our interest is in the pdf (i.e joint pdf of the elements) of

$$\mathbf{H} = \mathbf{K}^{-1} \in \mathbb{R}^{n \times n}. \quad (21)$$

- The elements of \mathbf{H} are complicated non-linear function of the elements of \mathbf{K} (i.e. even if the elements of \mathbf{K} are joint Gaussian, the elements of \mathbf{H} will not be joint Gaussian).
- \mathbf{H} may not have any banded structure even if \mathbf{K} is of banded nature.

Pdf transformation in matrix space

The procedure to obtain the pdf of \mathbf{H} is very similar to that of the univariate case:

$$p_{\mathbf{H}}(\mathbf{H})(d\mathbf{H}) = p_{\mathbf{K}}(\mathbf{K})(d\mathbf{K}) \quad (22)$$

$$\text{or } p_{\mathbf{H}}(\mathbf{H}) = \frac{1}{\left| \frac{d\mathbf{H}}{d\mathbf{K}} \right|} p_{\mathbf{K}}(\mathbf{K}) \quad (23)$$

$$\text{or } p_{\mathbf{H}}(\mathbf{H}) = \frac{1}{J(\mathbf{K} = \mathbf{H}^{-1})} p_{\mathbf{K}}(\mathbf{K} = \mathbf{H}^{-1}) \quad (24)$$

$$= |\mathbf{H}|^{-(n+1)} p_{\mathbf{K}}(\mathbf{H}^{-1}). \quad (25)$$

For the univariate case ($n = 1$) Eq. (24) reduces to the familiar equivalent expression obtained in Eq. (19).

Derivation of the Jacobian - 1

We have

$$\mathbf{K}\mathbf{K}^{-1} = \mathbf{K}\mathbf{H} = \mathbf{I}_n. \quad (26)$$

Taking the matrix differential

$$(\mathbf{dK})\mathbf{H} + \mathbf{K}(\mathbf{dH}) = \mathbf{O}_n \quad \text{or} \quad (\mathbf{dH}) = -\mathbf{K}^{-1}(\mathbf{dK})\mathbf{K}^{-1}. \quad (27)$$

Treat $(\mathbf{dH}), (\mathbf{dK}) \in \mathbb{R}^{n \times n}$ as variables and \mathbf{K} as constant since it does not contain (\mathbf{dH}) or (\mathbf{dK}) . Taking the vec of Eq. (26)

$$\text{vec}(\mathbf{dH}) = -\text{vec}(\mathbf{K}^{-1}(\mathbf{dK})\mathbf{K}^{-1}) = -(\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}) \text{vec}(\mathbf{dK}). \quad (28)$$

Derivation of the Jacobian - 2

Because $(d\mathbf{H})$ and $(d\mathbf{K})$ are symmetric matrices we need to eliminate the 'duplicate' variables appearing in the preceding linear transformation. This can be achieved in a systematic manner by using the translation matrix \mathbf{B}_n as

$$\text{vecp}(d\mathbf{H}) = \mathbf{B}_n^\dagger \text{vec}(d\mathbf{H}) = - [\mathbf{B}_n^\dagger (\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}) \mathbf{B}_n] \text{vecp}(d\mathbf{K}). \quad (29)$$

The Jacobian associated with the above linear transformation is simply the determinant of the matrix $\mathbf{B}_n^\dagger (\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}) \mathbf{B}_n$, that is

$$J = |\mathbf{B}_n^\dagger (\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}) \mathbf{B}_n| = |\mathbf{K}|^{-(n+1)}. \quad (30)$$

RM model for stiffness matrix

If the mean of \mathbf{K} is $\bar{\mathbf{K}}$, then $\mathbf{K} \sim W_n(p, \Sigma)$, where

$$p = n + 1 + \theta$$

$$\Sigma = \bar{\mathbf{K}}/\alpha$$

$$\theta = \frac{1}{\sigma_K^2} \left\{ 1 + \frac{\{\text{Trace}(\bar{\mathbf{K}})\}^2}{\text{Trace}(\bar{\mathbf{K}}^2)} \right\} - (n + 1)$$

and $\alpha = \sqrt{\theta(n + 1 + \theta)}$.

σ_K is the normalized standard-deviation of \mathbf{K} :

$$\sigma_K^2 = \frac{\mathbb{E} [\|\mathbf{K} - \mathbb{E}[\mathbf{K}]\|_{\text{F}}^2]}{\|\mathbb{E}[\mathbf{K}]\|_{\text{F}}^2}. \quad (31)$$

Pdf of \mathbf{K}^{-1}

The pdf $\mathbf{H} = \mathbf{K}^{-1}$, that is, the joint pdf of all the elements of \mathbf{H} can be obtained as

$$\begin{aligned} p_{\mathbf{H}}(\mathbf{H}) &= |\mathbf{H}|^{-(n+1)} p_{\mathbf{K}}(\mathbf{H}^{-1}) \\ &= |\mathbf{H}|^{-(n+1)} \left\{ 2^{\frac{1}{2}np} \Gamma_n \left(\frac{1}{2}p \right) |\boldsymbol{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{H}^{-1}|^{\frac{1}{2}(p-n-1)} \text{etr} \left\{ -\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{H}^{-1} \right\} \\ &= |\mathbf{H}|^{-(n+1+p)/2} \left\{ 2^{\frac{1}{2}np} \Gamma_n \left(\frac{1}{2}p \right) |\boldsymbol{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} \text{etr} \left\{ -\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{H}^{-1} \right\}. \end{aligned} \tag{32}$$

Using this exact pdf, the moments of the inverse matrix can be obtained.

Inverted Wishart matrix

A $n \times n$ symmetric positive definite random matrix \mathbf{V} is said to have an inverted Wishart distribution with parameters m and $\Psi \in \mathbb{R}_n^+$, if its pdf is given by (from Muirhead, 1982)

$$p_{\mathbf{V}}(\mathbf{V}) = \frac{2^{-\frac{1}{2}(m-n-1)n} |\Psi|^{\frac{1}{2}(m-n-1)}}{\Gamma_n\left(\frac{1}{2}(m-n-1)\right) |\mathbf{V}|^{m/2}} \text{etr} \left\{ -\frac{1}{2} \mathbf{V}^{-1} \Psi \right\}; \quad m > 2n, \quad \Psi > 0. \quad (33)$$

This distribution is usually denoted as $\mathbf{V} \sim IW_n(m, \Psi)$.

We can show that $\mathbf{K}^{-1} \sim IW_n(\theta + 2n + 2, \alpha \overline{\mathbf{K}}^{-1})$.

Moments of \mathbf{K}^{-1}

The first moment (mean), second-moment and the elements of the covariance tensor of \mathbf{K}^{-1} can be obtained²⁴ exactly in closed-form as

$$\mathbb{E} [\mathbf{K}^{-1}] = \frac{\Psi}{m - 2n - 2} = \frac{\alpha}{\theta} \overline{\mathbf{K}}^{-1} \quad (34)$$

$$\mathbb{E} [\mathbf{K}^{-2}] = \frac{\text{Trace}(\Psi) \Psi + (m - 2n - 2) \Psi^2}{(m - 2n - 1)(m - 2n - 2)(m - 2n - 4)}$$

$$\mathbb{E} [\mathbf{K}^{-1} \mathbf{A} \mathbf{K}^{-1}] = \frac{\text{Trace}(\mathbf{A} \Psi) \Psi + (m - 2n - 2) \Psi \mathbf{A} \Psi}{(m - 2n - 1)(m - 2n - 2)(m - 2n - 4)}$$

$$= \frac{\alpha^2 \left(\text{Trace}(\mathbf{A} \overline{\mathbf{K}}^{-1}) \overline{\mathbf{K}}^{-1} + \theta \overline{\mathbf{K}}^{-1} \mathbf{A} \overline{\mathbf{K}}^{-1} \right)}{\theta(\theta + 1)(\theta - 2)}.$$

(35)

Response Moments - 1

- The complete response vector is

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{f}. \quad (36)$$

- In many practical problems only few elements of \mathbf{u} or linear combinations of some elements of \mathbf{u} may be of interest. Therefore, we are interested in the quantity

$$\mathbf{y} = \mathbf{R}\mathbf{u} = \mathbf{R}\mathbf{K}^{-1}\mathbf{f}; \quad \mathbf{R} \in \mathbb{R}^{r \times n} \quad (37)$$

- The matrix \mathbf{R} can be also selected to ‘extract’ other physical quantities such as the stress components within one element or a group of elements.

Response Moments - 2

- The mean of \mathbf{y} :

$$\bar{\mathbf{y}} = \mathbb{E}[\mathbf{y}] = \mathbb{E}[\mathbf{R}\mathbf{K}^{-1}\mathbf{f}] = \mathbf{R} \mathbb{E}[\mathbf{K}^{-1}] \mathbf{f} = \frac{\alpha}{\theta} \mathbf{R}\bar{\mathbf{K}}^{-1} \mathbf{f}. \quad (38)$$

- The complete covariance matrix of \mathbf{y} :

$$\begin{aligned} \text{cov}(\mathbf{y}, \mathbf{y}) &= \mathbb{E}[(\mathbf{y} - \bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}})^T] = \mathbb{E}[\mathbf{y}\mathbf{y}^T] - \bar{\mathbf{y}}\bar{\mathbf{y}}^T \\ &= \mathbf{R} \mathbb{E}[\mathbf{K}^{-1}\mathbf{f}\mathbf{f}^T\mathbf{K}^{-1}] \mathbf{R}^T - \bar{\mathbf{y}}\bar{\mathbf{y}}^T \\ &= \frac{\alpha^2 \text{Trace}(\mathbf{f}\mathbf{f}^T\bar{\mathbf{K}}^{-1}) \mathbf{R}\bar{\mathbf{K}}^{-1} \mathbf{R}^T + \theta(\theta + 2)\bar{\mathbf{y}}\bar{\mathbf{y}}^T}{\theta(\theta + 1)(\theta - 2)}. \end{aligned} \quad (39)$$

Steps for complete analysis

- Obtain normalized standard deviation

$$\sigma_G^2 = \frac{\mathbb{E}[\|\mathbf{K} - \mathbb{E}[\mathbf{K}]\|_F^2]}{\|\mathbb{E}[\mathbf{K}]\|_F^2} = \frac{\text{Trace}(\text{cov}(\text{vec}(\mathbf{K})))}{\text{Trace}(\bar{\mathbf{K}}^2)}.$$

- Calculate the constants

$$\theta = \frac{1}{\sigma_K^2} \left\{ 1 + \frac{\{\text{Trace}(\bar{\mathbf{K}})\}^2}{\text{Trace}(\bar{\mathbf{K}}^2)} \right\} - (n + 1),$$

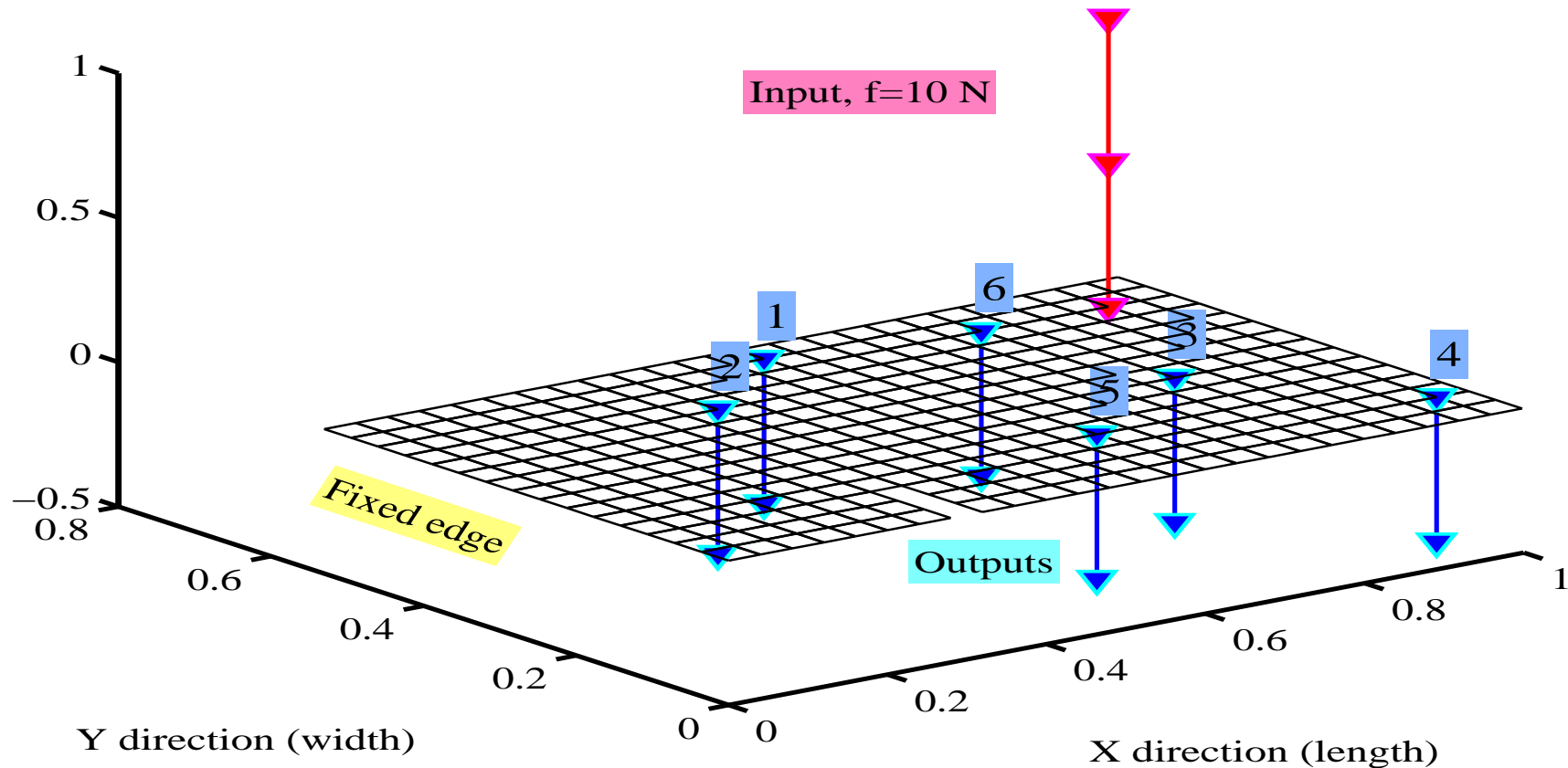
$$p = n + 1 + \theta, \quad \alpha = \sqrt{\theta(n + 1 + \theta)} \quad \text{and} \quad \Sigma = \bar{\mathbf{K}}/\alpha.$$

- The mean: $\bar{\mathbf{y}} = \frac{\alpha}{\theta} \mathbf{R} \bar{\mathbf{K}}^{-1} \mathbf{f}.$

- The covariance:

$$\text{cov}(\mathbf{y}, \mathbf{y}) = \frac{\alpha^2 \text{Trace}(\mathbf{f} \mathbf{f}^T \bar{\mathbf{K}}^{-1}) \mathbf{R} \bar{\mathbf{K}}^{-1} \mathbf{R}^T + \theta(\theta + 2) \bar{\mathbf{y}} \bar{\mathbf{y}}^T}{\theta(\theta + 1)(\theta - 2)}.$$

Example: A cantilever Plate



A steel cantilever plate with a slot; $\bar{E} = 200 \times 10^9 \text{ N/m}^2$, $\bar{\mu} = 0.3$, $\bar{t} = 7.5 \text{ mm}$, $L_x = 1.2 \text{ m}$,

$L_y = 0.8 \text{ m}$; 25×15 elements resulting $n = 1200$.

Stochastic Properties

- The Young's modulus, Poissons ratio and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E} (1 + \epsilon_E f_1(\mathbf{x})) \quad (40)$$

$$\mu(\mathbf{x}) = \bar{\mu} (1 + \epsilon_\mu f_2(\mathbf{x})) \quad (41)$$

$$\text{and } t(\mathbf{x}) = \bar{t} (1 + \epsilon_t f_4(\mathbf{x})) \quad (42)$$

- The strength parameters are: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.10$, and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x})$, $i = 1, 2, 3$ are delta-correlated homogenous Gaussian random fields.

Response calculation

- The value of σ_k (calculated using a 5000-sample Monte Carlo simulation of the random fields) is obtained as $\sigma_K = 0.2616$.

- From the 1200×1200 stiffness matrix we obtain

$$\text{Trace}(\overline{\mathbf{K}}) = 5.5225 \times 10^9 \quad \text{and} \quad \text{Trace}(\overline{\mathbf{K}}^2) = 9.6599 \times 10^{16}.$$

and

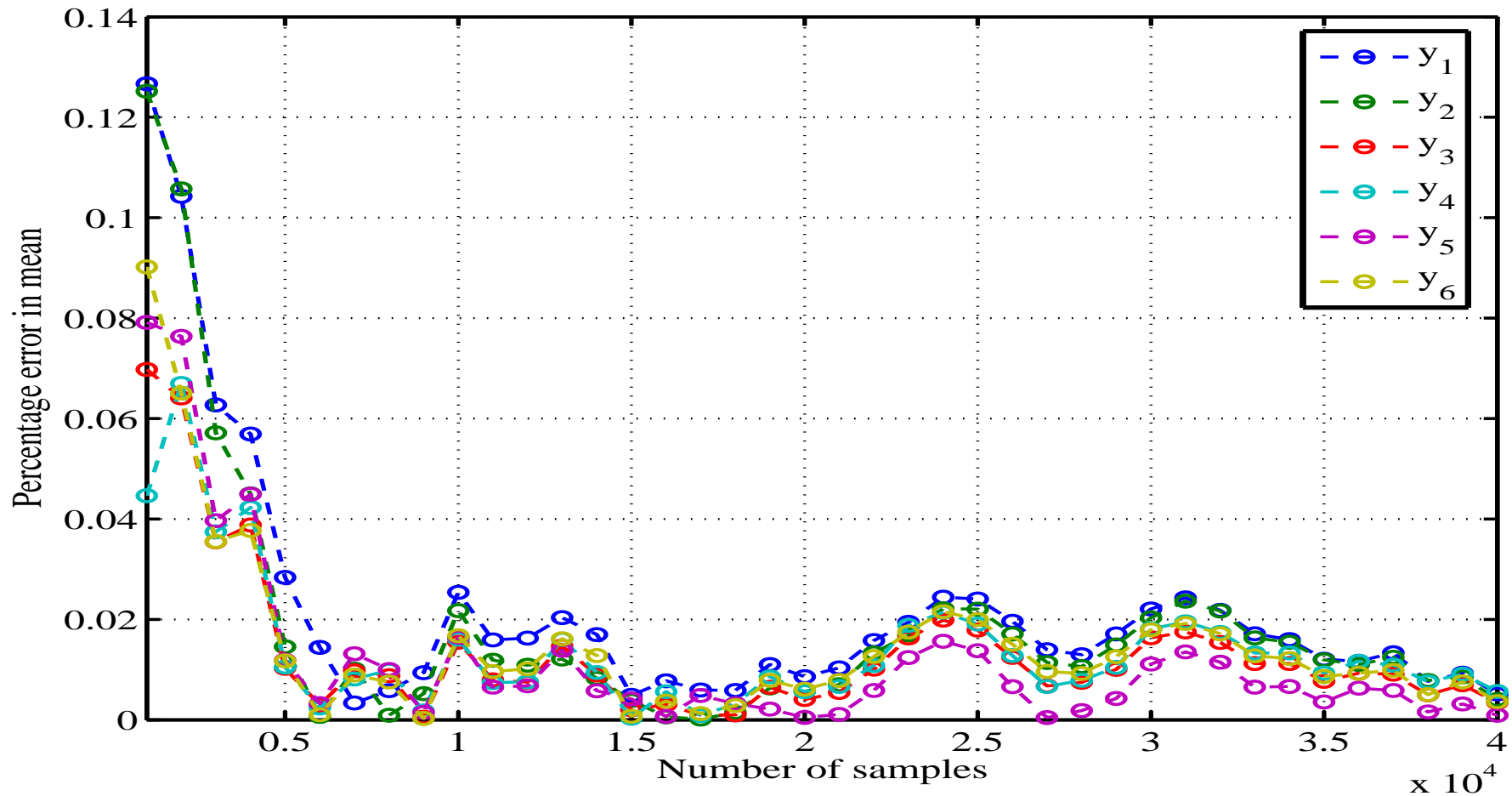
$$\theta = 3.4274 \times 10^3 \quad \text{and} \quad \alpha = 3.9827 \times 10^3.$$

Comparison of results

The mean and standard deviation of the response vector. The numbers in the parenthesis correspond to the percentage error in the Monte Carlo Simulation (with 1000 samples) results with respect to the exact analytical results.

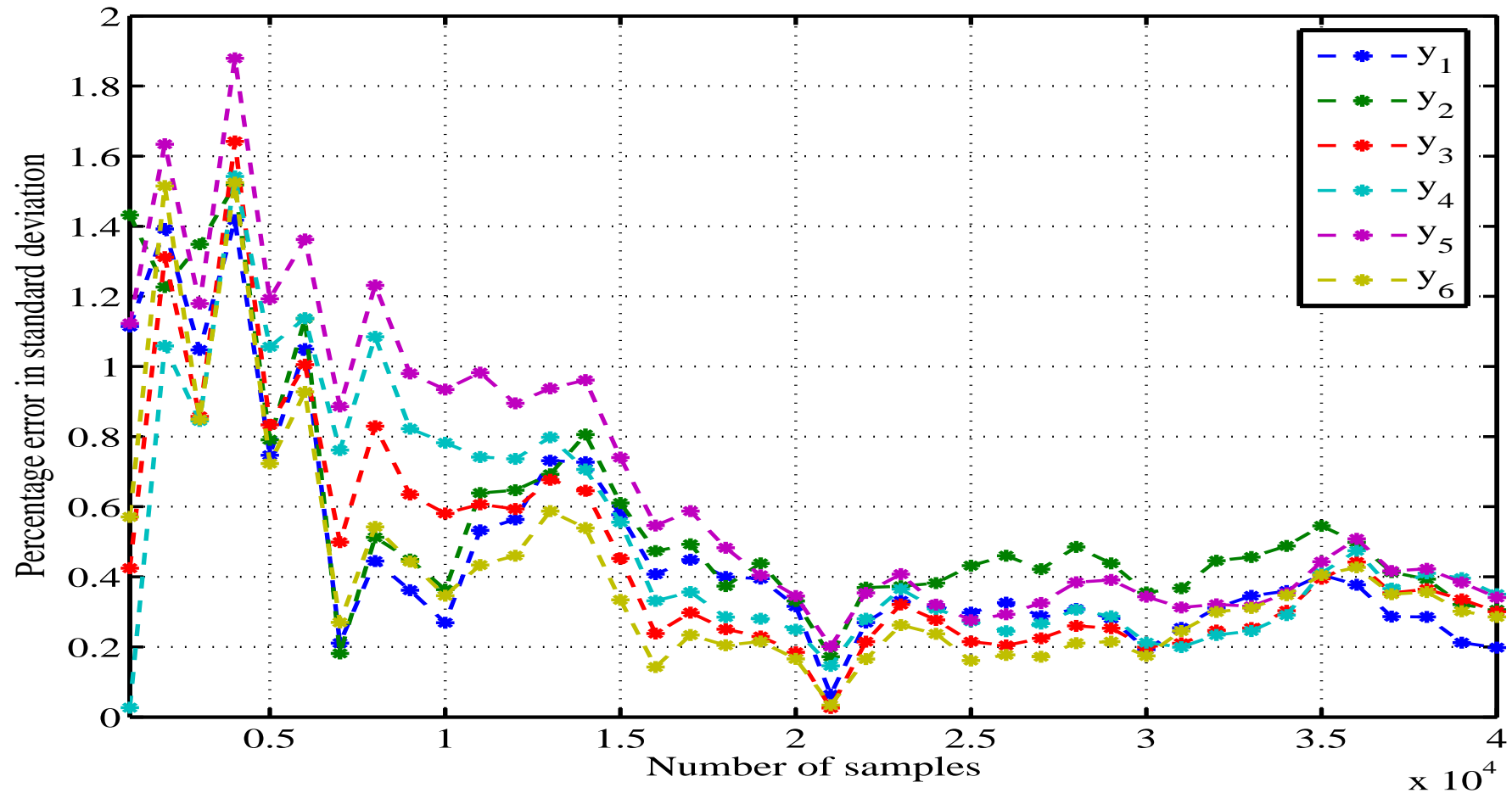
Response quantity	Analytical mean (mm)	MSC mean (mm)	Analytical standard deviation (mm)	MSC standard deviation (mm)
$y_1 = u_{112}$	5.5058	5.5178 (0.218 %)	0.1438	0.1459 (1.436 %)
$y_2 = u_{325}$	2.6420	2.6475 (0.208 %)	0.0734	0.0740 (0.818 %)
$y_3 = u_{658}$	10.2265	10.2485 (0.216 %)	0.2537	0.2561 (0.972 %)
$y_4 = u_{1045}$	12.6039	12.6317 (0.221 %)	0.3294	0.3313 (0.570 %)
$y_5 = u_{868}$	5.9608	5.9725 (0.197 %)	0.1586	0.1604 (1.155 %)
$y_6 = u_{205}$	10.2951	10.3169 (0.212 %)	0.2507	0.2547 (1.580 %)

Convergence of mean



Variation of the relative error in MCS mean with respect to the number of samples.

Convergence of std



Variation of the relative error in MCS standard deviation with respect to the number of samples.

Summary - 1

- The probabilistic characterization of the response of linear stochastic systems requires inverse of a real symmetric random matrix.
- An exact and simple closed-form expression of the joint probability density function of the elements of the inverse of a symmetric random matrix is given.
- A matrix itself is treated like a variable, as opposed to view it as a collection of many variables. This outlook significantly simplifies the calculation of the Jacobian involved in the non-linear matrix transformation.

Summary - 2

- The random matrices considered are in general non-Gaussian and of arbitrary dimensions.
- Moments of the response do not require a series/perturbation/PC expansion.
- The numerical implementation is straight-forward and non-intrusive.

Open Issues

- Any real matrix pdf can be used for $p_{\mathbf{K}}(\mathbf{K})$ and the pdf of $\mathbf{H} = \mathbf{K}^{-1}$ can be obtained. However, obtaining the response pdf (requires further transformation) or response moments from $p_{\mathbf{H}}(\mathbf{H})$ is a not trivial task.
- Selecting a matrix variate pdf to matrix data is a challenging task itself (topic of my later today).
- Once the pdf of the matrix is 'known', the method is exact. The approximation in the implementation often arises in 'fitting' a random matrix pdf.

References

- [1] Shinozuka M., Yamazaki F. S.T. Ariaratnam, G.I. Schueller, I. Elishakoff, eds., *Stochastic structural dynamics: Progress in theory and applications*. Elsevier Applied Science, London, 1998.
- [2] Kleiber M., Hien T.D. *The Stochastic Finite Element Method*. John Wiley, Chichester, 1992.
- [3] Yamazaki F., Shinozuka M., Dasgupta G. *ASCE Journal of Engineering Mechanics*, 114(8), 1335–1354, 1988.
- [4] Adhikari S., Manohar C.S. *International Journal for Numerical Methods in Engineering*, 44(8), 1157–1178, 1999.
- [5] Elishakoff I., Ren Y.J., Shinozuka M. *Chaos Solitons & Fractals*, 5(5), 833–846, 1995.
- [6] Ghanem R., Spanos P. *Stochastic Finite Elements: A Spectral Approach*. Springer-Verlag, New York, USA, 1991.
- [7] Nair P.B., Keane A.J. *AIAA Journal*, 40(8), 1653–1664, 2002.
- [8] Sachdeva S.K., Nair P.B., Keane A.J. *Computer Methods in Applied Mechanics and Engineering*, 195(19-22), 2371–2392, 2006.
- [9] —. *Probabilistic Engineering Mechanics*, 21(2), 182–192, 2006.
- [10] Xiu D.B., Karniadakis G.E. *Siam Journal on Scientific Computing*, 24(2), 619–644, 2002.
- [11] —. *Journal of Computational Physics*, 187(1), 137–167, 2003.
- [12] Wan X.L., Karniadakis G.E. *Journal of Scientific Computing*, 27((-3)), 455–464, 2006.
- [13] Sarkar A., Benabbou N., Ghanem R. *47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics & Materials Conference*. Newport, Rhode Island, USA, 2006.
- [14] —. *7th World Congress on Computational Mechanics*. Los Angeles, CA, USA, 2006.
- [15] Hurtado J.E., Barbat A.H. *Archives of Computational Methods in Engineering*, 5(1), 3–29, 1998.
- [16] Papadrakakis M., Papadopoulos V. *Computer Methods in Applied Mechanics and Engineering*, 134(3-4), 325–340, 1996.
- [17] Muscolino G., Ricciardi G., Impollonia N. *Probabilistic Engineering Mechanics*, 15(2), 199–212, 2000.
- [18] Impollonia N., Muscolino G. *Meccanica*, 37(1-2), 179–192, 2002.
- [19] Falsone G., Impollonia N. *Computer Methods in Applied Mechanics and Engineering*, 191(44), 5067–5085, 2002.
- [20] —. *Computer Methods in Applied Mechanics and Engineering*, 192(16-18), 2187–2188, 2003.
- [21] Impollonia N., Ricciardi G. *Probabilistic Engineering Mechanics*, 21(2), 171–181, 2006.
- [22] Elishakoff I., Ren Y.J., Shinozuka M. *International Journal of Solids and Structures*, 32(16), 2315–2327, 1995.
- [23] Elishakoff I., Ren Y.J. *Large Variation Finite Element Method for Stochastic Problems*. Oxford University Press, Oxford, U.K., 2003.
- [24] Gupta A., Nagar D. *Matrix Variate Distributions*. Monographs & Surveys in Pure & Applied Mathematics. Chapman & Hall/CRC, London, 2000.