Uncertainty Propagation in Linear Systems: An Exact Solution Using random Matrix Theory

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Outline

- Motivation
- Current methods for response-statistics calculation
- Matrix variate probability density functions
- Exact inverse of a general real symmetric random matrix
- Exact response moments of linear systems
- Numerical example
- Conclusions

Background

- In many stochastic mechanics problems we need to solve a system of linear stochastic equations:

$$\mathbf{K}\mathbf{u} = \mathbf{f}. \tag{1}$$

  $\mathbf{K} \in \mathbb{R}^{n \times n}$ is a $n \times n$ real non-negative definite random matrix, $\mathbf{f} \in \mathbb{R}^n$ is a n-dimensional real deterministic input vector and $\mathbf{u} \in \mathbb{R}^n$ is a n-dimensional real uncertain output vector which we want to determine.

- This typically arise due to the discretisation of stochastic partial differential equations (eg. in the stochastic finite element method)

Background

- In the context of linear structural mechanics, $\mathbf{K}$ is known as the stiffness matrix, $\mathbf{f}$ is the forcing vector and $\mathbf{u}$ is the vector of structural displacements.
- Often, the objective is to determine the probability density function (pdf) and consequently the cumulative distribution function (cdf) of $\mathbf{u}$. This will allow one to calculate the reliability of the system.
- It is generally difficult to obtain the probably density function (pdf) of the response. As a consequence, engineers often intend to obtain only the fist few moments (typically the fist two) of the response quantity.

Objectives

- We propose an exact analytical method for the inverse of a real symmetric (in general non-Gaussian) random matrix of arbitrary dimension.
- The method is based on random matrix theory and utilizes the Jacobian of the underlying nonlinear matrix transformation.
- Exact expressions for the mean and covariance of the response vector is obtained in closed-form.

Current Approaches

The random matrix can be represented as

$$\mathbf{K} = \mathbf{K}^0 + \mathbf{\Delta K} \qquad (2)$$

$\mathbf{K}^0 \in \mathbb{R}^{n\times n}$ is the deterministic part and the random part:

$$\mathbf{\Delta K} = \sum_{j=1}^{m} \xi_j \mathbf{K}_j^I + \sum_{j=1}^{m}\sum_{l=1}^{m} \xi_j \xi_l \mathbf{K}_{jl}^{II} + \cdots \qquad (3)$$

m is the number of random variables, $\mathbf{K}_j^I, \mathbf{K}_{jl}^{II} \in \mathbb{R}^{n\times n}, \forall\, j, l$ are deterministic matrices and $\xi_j, \forall\, j$ are real random variables.

Perturbation based approach

Represent the response as

$$\mathbf{u} = \mathbf{u}^0 + \xi_j \mathbf{u}_j^I + \sum_{j=1}^{m} \sum_{l=1}^{m} \xi_j \xi_l \mathbf{u}_{jl}^{II} + \cdots . \tag{4}$$

where

$$\mathbf{u}^0 = \mathbf{K}^{0^{-1}} \mathbf{f} \tag{5}$$

$$\mathbf{u}_j^I = -\mathbf{K}^{0^{-1}} \mathbf{K}_j^I \mathbf{u}^0, \quad \forall\, j \tag{6}$$

and $$\mathbf{u}_{jl}^{II} = -\mathbf{K}^{0^{-1}} [\mathbf{K}_{jl}^{II} \mathbf{u}^0 + \mathbf{K}_j^I \mathbf{u}_l^I + \mathbf{K}_l^I \mathbf{u}_j^I], \quad \forall\, j, l. \tag{7}$$

Neumann expansion

Provided $\left\|\mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K}\right\|_{\mathrm{F}} < 1$,

$$\begin{aligned}\mathbf{K}^{-1} &= \left[\mathbf{K}_0(\mathbf{I}_n + \mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K})\right]^{-1} \\ &= \mathbf{K}^{0^{-1}} - \mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K}\mathbf{K}^{0^{-1}} + \mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K}\mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K}\mathbf{K}^{0^{-1}} + \cdots .\end{aligned}$$

Therefore,

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{f} = \mathbf{u}^0 - \mathbf{T}\mathbf{u}_0 + \mathbf{T}^2\mathbf{u}_0 + \cdots \tag{8}$$

where $\mathbf{T} = \mathbf{K}^{0^{-1}}\boldsymbol{\Delta}\mathbf{K} \in \mathbb{R}^{n\times n}$ is a random matrix.

Projection methods

Here one ‘projects’ the solution vector onto a complete stochastic basis. Depending on how the basis is selected, several methods are proposed.

Using the classical Polynomial Chaos (PC) projection scheme

$$\mathbf{u} = \sum_{j=0}^{P-1} \mathbf{u}_j \Psi_j(\boldsymbol{\xi}) \tag{9}$$

where $\mathbf{u}_j \in \mathbb{R}^n, \forall j$ are unknown vectors and $\Psi_j(\boldsymbol{\xi})$ are multidimensional Hermite polynomials in ξ_r.

A partial summary

	Methods	Sub-methods
1.	Perturbation based methods	First and second order perturbation[1,2], Neumann expansion[3,4], improved perturbation method[5].
2.	Projection methods	Polynomial chaos expansion[6], random eigenfunction expansion[4], stochastic reduced basis method[7–9], Wiener–Askey chaos expansion[10–12], domain decomposition method[13,14].
3.	Monte carlo simulation and other methods	Simulation methods[15,16], Analytical method in references[17–21], Exact solutions for beams[22,23].

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} |\mathbf{\Sigma}|^{-p/2} |\mathbf{\Psi}|^{-n/2} \operatorname{etr}\left\{ -\frac{1}{2} \mathbf{\Sigma}^{-1} (\mathbf{X} - \mathbf{M}) \mathbf{\Psi}^{-1} (\mathbf{X} - \mathbf{M})^T \right\} \quad (10)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Symmetric Gaussian matrix

If $\mathbf{Y} \in \mathbb{R}^{n\times n}$ is a symmetric Gaussian random matrix then its pdf is given by

$$p_{\mathbf{Y}}(\mathbf{Y}) = (2\pi)^{-n(n+1)/4} \left|\mathbf{B}_n^T(\mathbf{\Sigma} \otimes \mathbf{\Psi})\mathbf{B}_n\right|^{-1/2} \mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{Y}-\mathbf{M})^T\right\}. \quad (11)$$

This is denoted as $\mathbf{Y} = \mathbf{Y}^T \sim SN_{n,n}\left(\mathbf{M}, \mathbf{B}_n^T(\mathbf{\Sigma} \otimes \mathbf{\Psi})\mathbf{B}_n\right)$. The elements of the translation matrix $\mathbf{B}_n \in \mathbb{R}^{n^2\times n(n+1)/2}$ are:

$$(B_n)_{ij,gh} = \frac{1}{2}\left(\delta_{ig}\delta_{jh} + \delta_{ih}\delta_{jg}\right), \quad i \le n, j \le n, g \le h \le n, \quad (12)$$

Matrix variate Gamma distribution

A $n \times n$ symmetric positive definite matrix random $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a)\left|\mathbf{\Psi}\right|^{-a}\right\}^{-1}\left|\mathbf{W}\right|^{a-\frac{1}{2}(n+1)} \operatorname{etr}\left\{-\mathbf{\Psi}\mathbf{W}\right\};\ \Re(a) > \frac{1}{2}(n-1)$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right];\ \text{for } \Re(a) > (n-1)/2$$

Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \Gamma_n\left(\frac{1}{2}p\right) |\mathbf{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \mathrm{etr}\left\{ -\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S} \right\} \tag{13}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$.

Note: $G_n(a, \mathbf{\Psi}) = W_n(2a, \mathbf{\Psi}^{-1}/2)$, so that Gamma and Wishart are equivalent distributions.

Inverse of a scalar

$$ku = f \tag{14}$$

where $k, u, f \in \mathbb{R}$. Suppose the pdf k is $p_k(k)$ and we are interested in deriving the pdf of

$$h = k^{-1}. \tag{15}$$

The Jacobian of the above transformation

$$J = \left| \frac{\partial h}{\partial k} \right| = \left| -k^{-2} \right| = |k|^{-2}. \tag{16}$$

Using the Jacobian, the pdf of h can be obtained as

$$p_h(h)(dh) = p_k(k)(dk) \tag{17}$$

$$\text{or} \quad p_h(h) = \frac{1}{\left| \frac{\partial h}{\partial k} \right|} p_k(k) \tag{18}$$

$$\text{or} \quad p_h(h) = \frac{1}{J\left(k = h^{-1}\right)} p_k\left(k = h^{-1}\right) = |h|^{-2} p_k\left(h^{-1}\right). \tag{19}$$

The case of $n \times n$ matrices

- Suppose the matrix variate probability density function of the non-singular matrix $\mathbf{K}$ is given by $p_{\mathbf{K}}(\mathbf{K}) : \mathbb{R}^{n\times n} \to \mathbb{R}$. Our interest is in the pdf (i.e joint pdf of the elements) of

$$\mathbf{H} = \mathbf{K}^{-1} \in \mathbb{R}^{n\times n}. \quad (20)$$

- The elements of $\mathbf{H}$ are complicated non-linear function of the elements of $\mathbf{K}$ (i.e. even if the elements of $\mathbf{K}$ are joint Gaussian, the elements of $\mathbf{H}$ will not be joint Gaussian).
- $\mathbf{H}$ may not have any banded structure even if $\mathbf{K}$ is of banded nature.

Pdf transforation in matrix space

The procedure to obtain the pdf of $\mathbf{H}$ is very similar to that of the univariate case:

$$p_{\mathbf{H}}\left(\mathbf{H}\right)\left(\mathbf{dH}\right) = p_{\mathbf{K}}(\mathbf{K})(\mathbf{dK}) \tag{21}$$

$$\text{or} \quad p_{\mathbf{H}}\left(\mathbf{H}\right) = \frac{1}{\left|\frac{\mathbf{dH}}{\mathbf{dK}}\right|} p_{\mathbf{K}}(\mathbf{K}) \tag{22}$$

$$\text{or} \quad p_{\mathbf{H}}\left(\mathbf{H}\right) = \frac{1}{J\left(\mathbf{K} = \mathbf{H}^{-1}\right)} p_{\mathbf{K}}\left(\mathbf{K} = \mathbf{H}^{-1}\right) \tag{23}$$

$$= \left|\mathbf{H}\right|^{-(n+1)} p_{\mathbf{K}}\left(\mathbf{H}^{-1}\right). \tag{24}$$

For the univariate case ($n = 1$) Eq. (24) reduces to the familiar equivalent expression obtained in Eq. (19).

Derivation of the Jacobian - 1

We have

$$\mathbf{K}\mathbf{K}^{-1} = \mathbf{K}\mathbf{H} = \mathbf{I}_n. \tag{25}$$

Taking the matrix differential

$$(\mathbf{dK})\,\mathbf{H} + \mathbf{K}\,(\mathbf{dH}) = \mathbf{O}_n \quad \text{or} \quad (\mathbf{dH}) = -\mathbf{K}^{-1}\,(\mathbf{dK})\,\mathbf{K}^{-1}. \tag{26}$$

Treat $(\mathbf{dH})\,,(\mathbf{dK}) \in \mathbb{R}^{n\times n}$ as variables and $\mathbf{K}$ as constant since it does not contain $(\mathbf{dH})$ or $(\mathbf{dK})$. Taking the vec of Eq. (26)

$$\mathrm{vec}\,(\mathbf{dH}) = -\mathrm{vec}\left(\mathbf{K}^{-1}\,(\mathbf{dK})\,\mathbf{K}^{-1}\right) = -\left(\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}\right)\mathrm{vec}\,(\mathbf{dK})\,. \tag{27}$$

Derivation of the Jacobian - 2

Because $(\mathbf{dH})$ and $(\mathbf{dK})$ are symmetric matrices we need to eliminate the 'duplicate' variables appearing in the preceding linear transformation. This can be achieved in a systematic manner by using the translation matrix $\mathbf{B}_n$ as

$$\mathrm{vecp}\left(\mathbf{dH}\right) = \mathbf{B}_n^{\dagger}\mathrm{vec}\left(\mathbf{dH}\right) = -\left[\mathbf{B}_n^{\dagger}\left(\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}\right)\mathbf{B}_n\right]\mathrm{vecp}\left(\mathbf{dK}\right). \tag{28}$$

The Jacobian associated with the above linear transformation is simply the determinant of the matrix $\mathbf{B}_n^{\dagger}\left(\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}\right)\mathbf{B}_n$, that is

$$J = \left|\mathbf{B}_n^{\dagger}\left(\mathbf{K}^{-1} \otimes \mathbf{K}^{-1}\right)\mathbf{B}_n\right| = \left|\mathbf{K}\right|^{-(n+1)}. \tag{29}$$

RM model for stiffness matrix

If the mean of $\mathbf{K}$ is $\overline{\mathbf{K}}$, then $\mathbf{K} \sim W_n\left(p, \boldsymbol{\Sigma}\right)$, where

$$p = n + 1 + \theta$$

$$\boldsymbol{\Sigma} = \overline{\mathbf{K}}/\alpha$$

$$\theta = \frac{1}{\delta_K^2}\left\{1 + \{\text{Trace}\left(\overline{\mathbf{K}}\right)\}^2/\text{Trace}\left(\overline{\mathbf{K}}^2\right)\right\} - (n+1)$$

and $\alpha = \sqrt{\theta(n+1+\theta)}$.

δ_K is the normalized standard-deviation of $\mathbf{K}$:

$$\delta_K^2 = \frac{\mathrm{E}\left[\|\mathbf{K} - \mathrm{E}\left[\mathbf{K}\right]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{K}\right]\|_{\mathrm{F}}^2}. \tag{30}$$

Pdf of $\mathbf{K}^{-1}$

The pdf $\mathbf{H} = \mathbf{K}^{-1}$, that is, the joint pdf of all the elements of $\mathbf{H}$ can be obtained as

$$
\begin{aligned}
p_{\mathbf{H}}\left(\mathbf{H}\right) &= |\mathbf{H}|^{-(n+1)} \, p_{\mathbf{K}}\left(\mathbf{H}^{-1}\right) \\
&= |\mathbf{H}|^{-(n+1)} \left\{ 2^{\frac{1}{2}np} \, \Gamma_n\left(\frac{1}{2}p\right) |\mathbf{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} \left|\mathbf{H}^{-1}\right|^{\frac{1}{2}(p-n-1)} \operatorname{etr}\left\{ -\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{H}^{-1} \right\} \\
&= |\mathbf{H}|^{-(n+1+p)/2} \left\{ 2^{\frac{1}{2}np} \, \Gamma_n\left(\frac{1}{2}p\right) |\mathbf{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} \operatorname{etr}\left\{ -\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{H}^{-1} \right\}.
\end{aligned}
\tag{31}
$$

Using this exact pdf, the moments of the inverse matrix can be obtained.

Inverted Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{V}$ is said to have an inverted Wishart distribution with parameters m and $\boldsymbol{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{V}}(\mathbf{V}) = \frac{2^{-\frac{1}{2}(m-n-1)n} \left|\boldsymbol{\Psi}\right|^{\frac{1}{2}(m-n-1)}}{\Gamma_n\left(\frac{1}{2}(m-n-1)\right) \left|\mathbf{V}\right|^{m/2}} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{V}^{-1}\boldsymbol{\Psi}\right\};\; m > 2n,\; \boldsymbol{\Psi} > 0. \tag{32}$$

This distribution is usually denoted as $\mathbf{V} \sim IW_n(m, \boldsymbol{\Psi})$.

We can show that $\mathbf{K}^{-1} \sim IW_n(\theta + 2n + 2, \alpha\overline{\mathbf{K}}^{-1})$.

Moments of $\mathbf{K}^{-1}$

The first moment (mean), second-moment and the elements of the covariance tensor of $\mathbf{K}^{-1}$ can be obtained[24] exactly in closed-form as

$$\mathrm{E}\left[\mathbf{K}^{-1}\right] = \frac{\mathbf{\Psi}}{m-2n-2} = \frac{\alpha}{\theta}\overline{\mathbf{K}}^{-1} \tag{33}$$

$$\mathrm{E}\left[\mathbf{K}^{-2}\right] = \frac{\mathrm{Trace}\left(\mathbf{\Psi}\right)\mathbf{\Psi} + (m-2n-2)\mathbf{\Psi}^2}{(m-2n-1)(m-2n-2)(m-2n-4)}$$

$$\begin{aligned}\mathrm{E}\left[\mathbf{K}^{-1}\mathbf{A}\mathbf{K}^{-1}\right] &= \frac{\mathrm{Trace}\left(\mathbf{A}\mathbf{\Psi}\right)\mathbf{\Psi} + (m-2n-2)\mathbf{\Psi}\mathbf{A}\mathbf{\Psi}}{(m-2n-1)(m-2n-2)(m-2n-4)} \\ &= \frac{\alpha^2\left(\mathrm{Trace}\left(\mathbf{A}\overline{\mathbf{K}}^{-1}\right)\overline{\mathbf{K}}^{-1} + \theta\overline{\mathbf{K}}^{-1}\mathbf{A}\overline{\mathbf{K}}^{-1}\right)}{\theta(\theta+1)(\theta-2)}.\end{aligned} \tag{34}$$

Response Moments - 1

- The complete response vector is

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{f}. \tag{35}$$

- In many practical problems only few elements of $\mathbf{u}$ or linear combinations of some elements of $\mathbf{u}$ may be of interest. Therefore, we are interested in the quantity

$$\mathbf{y} = \mathbf{R}\mathbf{u} = \mathbf{R}\mathbf{K}^{-1}\mathbf{f};\ \mathbf{R} \in \mathbb{R}^{r\times n} \tag{36}$$

- The matrix $\mathbf{R}$ can be also selected to 'extract' other physical quantities such as the stress components within one element or a group of elements.

Response Moments - 2

- The mean of $\mathbf{y}$:

$$\bar{\mathbf{y}} = \mathrm{E}\left[\mathbf{y}\right] = \mathrm{E}\left[\mathbf{R}\mathbf{K}^{-1}\mathbf{f}\right] = \mathbf{R}\,\mathrm{E}\left[\mathbf{K}^{-1}\right]\mathbf{f} = \frac{\alpha}{\theta}\mathbf{R}\overline{\mathbf{K}}^{-1}\mathbf{f}. \quad (37)$$

- The complete covariance matrix of $\mathbf{y}$:

$$\begin{aligned}\mathrm{cov}\left(\mathbf{y},\mathbf{y}\right) &= \mathrm{E}\left[(\mathbf{y}-\bar{\mathbf{y}})(\mathbf{y}-\bar{\mathbf{y}})^T\right] = \mathrm{E}\left[\mathbf{y}\mathbf{y}^T\right] - \bar{\mathbf{y}}\bar{\mathbf{y}}^T \\ &= \mathbf{R}\,\mathrm{E}\left[\mathbf{K}^{-1}\mathbf{f}\mathbf{f}^T\mathbf{K}^{-1}\right]\mathbf{R}^T - \bar{\mathbf{y}}\bar{\mathbf{y}}^T \\ &= \frac{\alpha^2\mathrm{Trace}\left(\mathbf{f}\mathbf{f}^T\overline{\mathbf{K}}^{-1}\right)\mathbf{R}\overline{\mathbf{K}}^{-1}\mathbf{R}^T + \theta(\theta+2)\bar{\mathbf{y}}\bar{\mathbf{y}}^T}{\theta(\theta+1)(\theta-2)}.\end{aligned} \quad (38)$$

Steps for complete analysis

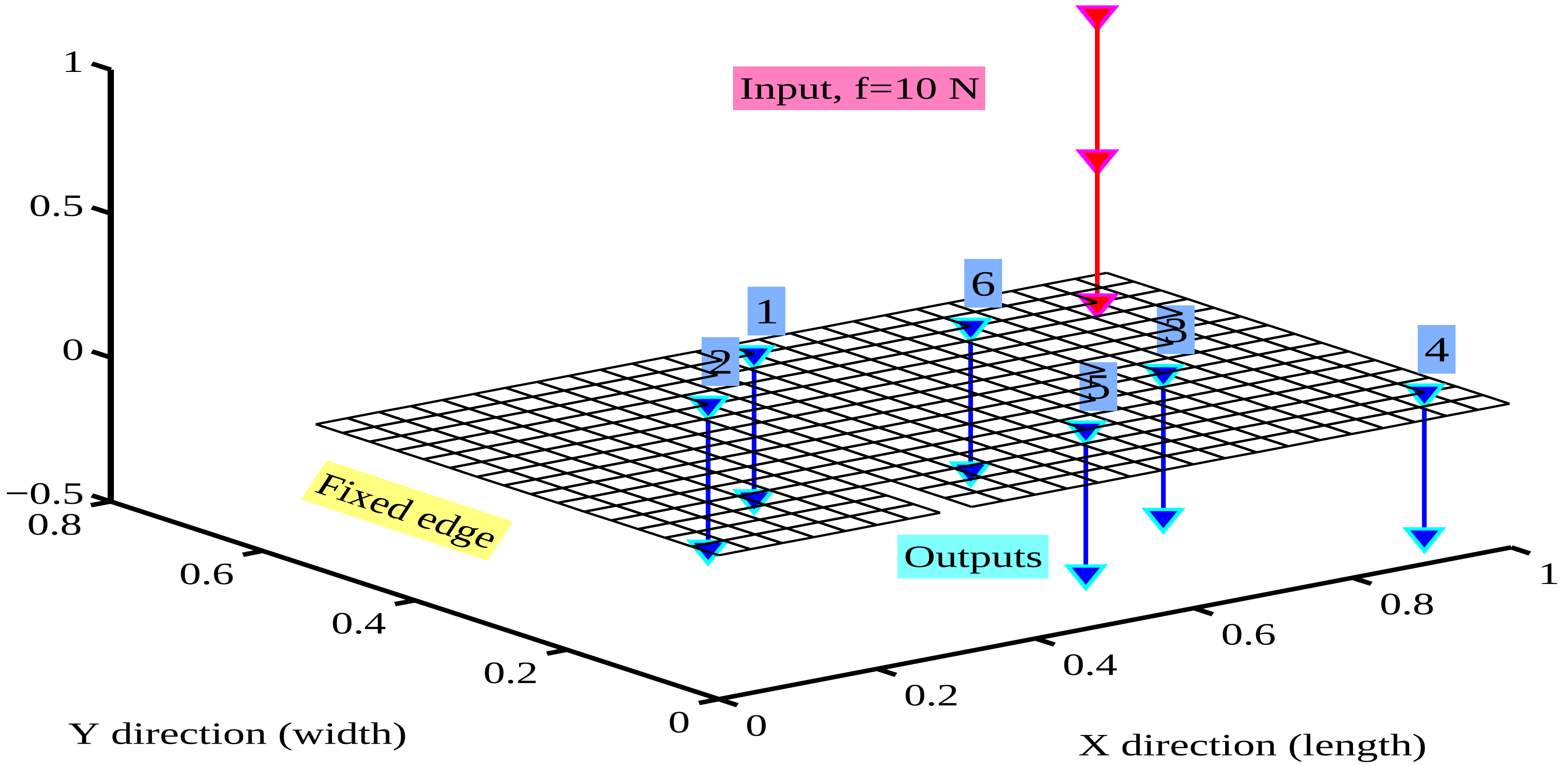
- Obtain normalized standard deviation
$$\delta_G^2 = \frac{\mathrm{E}\left[\| \mathbf{K} - \mathrm{E}[\mathbf{K}] \|_F^2\right]}{\| \mathrm{E}[\mathbf{K}] \|_F^2} = \frac{\mathrm{Trace}\left(\mathrm{cov}\left(\mathrm{vec}\left(\mathbf{K}\right)\right)\right)}{\mathrm{Trace}\left(\overline{\mathbf{K}}^2\right)}.$$

- Calculate the constants
$$\theta = \frac{1}{\delta_K^2}\left\{1 + \{\mathrm{Trace}\,(\overline{\mathbf{K}})\}^2/\mathrm{Trace}\left(\overline{\mathbf{K}}^2\right)\right\} - (n+1),$$
$p = n + 1 + \theta$, $\alpha = \sqrt{\theta(n+1+\theta)}$ and $\mathbf{\Sigma} = \overline{\mathbf{K}}/\alpha$.

- The mean: $\bar{\mathbf{y}} = \frac{\alpha}{\theta}\mathbf{R}\overline{\mathbf{K}}^{-1}\mathbf{f}$.

- The covariance:
$$\mathrm{cov}\left(\mathbf{y},\mathbf{y}\right) = \frac{\alpha^2 \mathrm{Trace}\left(\mathbf{f}\mathbf{f}^T\overline{\mathbf{K}}^{-1}\right)\mathbf{R}\overline{\mathbf{K}}^{-1}\mathbf{R}^T + \theta(\theta+2)\bar{\mathbf{y}}\bar{\mathbf{y}}^T}{\theta(\theta+1)(\theta-2)}.$$

Example: A cantilever Plate



A steel cantilever plate with a slot; $\bar{E} = 200 \times 10^9$N/m^2, $\bar{\mu} = 0.3$, $\bar{t} = 7.5$mm, $L_x = 1.2$m, $L_y = 0.8$m; 25×15 elements resulting $n = 1200$.

Stochastic Properties

- The Young's modulus, Poissons ratio and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\left(1 + \epsilon_E f_1(\mathbf{x})\right) \quad (39)$$

$$\mu(\mathbf{x}) = \bar{\mu}\left(1 + \epsilon_\mu f_2(\mathbf{x})\right) \quad (40)$$

$$\text{and} \quad t(\mathbf{x}) = \bar{t}\left(1 + \epsilon_t f_4(\mathbf{x})\right) \quad (41)$$

- The strength parameters are: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.10$, and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x}), i = 1, 2, 3$ are delta-correlated homogenous Gaussian random fields.

Response calculation

- The value of δ_k (calculated using a 5000-sample Monte Carlo simulation of the random fields) is obtained as $\delta_K = 0.2616$.
- From the 1200×1200 stiffness matrix we obtain

$$\mathrm{Trace}\left(\overline{\mathbf{K}}\right) = 5.5225{\times}10^{9} \quad \text{and} \quad \mathrm{Trace}\left(\overline{\mathbf{K}}^2\right) = 9.6599{\times}10^{16}.$$

and

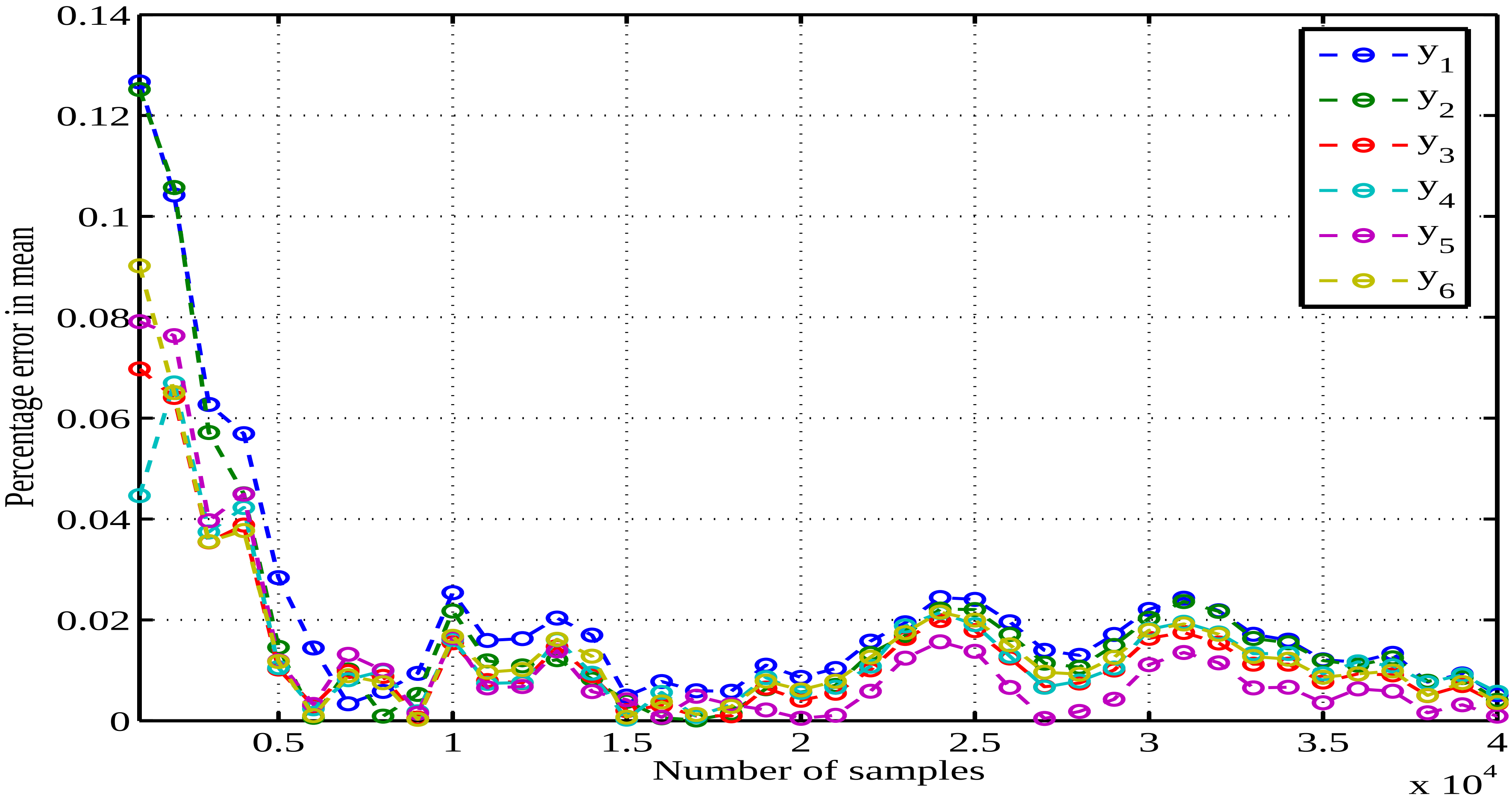
$$\theta = 3.4274 \times 10^3 \quad \text{and} \quad \alpha = 3.9827 \times 10^3.$$

Comparison of results

The mean and standard deviation of the response vector. The numbers in the parenthesis correspond to the percentage error in the Monte Carlo Simulation (with 1000 samples) results with respect to the exact analytical results.

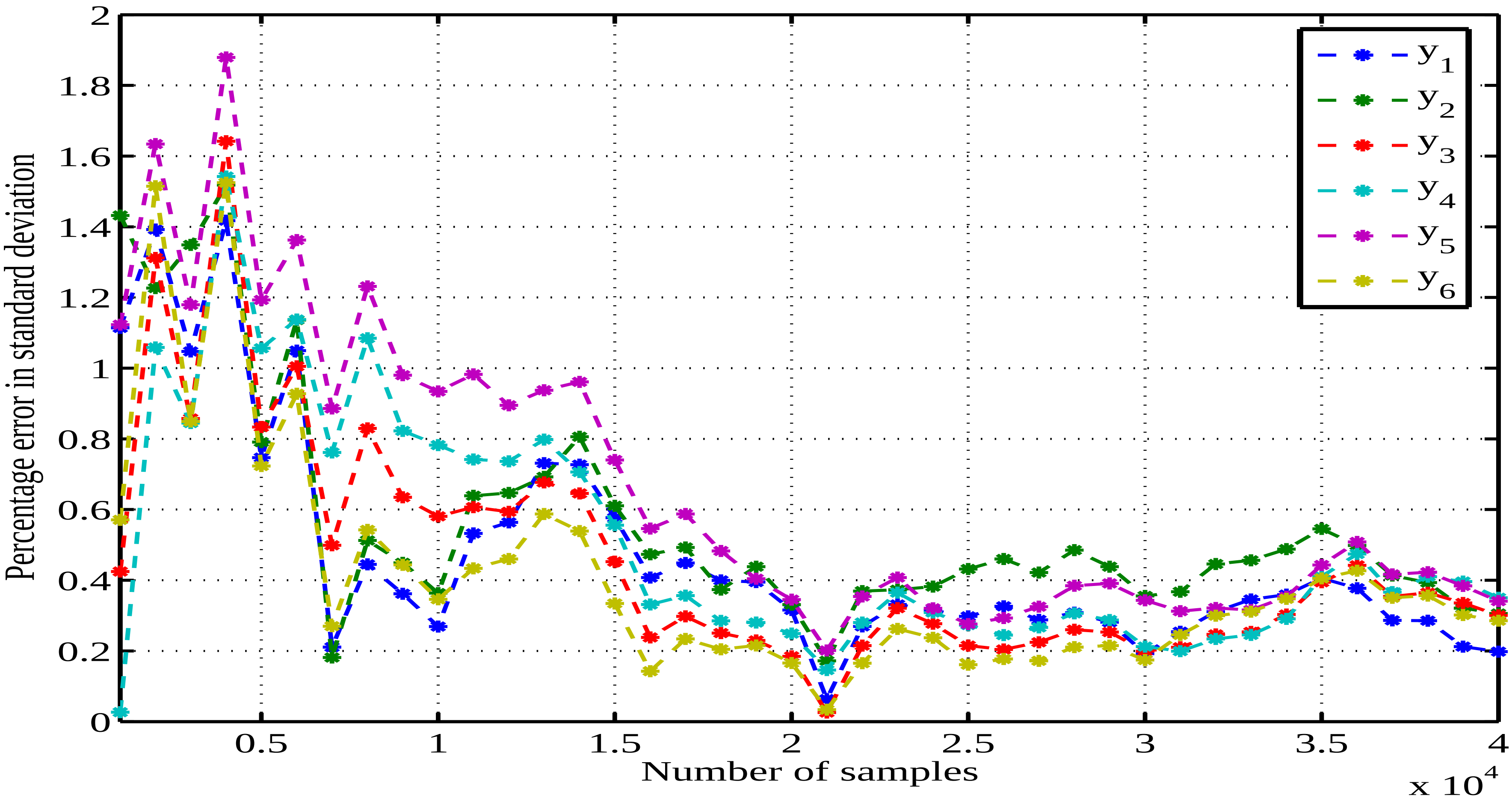
Response quantity	Analytical mean (mm)	MSC mean (mm)	Analytical standard deviation (mm)	MSC standard deviation (mm)
$y_1 = u_{112}$	5.5058	5.5178 (0.218 %)	0.1438	0.1459 (1.436 %)
$y_2 = u_{325}$	2.6420	2.6475 (0.208 %)	0.0734	0.0740 (0.818 %)
$y_3 = u_{658}$	10.2265	10.2485 (0.216 %)	0.2537	0.2561 (0.972 %)
$y_4 = u_{1045}$	12.6039	12.6317 (0.221 %)	0.3294	0.3313 (0.570 %)
$y_5 = u_{868}$	5.9608	5.9725 (0.197 %)	0.1586	0.1604 (1.155 %)
$y_6 = u_{205}$	10.2951	10.3169 (0.212 %)	0.2507	0.2547 (1.580 %)

Convergence of results - 1



Variation of the relative error in MCS mean with respect to the number of samples.

Convergence of results - 2



Variation of the relative error in MCS standard deviation with respect to the number of samples.

Summary - 1

- The probabilistic characterization of the response of linear stochastic systems requires inverse of a real symmetric random matrix (an outstanding problem for more than four decades).
- An exact and simple closed-form expression of the joint probability density function of the elements of the inverse of a symmetric random matrix is derived.
- A matrix itself is treated like a variable, as opposed to view it as a collection of many variables. This outlook significantly simples the calculation of the Jacobian involved in the non-linear matrix transformation.

Summary - 2

- The random matrices considered are in general non-Gaussian and of arbitrary dimensions.
- Moments of the response do not require a series/perturbation/PC expansion.
- The numerical implementation is straight-forward and non-intrusive.

Open Issues

- Any real matrix pdf can be used for $p_{\mathbf{K}}(\mathbf{K})$ and the pdf of $\mathbf{H} = \mathbf{K}^{-1}$ can be obtained. However, obtaining the response pdf (requires further transformation) or response moments from $p_{\mathbf{H}}(\mathbf{H})$ is a not trivial task.
- Selecting a matrix variate pdf to matrix data is a challenging task itself (topic of my Thursdays paper).
- The inverse of a complex symmetric random matrix cannot be obtained easily from the proposed formulation.
- As a result, it is applicable to static or undamped systems only and therefore is of somewhat limited applicability.

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