Can the Spatial Distribution of Damping be Measured?

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Outline of the Talk

- Introduction
- Models of damping: Viscous and non-viscous damping
- Complex frequencies and modes
- Theory of damping identification
- Numerical Results
- Experimental Results
- Conclusions

Viscously Damped Systems

Equations of motion:

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \mathbf{0}. \qquad (1)$$

Approximate Complex frequency and modes:

$$\lambda_j \approx \pm\omega_j + \mathrm{i}\, C'_{jj}/2, \quad \mathbf{z}_j \approx \mathbf{x}_j + \mathrm{i} \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{\omega_j C'_{kj}}{(\omega_j^2 - \omega_k^2)} \mathbf{x}_k.$$

ω_j: Undamped natural frequency, $\mathbf{x}_k$: Undamped modes

$C'_{kl} = \mathbf{x}_k^T \mathbf{C} \mathbf{x}_l$ are the elements of the damping matrix in modal coordinates.

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Some General Questions of Interest

1. From experimentally determined complex modes can one identify the *underlying damping mechanism*? Is it viscous or non-viscous? Can the correct model parameters be found experimentally?
2. Is it possible to establish experimentally the *spatial distribution* of damping?
3. Is it possible that more than one damping model with corresponding correct sets of parameters may represent the system response equally well, so that the identified model becomes *non-unique*?
4. Does the selection of damping model matter from an engineering point of view? Which aspects of behaviour are *wrongly predicted* by an incorrect damping model?

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Non-viscously Damped Systems

Equations of motion:

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \int_{-\infty}^{t} \boldsymbol{\mathcal{G}}(t-\tau)\,\dot{\mathbf{y}}(\tau)\,\mathrm{d}\tau + \mathbf{K}\mathbf{y}(t) = \mathbf{0} \qquad (2)$$

$\boldsymbol{\mathcal{G}}(t)$ is $N \times N$ matrix of kernel functions.
Approximate Complex frequency and modes:

$$\lambda_j \approx \pm\omega_j + \mathrm{i}\, G'_{jj}(\pm\omega_j)/2, \qquad \mathbf{z}_j \approx \mathbf{x}_j + \mathrm{i} \sum_{\substack{k=1\\k\neq j}}^{N} \frac{\omega_j G'_{kj}(\omega_j)}{(\omega_j^2 - \omega_k^2)} \mathbf{x}_k.$$

$\mathbf{G}(\omega)$ is the Fourier transform of $\boldsymbol{\mathcal{G}}(t)$, $G'_{kl}(\omega_j) = \mathbf{x}_k^T \mathbf{G}(\omega_j) \mathbf{x}_l$ in modal coordinates.

Non-viscous Damping Identification

- Damping model used for fitting:
  $\mathcal{G}(t) = \mu e^{-\mu t}\,\mathbf{C}$
- Determine the complex natural frequencies, $\hat{\lambda}_j$, and complex mode shapes, $\hat{\mathbf{z}}_j$, from a set of measured transfer functions. Denote $\hat{\mathbf{\Lambda}} = \mathrm{diag}(\lambda_j) \in \mathbb{C}^{m\times m}$, $\hat{\mathbf{Z}} = [\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2, \cdots \hat{\mathbf{z}}_m] \in \mathbb{C}^{N\times m}$.
- Set $\hat{\mathbf{\Omega}} = \Re(\hat{\mathbf{\Lambda}})$, $\hat{\mathbf{U}} = \Re\left[\hat{\mathbf{Z}}\right]$ and $\hat{\mathbf{V}} = \Im\left[\hat{\mathbf{Z}}\right]$.
- Obtain the relaxation parameter $\hat{\mu} = \dfrac{\hat{\omega}_1 \hat{\mathbf{v}}_1^T \mathbf{M} \hat{\mathbf{v}}_1}{\hat{\mathbf{v}}_1^T \mathbf{M} \hat{\mathbf{u}}_1}$.

Non-viscous Damping Identification

- Obtain undamped modal matrix $\hat{\mathbf{X}} = \hat{\mathbf{U}} - \frac{1}{\hat{\mu}}\left[\hat{\mathbf{V}}\hat{\mathbf{\Omega}}\right]$.
- Evaluate the matrix $\widetilde{\mathbf{B}} = \left[\hat{\mathbf{X}}^T\hat{\mathbf{X}}\right]^{-1}\hat{\mathbf{X}}^T\hat{\mathbf{V}}$.
- From the $\widetilde{\mathbf{B}}$ matrix, get $\mathbf{C}' = \widetilde{\mathbf{B}}\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}^2\widetilde{\mathbf{B}}\hat{\mathbf{\Omega}}^{-1}$ and $\mathrm{diag}(\mathbf{C}') = \Im(\hat{\mathbf{\Lambda}})$
- Use $\mathbf{C} = \left[\left(\hat{\mathbf{X}}^T\hat{\mathbf{X}}\right)^{-1}\hat{\mathbf{X}}^T\right]^T \mathbf{C}' \left[\left(\hat{\mathbf{X}}^T\hat{\mathbf{X}}\right)^{-1}\hat{\mathbf{X}}^T\right]$ to get the coefficient matrix in physical coordinates.

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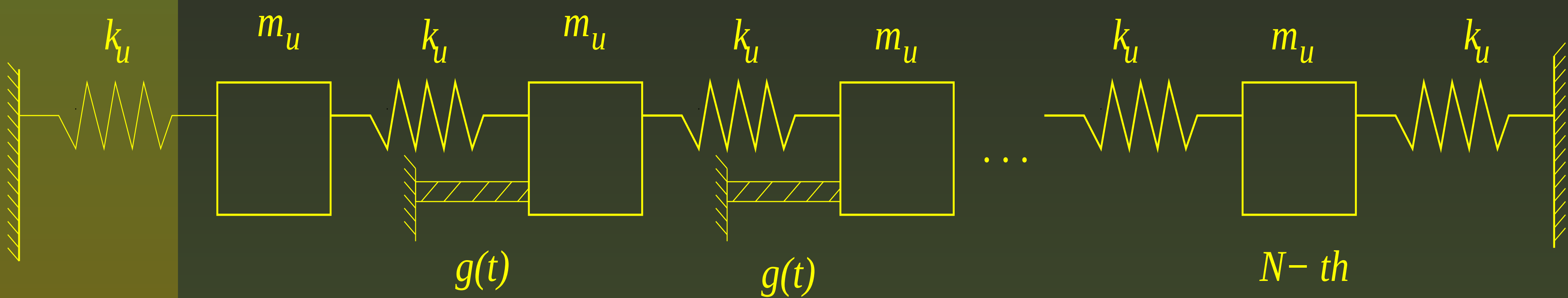
Simulation Example



Linear array of N spring-mass oscillators, $N = 30$, $m_u = 1\,Kg$, $k_u = 4 \times \times 10^3 N/m$.

Simplest case: the kernel functions have the form

$$\boldsymbol{\mathcal{G}}(t) = \mathbf{C}\, g(t) \tag{3}$$

Here $g(t)$ is some damping function and **C** is a positive definite constant matrix.

Models of Non-viscous Damping

- MODEL 1 (exponential): $g^{(1)}(t) = \mu_1 e^{-\mu_1 t}$
- MODEL 2 (Gaussian): $g^{(2)}(t) = 2\sqrt{\frac{\mu_2}{\pi}} e^{-\mu_2 t^2}$

The damping models are normalized such that the damping functions have unit area when integrated to infinity, *i.e.*,

$$\int_0^\infty g^{(j)}(t)\,\mathrm{d}t = 1.$$

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Characteristic Time Constant

For each damping function the *characteristic time constant* is defined via the first moment of $g(t)$ as

$$\theta = \int_0^\infty t\, g(t)\, \mathrm{d}t.$$

Express θ as: $\theta = \gamma\, T_{min}$.
The constant γ is the *non-dimensional characteristic time constant* and T_{min} is the minimum time period.
Expect:

$$\gamma \ll 1 : \quad \text{near viscous}$$

$$\gamma \to \mathcal{O}(1) : \quad \text{strongly non-viscous}$$

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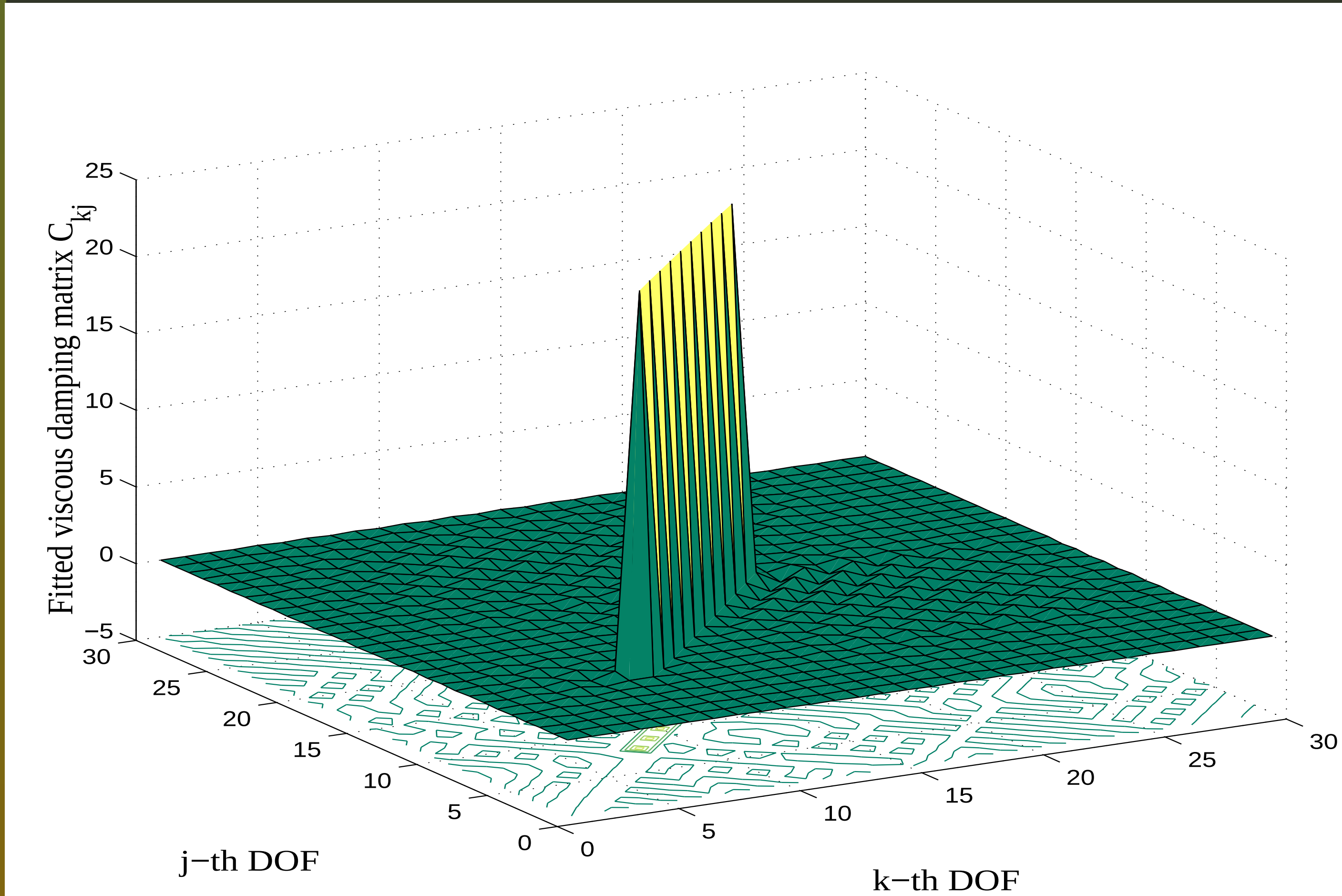
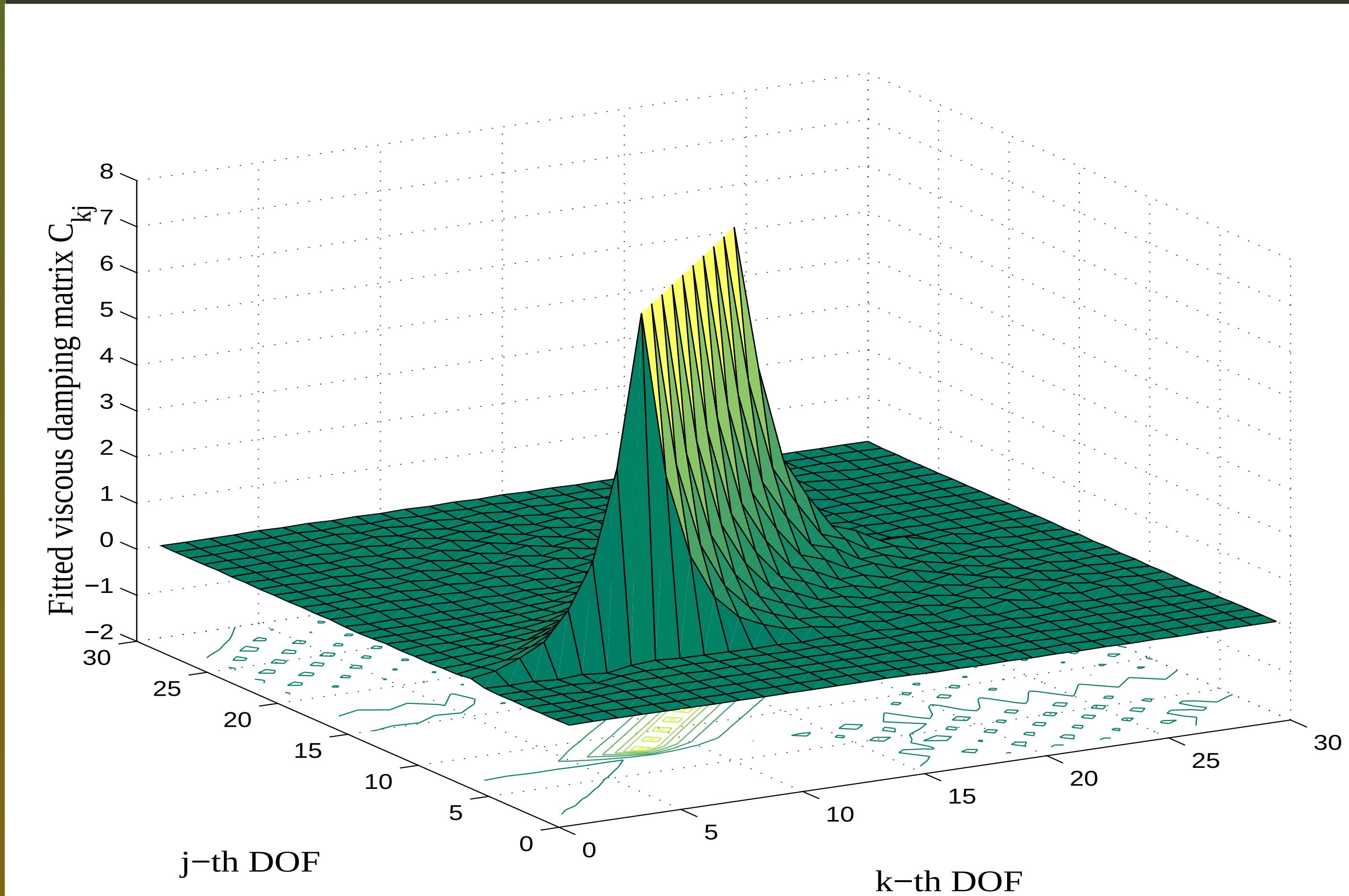
Viscous Damping Identification



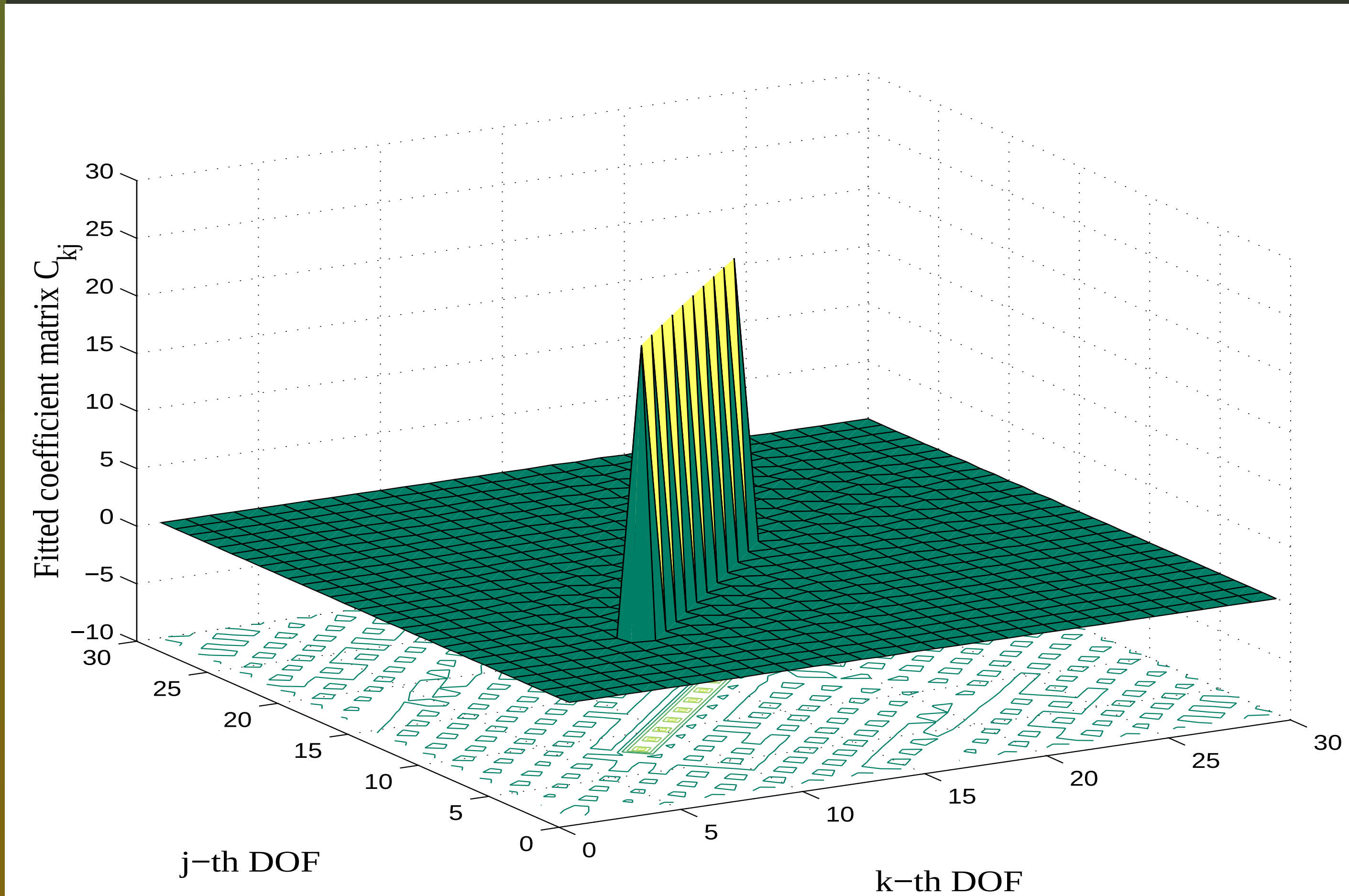
Fitted viscous damping matrix for $\gamma = 0.02$, damping model 2

Viscous Damping Identification



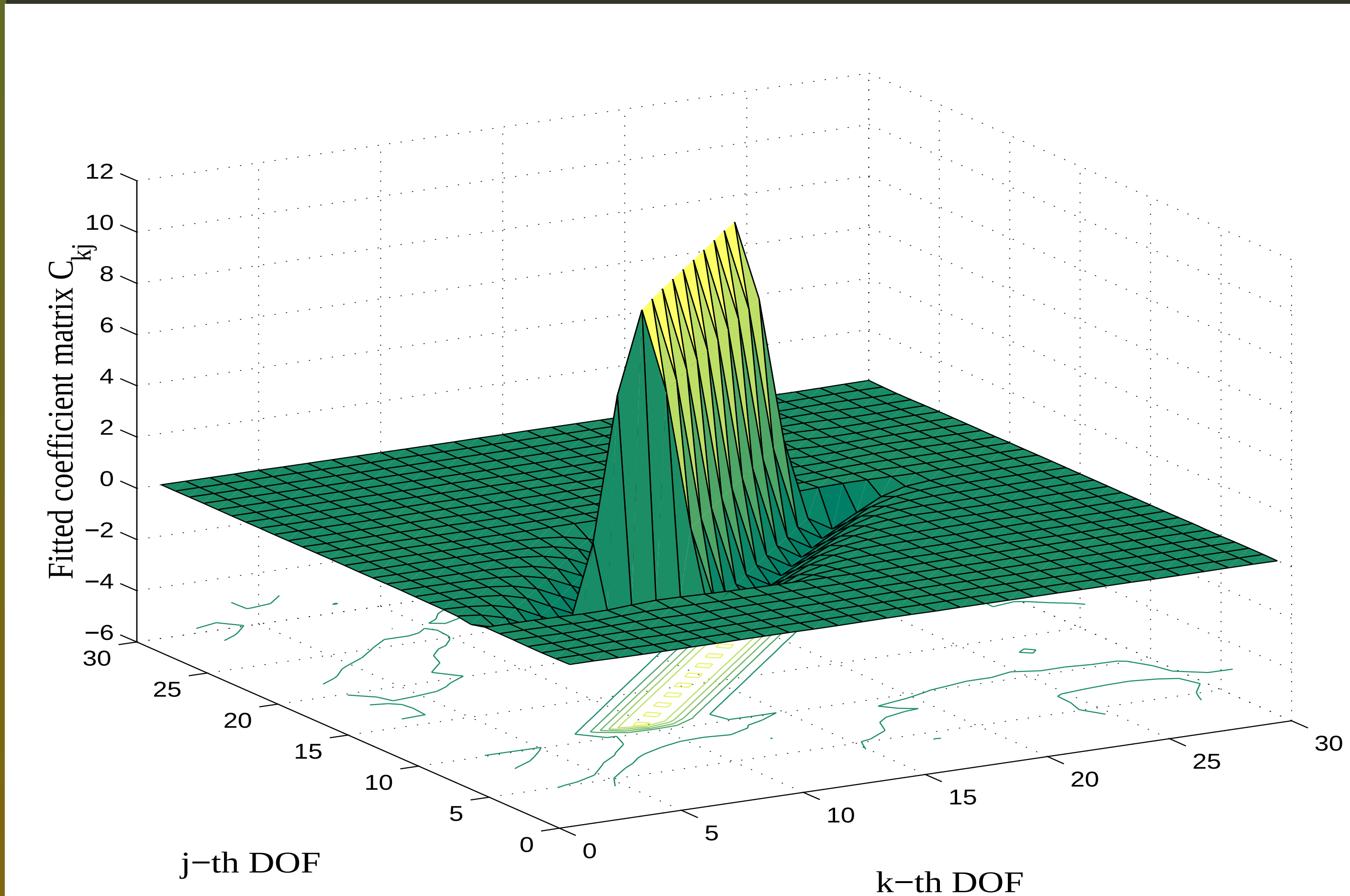
Fitted viscous damping matrix for $\gamma = 0.5$, damping model 1

Non-viscous Damping Identification



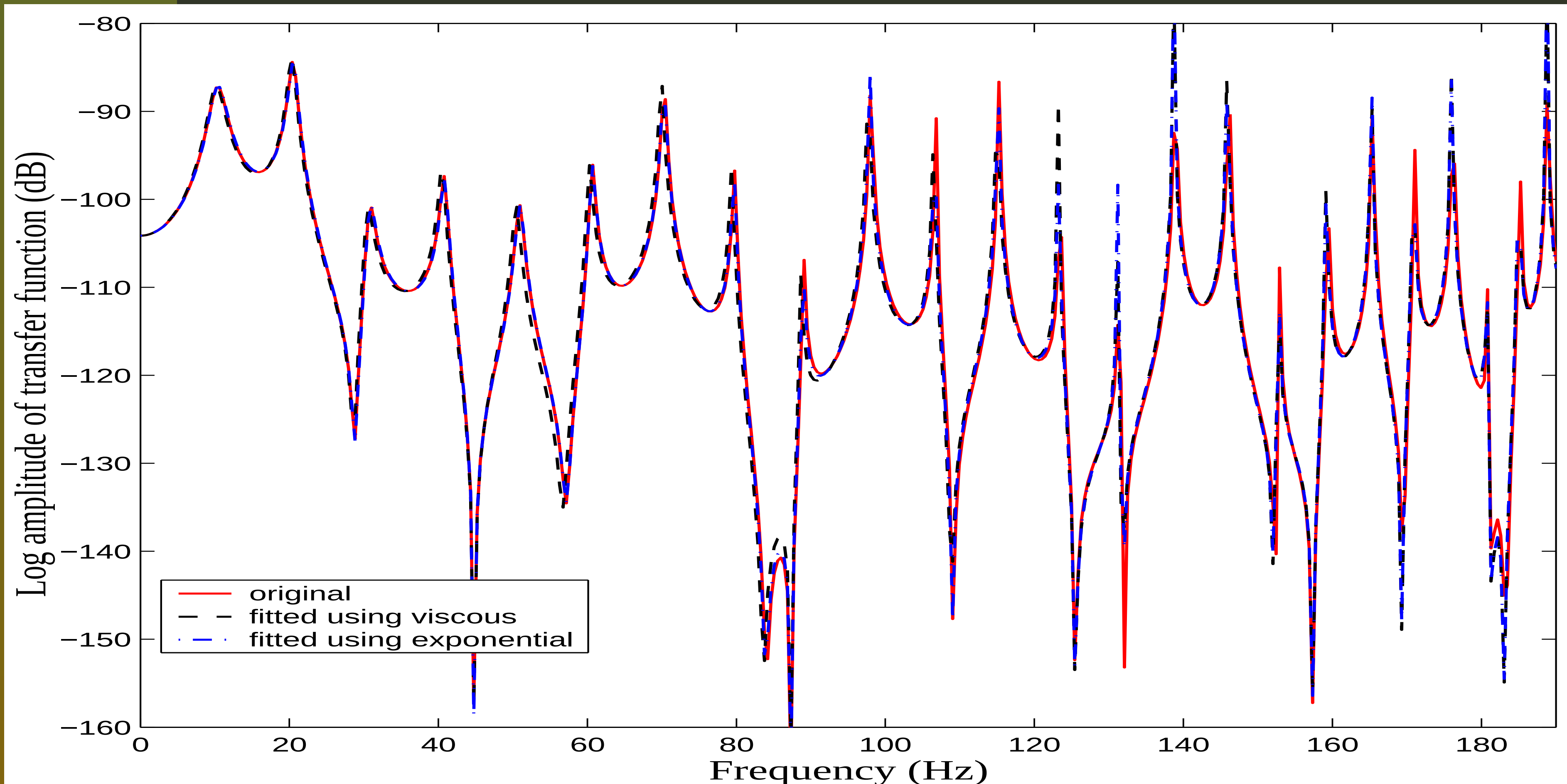
Fitted coefficient matrix of exponential model for $\gamma = 0.5$, damping model 1; $\gamma_{\text{fit}} = 0.49903$

Non-viscous Damping Identification



Fitted coefficient matrix of exponential model for $\gamma = 0.5$, damping model 2; $\gamma_{\rm fit} = 0.6366$

Non-viscous Damping Identification



Original and fitted transfer function $H_{kj}(\omega)$ $(k = 1, j = 24\ )$ for $\gamma = 0.5$, damping model 2

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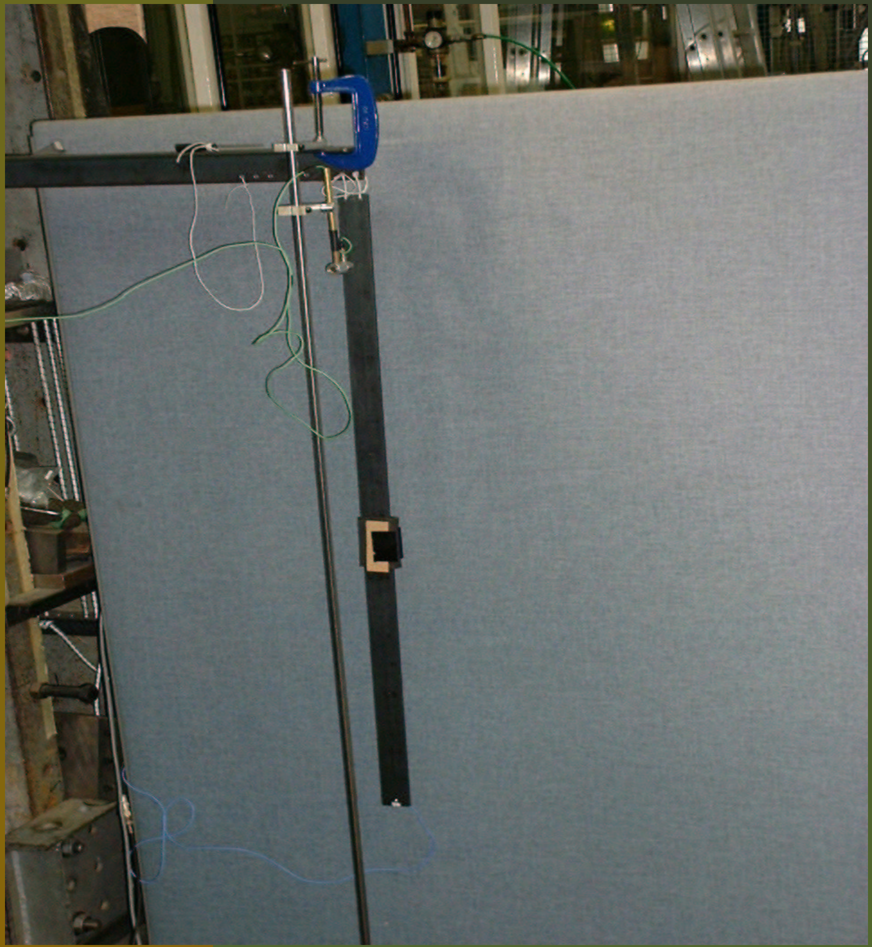
Experimental Setup

Damped free-free beam:

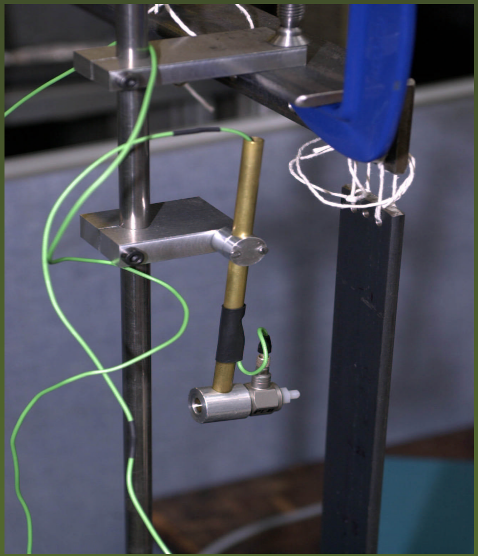
$L = 1$m, width = 39.0 mm

thickness = 5.93 mm



Clamped damping mechanism

Instrumented hammer for impulse input



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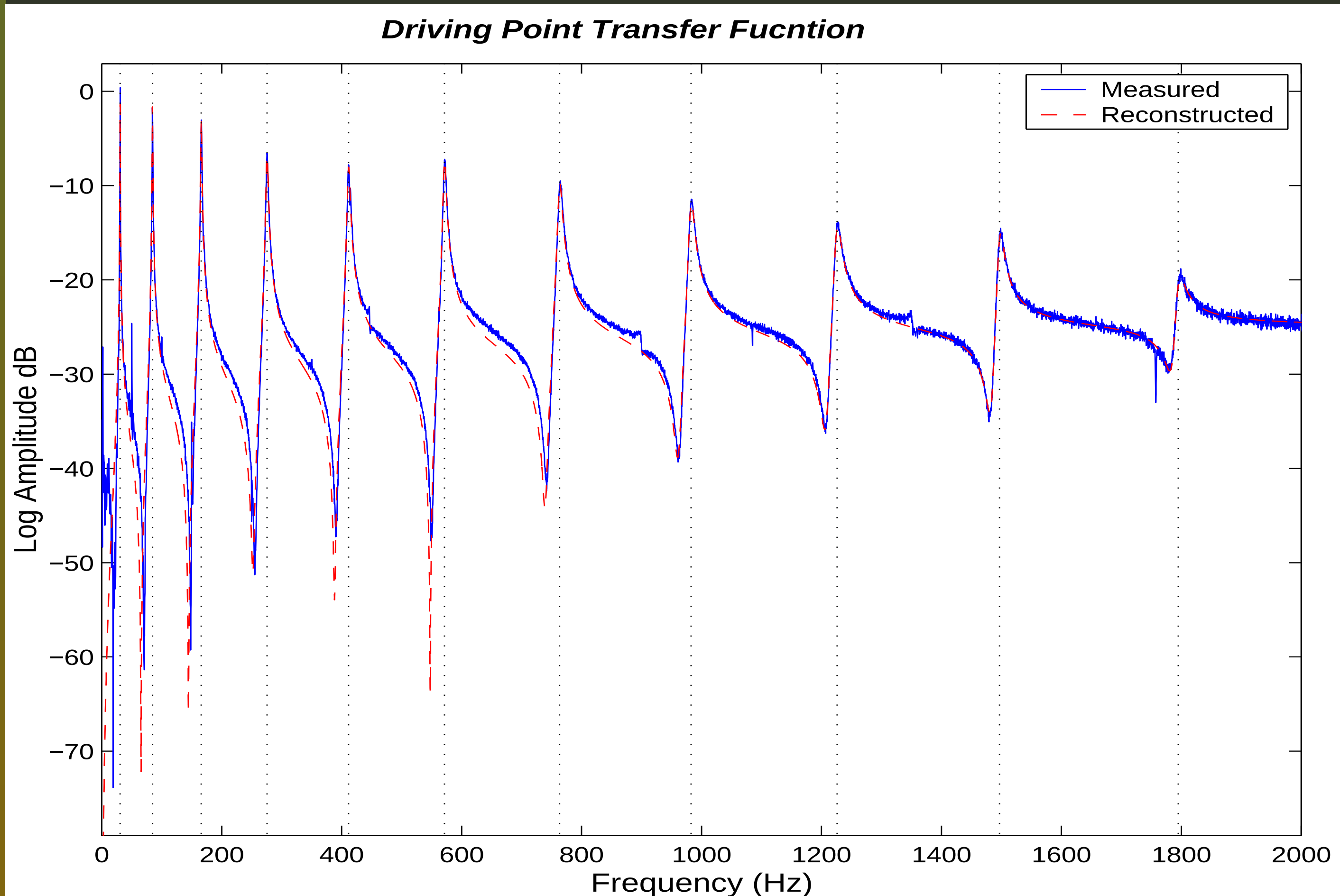
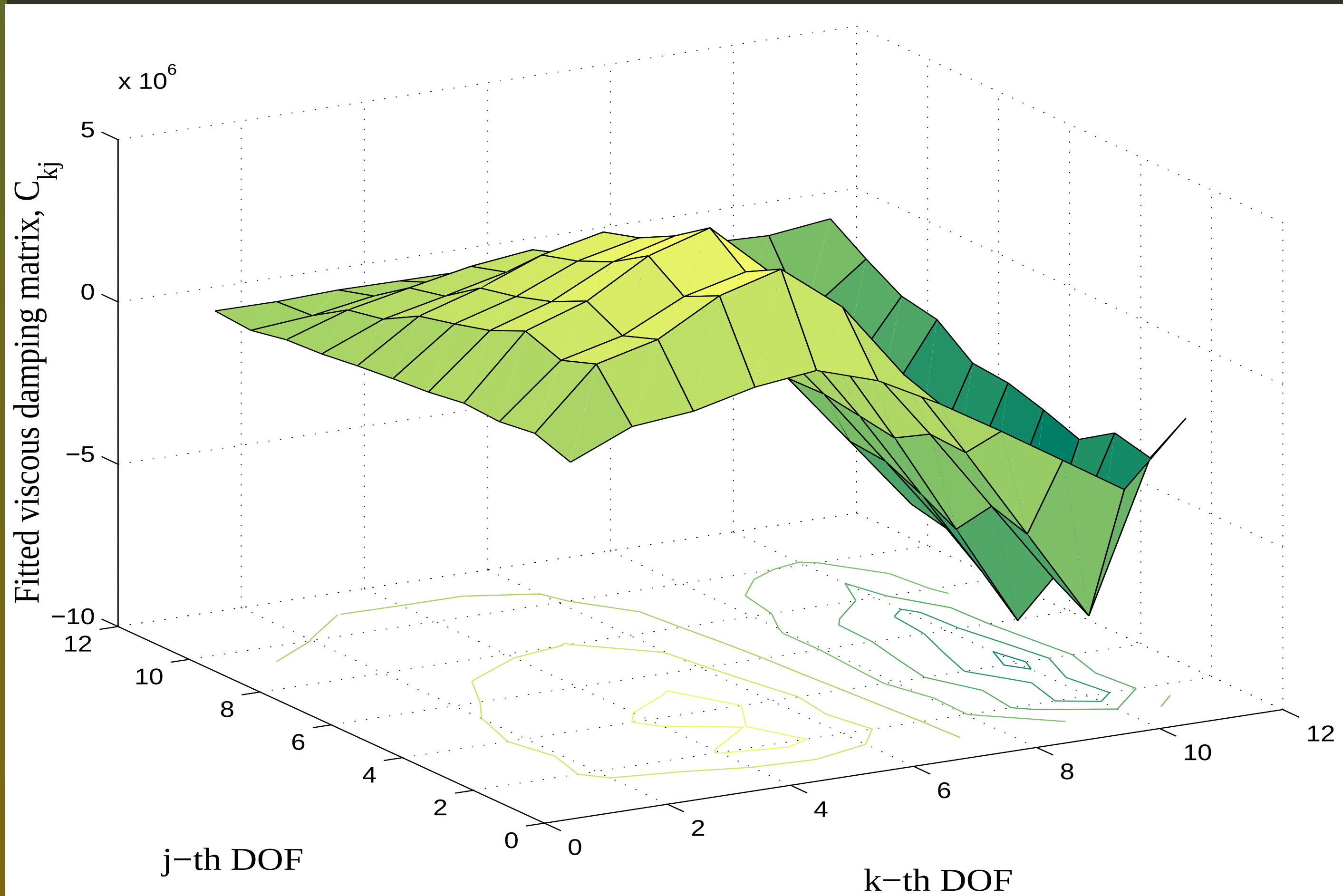
Measured Transfer Functions



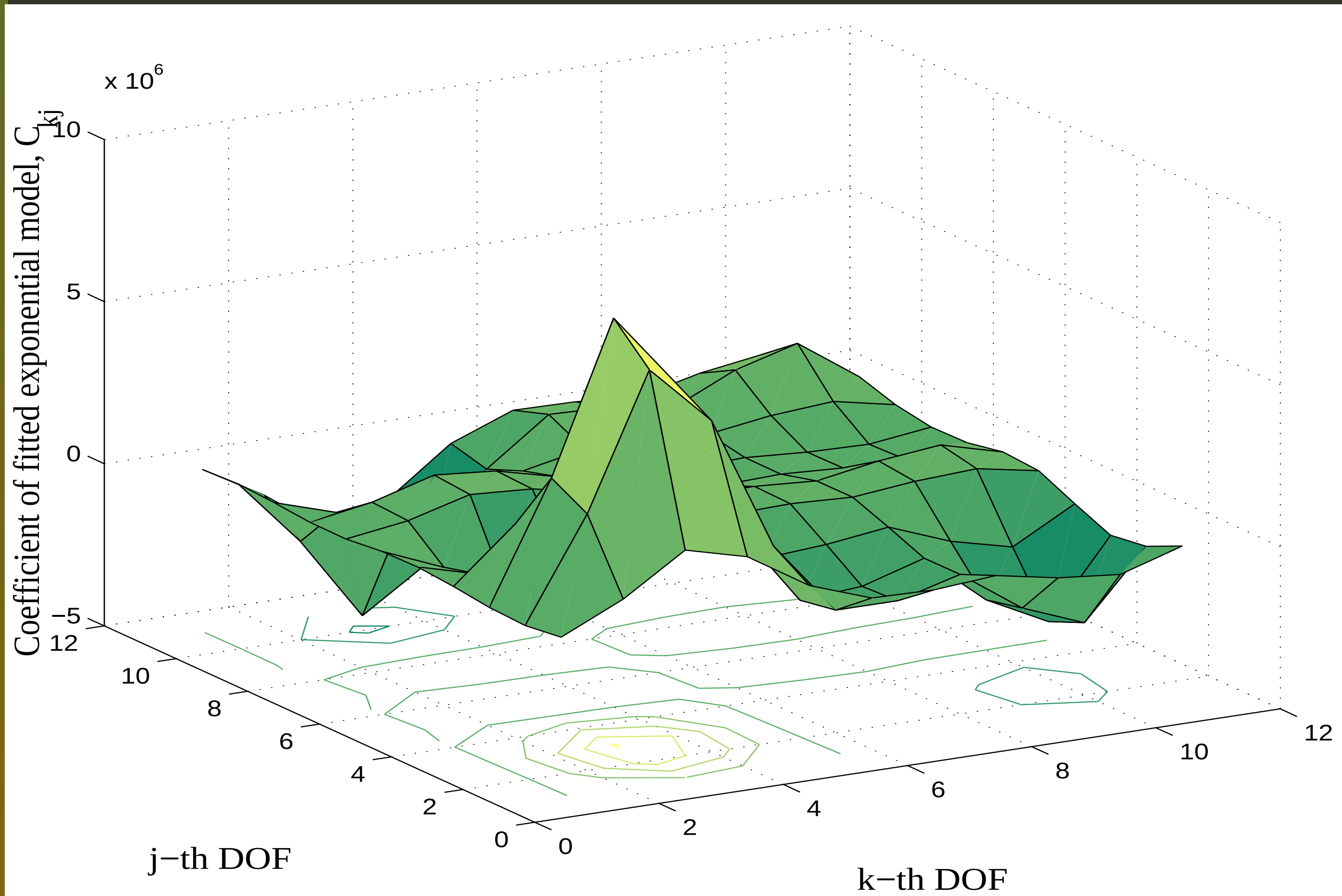
Measured and fitted transfer function of the beam

Viscous Damping Fitting



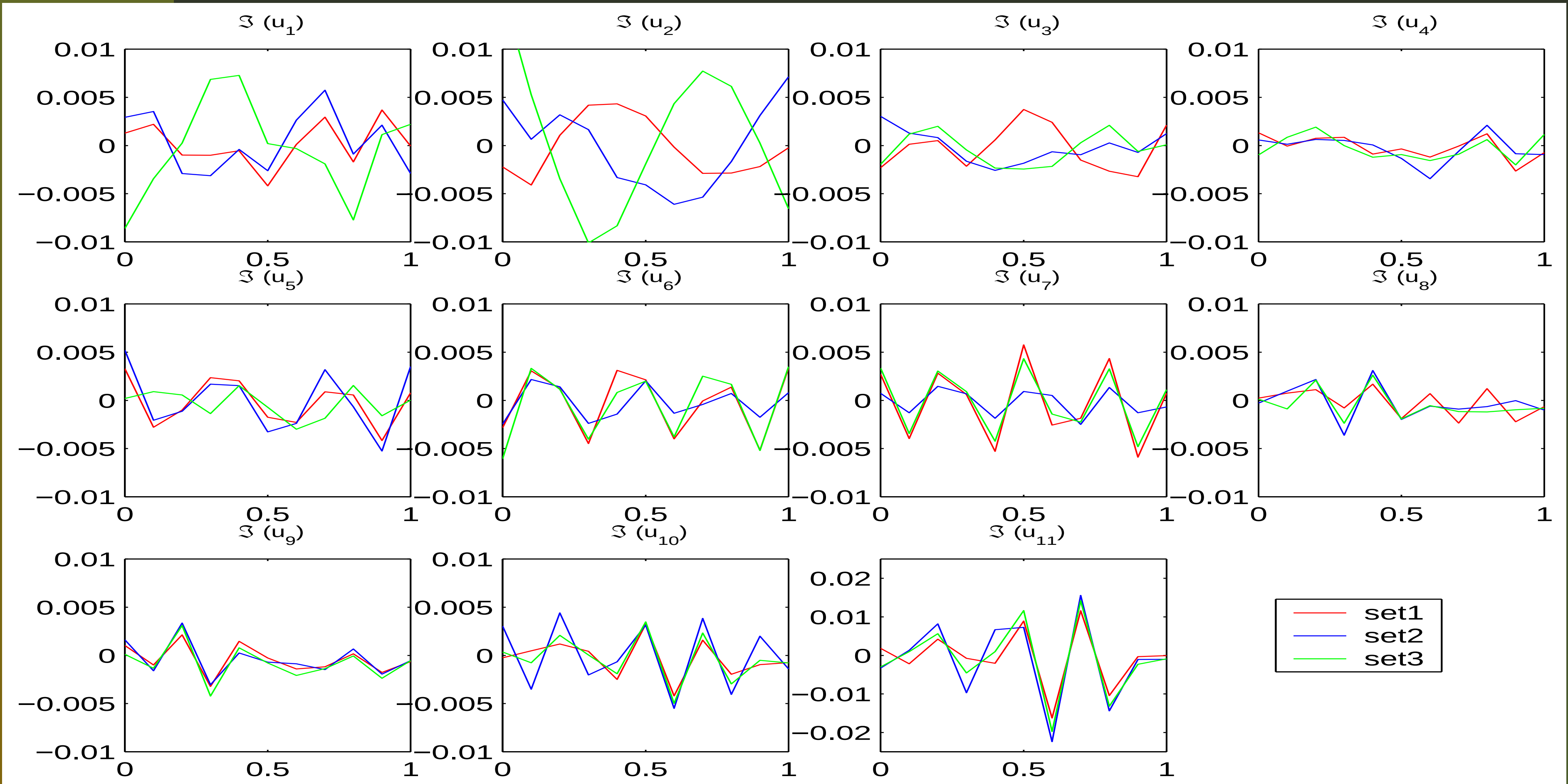
Fitted viscous damping matrix for damping between 4-5 nodes

Non-viscous Damping Fitting



Fitted coefficient matrix of exponential model for damping between 4-5 nodes; $\gamma_{\text{fit}} = 1.3061$

Measured Complex Modes



Imaginary parts of the identified complex modes

Summary and Conclusions

- A method is proposed to identify a non-proportional non-viscous damping model in vibrating systems from complex modes and natural frequencies.
- Numerical results show that the method generally predicts the spatial location of the damping with good accuracy.
- If the fitted damping model is wrong, the procedure yields a non-physical result by fitting a non-symmetric coefficient matrix. That is, the procedure gives an indication that a wrong model is selected for fitting.

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Summary and Conclusions

- It is possible that more than one damping model with corresponding correct sets of parameters may represent the system response equally well. This means that by measuring transfer functions it is not possible to identify the governing damping mechanism uniquely.
- Different damping models can be fitted with the identified poles and residues of the transfer functions so that they are approximated accurately by all models.
- *Can the spatial distribution of damping be measured?* — Yes! – provided the complex modes are known with sufficient accuracy.