THE NATURE OF RANDOM SYSTEM MATRICES IN STRUCTURAL DYNAMICS

S. Adhikari

Department of Engineering
University of Cambridge
Trumpington Street
Cambridge CB2 1PZ (U.K.)

May 2001

Outline of the Talk

- Introduction
- System randomness: Probabilistic approach
- Parametric and non-parametric modeling
- Maximum entropy principle
- Gaussian Orthogonal Ensembles (GOE)
- Random rod example
- Conclusions

Random Systems

Equations of motion:

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \mathbf{p}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are respectively the mass, damping and stiffness matrices, $\mathbf{y}(t)$ is the vector of generalized coordinates and $\mathbf{p}(t)$ is the applied forcing function.

We consider randomness of the system matrices as

$$\begin{aligned} \mathbf{M} &= \overline{\mathbf{M}} + \delta\mathbf{M} \\ \mathbf{C} &= \overline{\mathbf{C}} + \delta\mathbf{C} \\ \text{and} \quad \mathbf{K} &= \overline{\mathbf{K}} + \delta\mathbf{K}. \end{aligned} \tag{2}$$

Here, $\overline{(\bullet)}$ and $\delta(\bullet)$ denotes the nominal (deterministic) and random parts of $(\bullet)$ respectively.

Probabilistic Approach

1. Parametric modeling:
   The Stochastic Finite Element Method (SFEM)

   - Probability density function $p_{\mathbf{q}}(\mathbf{q})$ of random vectors $\mathbf{q} \in \mathbb{R}^l$ have to be constructed from the *random fields* describing the geometry, boundary conditions and constitutive equations by discretization of the fields.

   - Mappings $\mathbf{q} \rightarrow \mathbf{G}(\bar{\mathbf{q}} + \mathbf{q}); \mathbb{R}^l \rightarrow \mathbb{R}^{N \times N}$, where $\mathbf{G}$ denotes $\mathbf{M}, \mathbf{C}$ or $\mathbf{K}$, have to be explicitly constructed. For an analytical approach, this step often requires linearization of the functions.

   - For Monte-Carlo-Simulation:
     Re-assembly of the element matrices is required for each sample.

2. Non-parametric modeling:
   Direct construction of pdf of $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ without having to determine the uncertain local parameters of a FE model.

Maximum Entropy Principle

What is entropy?

A *measure* of uncertainty.

For a continuous random variable $x \in \mathcal{D}$, Shannon's Measure of Entropy (1948):

$$S(p(x)) = -\int_{\mathcal{D}} p(x) \ln p(x) \mathrm{d}x$$

Constraint:

$$\int_{\mathcal{D}} p(x) \mathrm{d}x = 1$$

Philosophy of Jayne's Maximum Entropy Principle (1957):

- Speak the truth and nothing but the truth
- Make use of all the information that is given and scrupulously avoid making assumptions about information that is not available.

Maximum Entropy Principle

Only mean is known:

Additional constraint:

$$\int_{\mathcal{D}} xp(x)\mathrm{d}x = m$$

Construct the Lagrangian as

$$\begin{aligned}\mathcal{L} =& -\int_{\mathcal{D}} p(x)\ln p(x)\mathrm{d}x - \lambda_0\left[\int_{\mathcal{D}} p(x)\mathrm{d}x - 1\right] \\ & - \lambda_1\left[\int_{\mathcal{D}} xp(x)\mathrm{d}x - m\right] \\ =& \int_{\mathcal{D}} g(p(x))\mathrm{d}x\end{aligned}$$

where

$$g(p(x)) = -p(x)\ln p(x) - \lambda_0 p(x) - \lambda_1 xp(x) + \lambda_0 + m\lambda_1 \tag{3}$$

Maximum Entropy Principle

From the calculus of variation, for $\delta\mathcal{L} = 0$ it is required that $g(p(x))$ must satisfy the Euler-Lagrange equation

$$\frac{\partial g(p(x))}{\partial p(x)} - \frac{\partial}{\partial x}\left[\frac{\partial g(p(x))}{\partial p(x)}\right] = 0 \qquad (4)$$

Substituting $g(p(x))$ from (3), equation (4) results

$$-\ln p(x) - 1 - \lambda_0 - \lambda_1 x + \lambda_1 = 0$$
$$\text{or} \quad p(x) = Ae^{-\lambda_1 x}$$

That is, *exponential* distribution.

A and λ_1 should be determined from the constraint equations. The analysis can be extended to vector valued random variables and random processes.

If mean is unknown then $p(x)$ is constant, ie, *uniform* distribution. This is also known as the *Laplace's principle of insufficient reason*.

Maximum Entropy Principle

Mean and standard deviation is known:

Additional constraint:

$$\int_{\mathcal{D}} (x-m)^2 p(x) \mathrm{d}x = \sigma^2$$

Following previous steps

$$p(x) = Ae^{-\lambda_1 x - \lambda_2 x^2} \tag{5}$$

That is, *Gaussian* distribution.

Soize Model (2000)

The probability density function of any system matrix (say $\mathcal{G}$) is defined as

$$p_{[\mathcal{G}]}([\mathcal{G}]) = \mathbb{I}_{\mathbb{M}_N^+(\mathbb{R})}([\mathcal{G}]) c_{\mathcal{G}} (\det[\mathcal{G}])^{\lambda_{\mathcal{G}}-1} \times \exp\left(-\frac{(N-1+2\lambda_{\mathcal{G}})}{2}\mathrm{Trace}(\mathcal{G})\right)$$

where

$$c_{\mathcal{G}} = \frac{(2\pi)^{-N(N-1)/4}\left(\frac{N-1+2\lambda_{\mathcal{G}}}{2}\right)^{N(N-1+2\lambda_{\mathcal{G}})/2}}{\left\{\prod_{l=1}^{N}\Gamma\left(\frac{(N-1+2\lambda_{\mathcal{G}})}{2}\right)\right\}}$$

The ‘dispersion’ parameter

$$\lambda_{\mathcal{G}} = \frac{1}{2\delta_{\mathcal{G}}^2}\left(1-\delta_{\mathcal{G}}^2(N-1)+\frac{(\mathrm{Trace}[\mathcal{G}])^2}{\mathrm{Trace}([\mathcal{G}^2])}\right)$$

and

$$\delta_{\mathcal{G}} = \left\{\frac{E\|[\mathcal{G}]-[\overline{\mathcal{G}}]\|}{\|[\overline{\mathcal{G}}]\|}\right\}^{1/2}$$

$\mathbb{I}_{\mathbb{M}_N^+(\mathbb{R})}([\mathcal{G}]) = 1$ if $[\mathcal{G}] \in \mathbb{M}_N^+(\mathbb{R})$ otherwise 0. Here $\mathbb{M}_N^+(\mathbb{R})$ is the subspace of $\mathbb{M}_N(\mathbb{R})$ constituted of all $N \times N$ *positive definite* symmetric real matrices.

Gaussian Orthogonal Ensembles (GOE)

1. The ensemble (say $\mathbf{H}$) is invariant under every transformation $\mathbf{H} \rightarrow \mathbf{W}^T\mathbf{H}\mathbf{W}$ where $\mathbf{W}$ is any orthogonal matrix.

2. The various elements $H_{jk}, k \leq j$ are statistically independent.

3. Standard deviation of diagonals are twice that of the off-diagonal terms, $\sigma_{H_{jj}} = 2\sigma_{H_{jk}} = \sigma$, $\forall j \neq k$, where σ is some constant.

The probability density function

$$p_{\mathbf{H}}(\mathbf{H}) = \exp\left(-a\text{Trace}(\mathbf{H}^2) + b\text{Trace}(\mathbf{H}) + c\right)$$

Probability density function of the eigenvalues of $\mathbf{H}$

$$p_(x_1, x_2, \cdots, x_N) = C_N \exp\left(-\frac{1}{2}\sum_{j=1}^{N} x_j^2\right) \prod |x_j - x_k|$$

GOE in structural dynamics

The equations of motion describing free vibration of a linear undamped system in the state-space

$$\mathbf{A}\mathbf{y} = \mathbf{0}$$

where $\mathbf{A} \in \mathbb{R}^{2N \times 2N}$ is the system matrix. Transforming into the modal coordinates

$$\boldsymbol{\mathcal{A}}\mathbf{u} = \mathbf{0}$$

where $\boldsymbol{\mathcal{A}} \in \mathbb{R}^{2N \times 2N}$ is a diagonal matrix.

Suppose the system is now subjected to n constraints of the form

$$(\mathbf{C} - \mathbf{I}) \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \mathbf{0}$$

where $\mathbf{C} \in \mathbb{R}^{n \times (2N-n)}$ constraint matrix, $\mathbf{I}$ is the $n \times n$ identity matrix, $\mathbf{u}_1$ and $\mathbf{u}_1$ are partition of $\mathbf{u}$.

If the entries of $\mathbf{C}$ are independent, then it can be shown (Langley, 2001) that the random part of the system matrix of the constrained system approaches to GOE.

Random Rod

Equations of motion:

$$\frac{\partial}{\partial x}\left[AE(x)\frac{\partial U}{\partial x}\right] = m(x)\frac{\partial^2 U}{\partial t^2} \tag{6}$$

Boundary condition: fixed-fixed (U(0)=U(L)=0)

$$m(x) = m_0\left[1 + \epsilon_1 f_1(x)\right]$$
$$AE(x) = AE_0\left[1 + \epsilon_2 f_2(x)\right]$$

$f_i(x)$ are zero mean random fields.

Deterministic mode shapes:
$\phi_k(x) = a\sin(k\pi x/L)$ where $a = \sqrt{2/Lm_0}$

Consider the mass matrix in the deterministic modal coordinates:

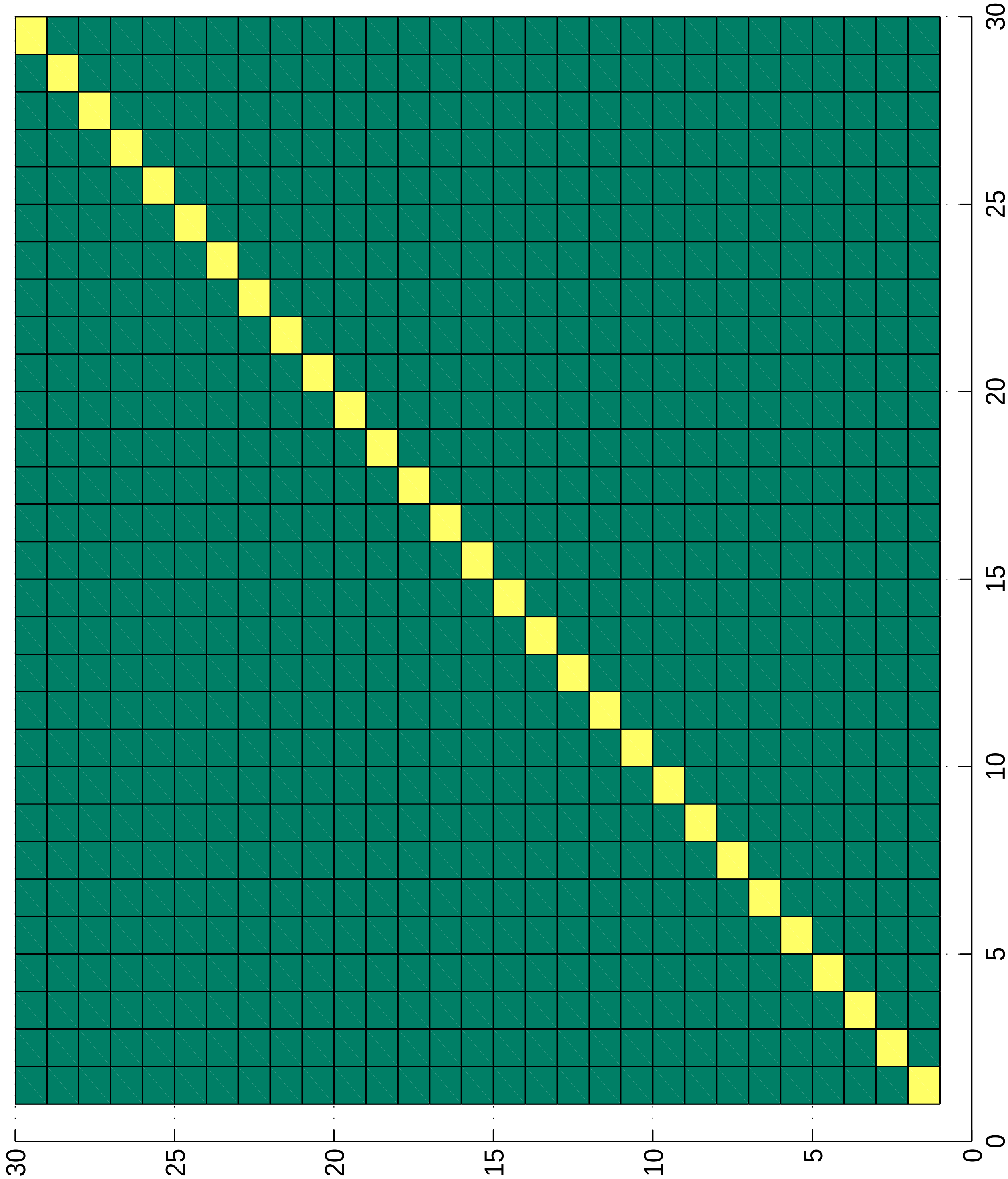
$$\begin{aligned} m'_{jk} &= \int_0^L \phi_j(x) m_0 \phi_k(x) \mathrm{d}x + \epsilon_1 \int_0^L \phi_j(x) f_1(x) \phi_k(x) \mathrm{d}x \\ &= m'_{0_{jk}} + \epsilon_1 \Delta m'_{jk} \end{aligned}$$

The random part

$$\Delta m'_{jk} = \int_0^L \phi_j(x) f_1(x) \phi_k(x) \mathrm{d}x$$

$$< \Delta m'_{jk} \Delta m'_{rs} >= \int_0^L \int_0^L \phi_j(x_1)\phi_k(x_1)\phi_r(x_2)\phi_s(x_2) R_{f_1}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2$$

Mass Matrix



Random Rod

Case 1: $f_1(x)$ is δ-correlated (white noise):
$R_{f_1}(x_1, x_2) = Q_1 \delta(x_1 - x_2)$

Results:

- $< \Delta m'_{jj} \Delta m'_{rr} >= \frac{1}{4} a^4 Q_1 L$, $j \neq r$
- $< \Delta m'_{jj} \Delta m'_{jj} >= \frac{3}{8} a^4 Q_1 L$
- $< \Delta m'_{kj} \Delta m'_{kj} >= \frac{1}{4} a^4 Q_1 L$, $k \neq j$
- $< \Delta m'_{kj} \Delta m'_{rs} >= 0$
- $< \Delta m'_{kk} \Delta m'_{kr} >= 0$, $k \neq r$

Random Rod

Case 2: $f_1(x)$ is fully correlated:
$R_{f_1}(x_1, x_2) = Q_2$ for $x_1, x_2 \in [0, L]$

Results:

- $< \Delta m'_{jj} \Delta m'_{rr} >= \frac{1}{4} a^4 Q_2 L^2$, $j \neq r$
- $< \Delta m'_{jj} \Delta m'_{jj} >= \frac{3}{8} a^4 Q_2 L^2$
- $< \Delta m'_{kj} \Delta m'_{kj} >= 0$, $k \neq j$
- $< \Delta m'_{kj} \Delta m'_{rs} >= 0$
- $< \Delta m'_{kk} \Delta m'_{kr} >= 0$, $k \neq r$

Conclusions and Future Research

- Although mathematically optimal given knowledge of only the mean values of the matrices, it is not entirely clear how well the results obtained from Soize model will match the statistical properties of a physical system.

- Analytical works show that GOE may be a possible model for the random system matrices in the modal coordinates for very large and complex systems.

- The random rod analysis has shown that the system matrices in the modal coordinates is close to GOE (but not exactly GOE) rather than the Soize model.

- Future research will address more complicated systems and explore the possibility of using GOE (or close to that, due to non-negative definiteness) as a model of the random system matrices. Such a model would enable us to develop a general Monte-Carlo simulation technique to be used in conjunction with FE methods.