COMPLEX MODES IN LINEAR STOCHASTIC SYSTEMS

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Outline of the Talk

- Introduction
- Viscously Damped Systems
- Complex frequencies and modes
- System Randomness
- Derivatives of Complex Eigensolutions
- Statistics of Complex Eigensolutions
- Numerical examples
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Viscously Damped Systems

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}. \tag{1}$$

where **M**, **C** and **K** are the mass, damping and stiffness matrices respectively. $\mathbf{q}(t)$ is the vector of generalized coordinates.

Complex Frequencies and Modes

The eigenvalue problem associated with equation (1) can be represented by

$$\lambda_k^2 \mathbf{M}\mathbf{u}_k + \lambda_k \mathbf{C}\mathbf{u}_k + \mathbf{K}\mathbf{u}_k = \mathbf{0}.$$

The eigenvalues, λ_k, are the roots of the characteristic polynomial

$$\det\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right] = 0.$$

The order of the polynomial is $2N$ and the roots appear in complex conjugate pairs.

The eigenvalues are arranged as

$$s_1, s_2, \cdots, s_N, s_1^*, s_2^*, \cdots, s_N^*.$$

Each complex mode satisfies the normalization relationship

$$\mathbf{u}_j^T\left[2s_j\mathbf{M} + \mathbf{C}\right]\mathbf{u}_j = \frac{1}{\gamma_j}, \quad \forall k = 1, \cdots, 2N$$

System Randomness

Randomness of the system matrices has the following form:

$$\begin{aligned} \mathbf{M} &= \overline{\mathbf{M}} + \delta\mathbf{M}, \\ \mathbf{C} &= \overline{\mathbf{C}} + \delta\mathbf{C}, \\ \text{and} \quad \mathbf{K} &= \overline{\mathbf{K}} + \delta\mathbf{K}. \end{aligned}$$

Here, $\overline{(\bullet)}$ and $\delta(\bullet)$ denotes the nominal (deterministic) and random parts of $(\bullet)$ respectively. It is assumed that $\delta\mathbf{M}$, $\delta\mathbf{C}$ and $\delta\mathbf{K}$ are zero-mean random matrices.

The random parts are small and also they are such that

1. symmetry of the system matrices is preserved,
2. the mass matrix $\mathbf{M}$ is positive definite, and
3. $\mathbf{C}$ and $\mathbf{K}$ are non-negative definite.

Statistics of the Eigenvalues

If the random perturbations of the system matrices are small, s_j can be approximated by a first-order Taylor expansion as

$$s_j = \bar{s}_j + \sum_{r=1}^{N}\sum_{s=1}^{N} \frac{\partial s_j}{\partial K_{rs}} \delta K_{rs} + \sum_{r=1}^{N}\sum_{s=1}^{N} \frac{\partial s_j}{\partial C_{rs}} \delta C_{rs} + \sum_{r=1}^{N}\sum_{s=1}^{N} \frac{\partial s_j}{\partial M_{rs}} \delta M_{rs}$$

or in a matrix form as

$$\mathbf{s} - \bar{\mathbf{s}} = \mathcal{D}_{\mathbf{s}} \begin{Bmatrix} \delta\mathcal{K} \\ \delta\mathcal{C} \\ \delta\mathcal{M} \end{Bmatrix}$$

where

$$\mathcal{D}_{\mathbf{s}}^T = \begin{bmatrix} \frac{\partial s_1}{\partial \mathcal{K}} & \frac{\partial s_2}{\partial \mathcal{K}} & \cdots & \frac{\partial s_N}{\partial \mathcal{K}} \\ \frac{\partial s_1}{\partial \mathcal{C}} & \frac{\partial s_2}{\partial \mathcal{C}} & \cdots & \frac{\partial s_N}{\partial \mathcal{C}} \\ \frac{\partial s_1}{\partial \mathcal{M}} & \frac{\partial s_2}{\partial \mathcal{M}} & \cdots & \frac{\partial s_N}{\partial \mathcal{M}} \end{bmatrix} \in \mathbb{R}^{3N^2 \times N}$$

Derivatives of Complex Eigensolutions

From Adhikari (1999): [*AIAA Journal*, 37(11), pp. 1152–1158]

Derivative of the j-th complex eigenvalue

$$\frac{\partial s_j}{\partial \alpha} = -\gamma_j \mathbf{u}_j^T \left[s_j^2 \frac{\partial \mathbf{M}}{\partial \alpha} + s_j \frac{\partial \mathbf{C}}{\partial \alpha} + \frac{\partial \mathbf{K}}{\partial \alpha} \right] \mathbf{u}_j.$$

Derivative of the j-th complex eigenvector

$$\frac{\partial \mathbf{u}_j}{\partial \alpha} = \sum_{k=1}^{2N} a_{jk}^{(\alpha)} \mathbf{u}_k$$

where

$$a_{jk}^{(\alpha)} = -\frac{\gamma_j}{s_j - s_k} \mathbf{u}_k^T \left[s_j^2 \frac{\partial \mathbf{M}}{\partial \alpha} + s_j \frac{\partial \mathbf{C}}{\partial \alpha} + \frac{\partial \mathbf{K}}{\partial \alpha} \right] \mathbf{u}_j \quad \forall k = 1, 2, \cdots, 2N, \neq j$$

$$\text{and} \quad a_{jj}^{(\alpha)} = -\frac{\gamma_j}{2} \mathbf{u}_j^T \left[2 s_j \frac{\partial \mathbf{M}}{\partial \alpha} + \frac{\partial \mathbf{C}}{\partial \alpha} \right] \mathbf{u}_j.$$

Derivatives w.r.t. the System Matrices

For the eigenvalues:

$$\frac{\partial s_j}{\partial K_{rs}} = -\gamma_j \left(U_{rj} U_{sj} \right)$$

$$\frac{\partial s_j}{\partial C_{rs}} = s_j \frac{\partial s_j}{\partial K_{rs}}$$

and $$\frac{\partial s_j}{\partial M_{rs}} = s_j^2 \frac{\partial s_j}{\partial K_{rs}}.$$

For the eigenvectors:

$$\frac{\partial U_{lj}}{\partial K_{rs}} = -\gamma_j \sum_{\substack{k=1 \\ k \neq j}}^{2N} \frac{\left(U_{rk} U_{sj} \right)}{s_j - s_k} U_{lk}$$

$$\frac{\partial U_{lj}}{\partial C_{rs}} = -\frac{\gamma_j}{2} \left(U_{rj} U_{sj} \right) U_{lj} + s_j \frac{\partial U_{lj}}{\partial K_{rs}}$$

and $$\frac{\partial U_{lj}}{\partial M_{rs}} = -\gamma_j s_j \left(U_{rj} U_{sj} \right) U_{lj} + s_j^2 \frac{\partial U_{lj}}{\partial K_{rs}}.$$

Statistics of the Eigenvalues

The covariance matrix of the eigenvalues, $\boldsymbol{\Sigma}_{\mathbf{s}}$ is obtained as

$$\boldsymbol{\Sigma}_{\mathbf{s}} = < (\mathbf{s}-\bar{\mathbf{s}})(\mathbf{s}-\bar{\mathbf{s}})^{*^T} > \\ = \boldsymbol{\mathcal{D}}_{\mathbf{s}} \left\langle \begin{Bmatrix} \delta\boldsymbol{\mathcal{K}} \\ \delta\boldsymbol{\mathcal{C}} \\ \delta\boldsymbol{\mathcal{M}} \end{Bmatrix} \begin{Bmatrix} \delta\boldsymbol{\mathcal{K}} \\ \delta\boldsymbol{\mathcal{C}} \\ \delta\boldsymbol{\mathcal{M}} \end{Bmatrix}^T \right\rangle \boldsymbol{\mathcal{D}}_{\mathbf{s}}^{*^T} = \boldsymbol{\mathcal{D}}_{\mathbf{s}} \Sigma_{kcm} \boldsymbol{\mathcal{D}}_{\mathbf{s}}^{*^T}.$$

$\boldsymbol{\Sigma}_{kcm} \in \mathbb{R}^{3N^2 \times 3N^2}$, the joint covariance matrix of **M**, **C** and **K** is defined as

$$\boldsymbol{\Sigma}_{kcm} = \begin{bmatrix} < \delta\boldsymbol{\mathcal{K}}\delta\boldsymbol{\mathcal{K}}^T > & < \delta\boldsymbol{\mathcal{K}}\delta\boldsymbol{\mathcal{C}}^T > & < \delta\boldsymbol{\mathcal{K}}\delta\boldsymbol{\mathcal{M}}^T > \\ < \delta\boldsymbol{\mathcal{C}}\delta\boldsymbol{\mathcal{K}}^T > & < \delta\boldsymbol{\mathcal{C}}\delta\boldsymbol{\mathcal{C}}^T > & < \delta\boldsymbol{\mathcal{C}}\delta\boldsymbol{\mathcal{M}}^T > \\ < \delta\boldsymbol{\mathcal{M}}\delta\boldsymbol{\mathcal{K}}^T > & < \delta\boldsymbol{\mathcal{M}}\delta\boldsymbol{\mathcal{C}}^T > & < \delta\boldsymbol{\mathcal{M}}\delta\boldsymbol{\mathcal{M}}^T > \end{bmatrix}.$$

Statistics of the Eigenvectors

For small random perturbations of the system matrices, $\mathbf{u}_j$ can be approximated by a first-order Taylor expansion

$$\mathbf{u}_j - \bar{\mathbf{u}}_j = \boldsymbol{\mathcal{D}}_{\mathbf{u}_j} \begin{Bmatrix} \delta\boldsymbol{\mathcal{K}} \\ \delta\boldsymbol{\mathcal{C}} \\ \delta\boldsymbol{\mathcal{M}} \end{Bmatrix}.$$

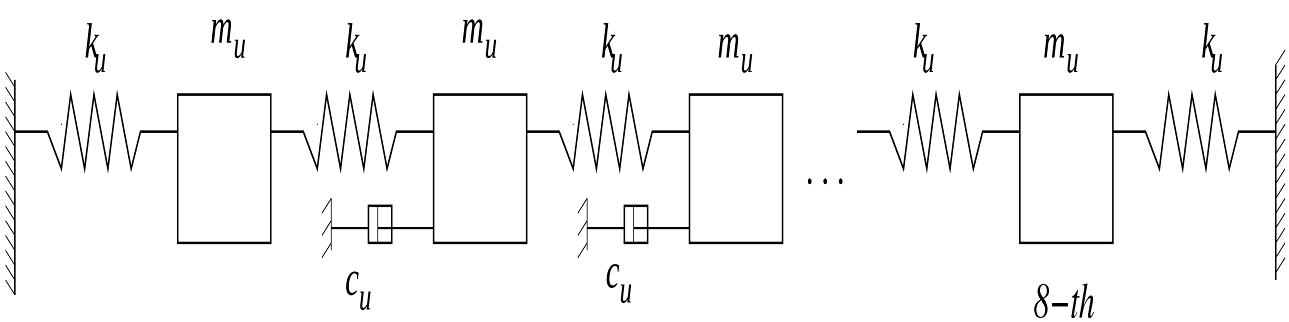
$\boldsymbol{\mathcal{D}}_{\mathbf{u}_j}$, the matrix containing derivatives of $\mathbf{u}_j$ with respect to elements of the system matrices, is given by

$$\boldsymbol{\mathcal{D}}_{\mathbf{u}_j}^T = \begin{bmatrix} \frac{\partial U_{1j}}{\partial \boldsymbol{\mathcal{K}}} & \frac{\partial U_{2j}}{\partial \boldsymbol{\mathcal{K}}} & \dots & \frac{\partial U_{Nj}}{\partial \boldsymbol{\mathcal{K}}} \\ \frac{\partial U_{1j}}{\partial \boldsymbol{\mathcal{C}}} & \frac{\partial U_{2j}}{\partial \boldsymbol{\mathcal{C}}} & \dots & \frac{\partial U_{Nj}}{\partial \boldsymbol{\mathcal{C}}} \\ \frac{\partial U_{1j}}{\partial \boldsymbol{\mathcal{M}}} & \frac{\partial U_{2j}}{\partial \boldsymbol{\mathcal{M}}} & \dots & \frac{\partial U_{Nj}}{\partial \boldsymbol{\mathcal{M}}} \end{bmatrix} \in \mathbb{R}^{3N^2 \times N}$$

The covariance matrix of j-th and k-th eigenvectors

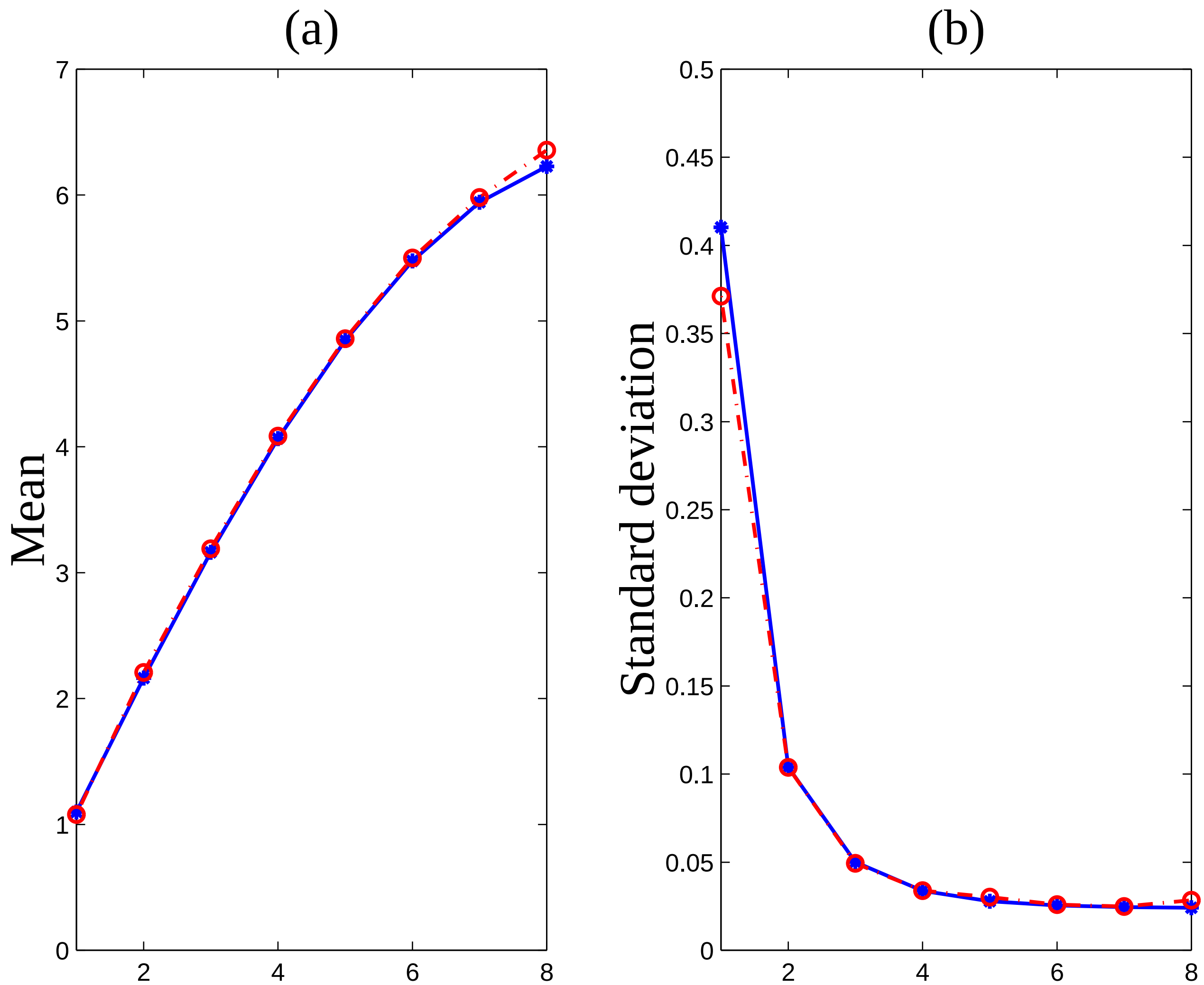
$$\boldsymbol{\Sigma}_{\mathbf{u}_j \mathbf{u}_k} = < \left(\mathbf{u}_j - \bar{\mathbf{u}}_j\right) \left(\mathbf{u}_k - \bar{\mathbf{u}}_k\right)^{*^T} > = \boldsymbol{\mathcal{D}}_{\mathbf{u}_j} \boldsymbol{\Sigma}_{kcm} \boldsymbol{\mathcal{D}}_{\mathbf{u}_k}^{*^T}.$$

Numerical example



Linear array of 8 spring-mass oscillators; nominal system: $m_u = 1$ Kg, $k_u = 10$ N/m and $c_u = 0.1$ Nm/s

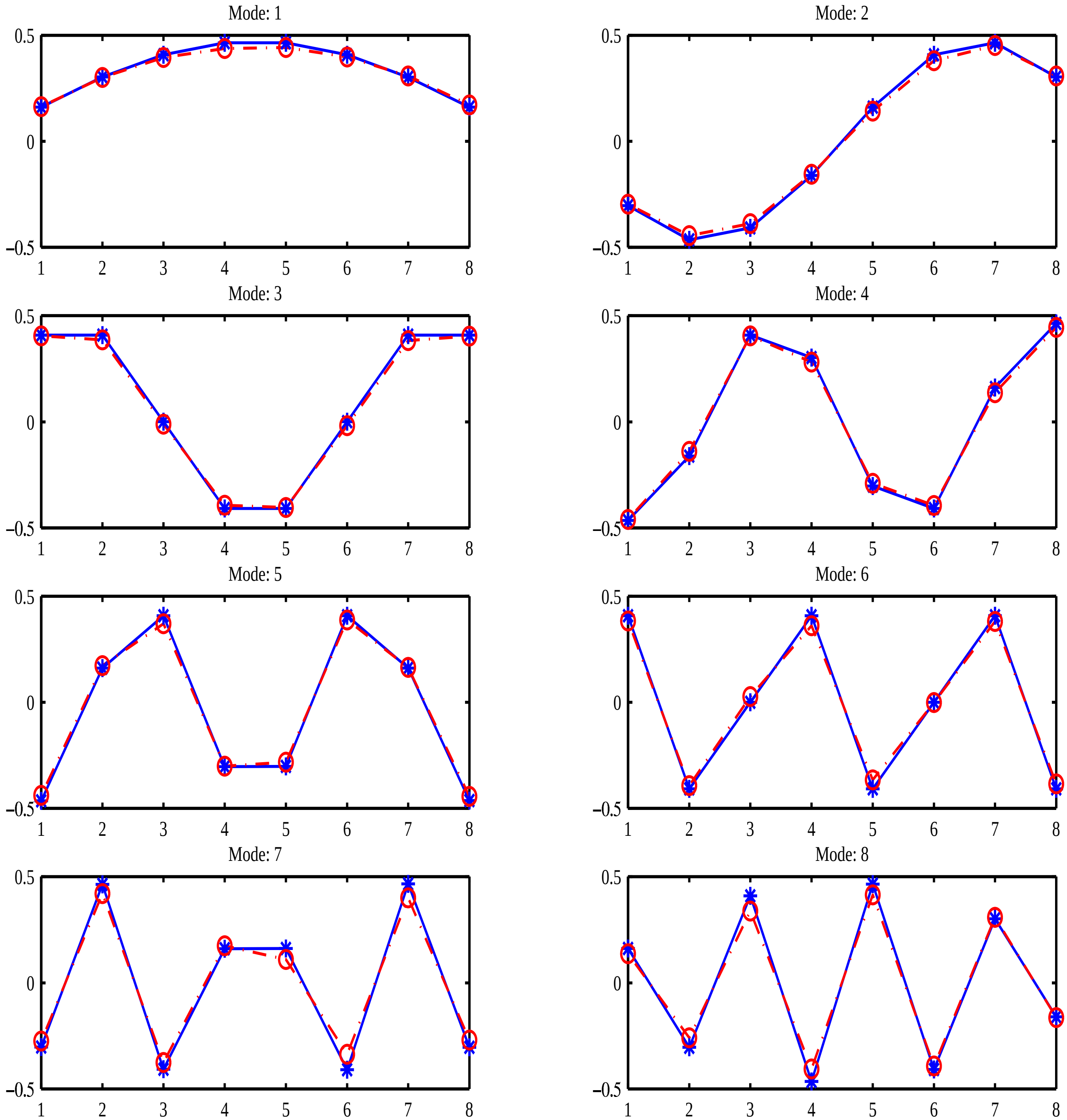
Statistics of the Eigenvalues



(a) Absolute value of mean of complex natural frequencies

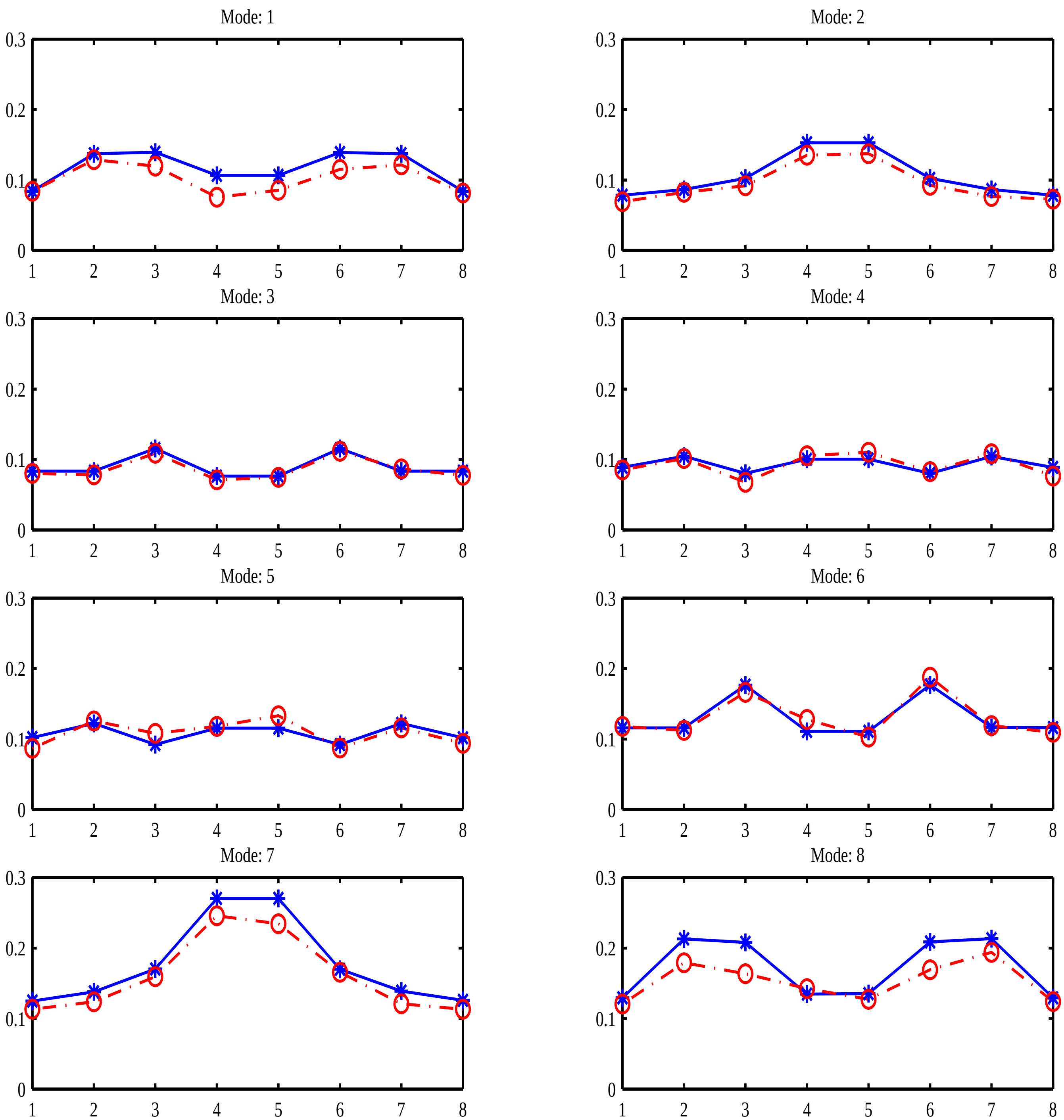
(b) Standard deviation of complex natural frequencies; ‘X-axis’ Mode number; ‘—’ Analytical; ‘-.-.-’ MCS

Statistics of the Eigenvectors



Real part of mean of the complex modes, 'X-axis' DOF; '—' Analytical; '-.- -' MCS

Statistics of the Eigenvectors



Standard deviation of the complex modes, 'X-axis' DOF; '—' Analytical; '-.-.-' MCS

Summary and Conclusions

- An approach has been proposed to obtain the second-order statistics of complex eigenvalues and eigenvectors of non-proportionally damped linear stochastic systems.

- It is assumed that the randomness is small so that the first-order perturbation method can be applied.

- The covariance matrices of the complex eigensolutions are expressed in terms of the covariance matrices of the system properties and derivatives of the eigensolutions with respect to the system parameters.

- The proposed method does not require conversion of the equations of motion into the first-order form.