Dynamics of Non-viscously Damped Distributed Parameter Systems

S Adhikari, Y Lei and M I Friswell

Department of Aerospace Engineering, University of Bristol, Bristol, U.K.

URL: http://www.aer.bris.ac.uk/contact/academic/adhikari/home.html

Outline of the Presentation

- Introduction
- Models of damping
- Equation of motion
- Outline of the solution method
- Incorporation of boundary conditions
- Numerical examples & results
- Conclusions & future works

Introduction (1)

Modelling and analysis of damping properties are not as advanced as mass and stiffness properties. The reasons:

- by contrast with inertia and stiffness forces, it is not in general clear which *variables* are relevant to determine the damping forces
- the *spatial location* of the damping sources are generally unclear - often the structural joints are more responsible for the energy dissipation than the (solid) material

Introduction (2)

- the *functional form* of the damping model is difficult to establish experimentally, and finally
- even if one manages to address the previous issues, what parameters should be used in a chosen model is still very much an open problem

The ‘solution’ over the past 100 years:

- Use viscous damping model

Viscous Damping Model

- Introduced by Lord Rayleigh in 1877
- instantaneous generalized velocities are the only relevant variables that determine damping

However,

- viscous damping is not the only damping model within the scope of linear analysis.

Non-viscous Damping Model

- Any causal model which makes the energy dissipation functional non-negative is a possible candidate for a damping model
- non-viscous damping models in general have more parameters and therefore are more likely to have a better match with experimental measurements

Question:

- What non-viscous damping model should be used?

Equation of Motion

$$\rho(\mathbf{r})\ddot{u}(\mathbf{r},t) + \mathcal{L}_1\dot{u}(\mathbf{r},t) + \mathcal{L}_2 u(\mathbf{r},t) = p(\mathbf{r},t) \qquad (1)$$

specified in some domain $\mathcal{D}$ with homogeneous linear boundary condition of the form

$$\mathcal{M}u(\mathbf{r},t) = 0;\ \mathbf{r} \in \Gamma$$

specified on some boundary surface Γ.

$u(\mathbf{r},t)$: displacement variable

$\rho(\mathbf{r})$: mass distribution of the system

$p(\mathbf{r},t)$: distributed time-varying forcing function

$\mathcal{L}_2$: spatial self-adjoint stiffness operator

$\mathcal{M}$: linear operator acting on the boundary

The Damping Operator

The damping operator $\mathcal{L}_1$ can be written in the form

$$\mathcal{L}_1 \dot{u}(\mathbf{r}, t) = \int_{\mathcal{D}} \int_{-\infty}^{t} C_1(\mathbf{r}, \boldsymbol{\xi}, t - \tau) \dot{u}(\boldsymbol{\xi}, \tau) \, d\tau \, d\boldsymbol{\xi} \qquad (2)$$

where $C_1(\mathbf{r}, \boldsymbol{\xi}, t)$ is the kernel function.

- The velocities $\dot{u}(\boldsymbol{\xi}, \tau)$ at different time instants and spatial locations are coupled through the kernel function
- Eq. (1) together with the damping operator (2) represents a partial integro-differential equation

The Damping Operator

Any function that makes the energy dissipation function

$$\mathcal{F}(t) = \frac{1}{2}\int_{\mathcal{D}}\left\{\int_{\mathcal{D}}\int_{-\infty}^{t} C_1(\mathbf{r},\boldsymbol{\xi},t-\tau)\,\dot{u}(\boldsymbol{\xi},\tau)d\tau\, d\boldsymbol{\xi}\right\}\dot{u}(\mathbf{r},t)\,d\mathbf{r} \tag{3}$$

non-negative can be used as a kernel function.

The main assumption:

- the damping kernel function $C_1(\mathbf{r},\boldsymbol{\xi},t)$ is separable in space and time

Viscous Damping

The kernel function is a delta function in both space and time:

$$C_1(\mathbf{r}, \boldsymbol{\xi}, t - \tau) = C(\mathbf{r})\delta(\mathbf{r} - \boldsymbol{\xi})\delta(t - \tau) \qquad (4)$$

- the spatial delta function means that the damping force is ‘locally reacting’ and the time delta function implies that the force depends only on the instantaneous value of the motion
- in general this represents the non-proportional viscous damping model

Viscoelastic Damping

The kernel function is a delta function in space but depends on the past time histories:

$$C_1(\mathbf{r}, \boldsymbol{\xi}, t-\tau) = C(\mathbf{r}) g(t-\tau) \delta(\mathbf{r}-\boldsymbol{\xi}) \tag{5}$$

- Represents a locally reacting viscoelastic damping model where the damping force depends on the past velocity time histories through a convolution integral over the kernel function $g(t)$
- $g(t)$ is known as retardation function, heredity function or relaxation function

Non-local Viscous Damping

The kernel function is a delta function in time but depends on the spatial distribution of the velocities:

$$C_1(\mathbf{r}, \boldsymbol{\xi}, t-\tau) = C(\mathbf{r})c(\mathbf{r}-\boldsymbol{\xi})\delta(t-\tau) \tag{6}$$

- velocities at different points can affect the damping force at a given point via a convolution integral

Non-local Viscoelastic Damping

- This is the most general form of damping model
- the only assumption is that the kernel function is separable in space and time:

$$C_1(\mathbf{r}, \boldsymbol{\xi}, t - \tau) = C(\mathbf{r}) c(\mathbf{r} - \boldsymbol{\xi}) g(t - \tau) \tag{7}$$

- all the previous three damping models can be identified as special cases of this model

Parametrization of Models (1)

Plausible functional form of the kernel functions in space and time is required

Requirement:

For a physically realistic model of damping

$$\Re\left[G(\omega)\int_{\mathcal{D}}\int_{\mathcal{D}} C(\mathbf{r})c(\mathbf{r}-\boldsymbol{\xi})U^*(\boldsymbol{\xi},\omega)U(\mathbf{r},\omega)\,d\boldsymbol{\xi}\,d\mathbf{r}\right]\geq 0$$

for all ω

Non-viscous Damping Functions

Damping functions (in Laplace domain)	Author, Year
$G(s) = \sum_{k=1}^{n} \frac{a_k s}{s + b_k}$	Biot (1955, 1958)
$G(s) = \frac{E_1 s^{\alpha} - E_0 b s^{\beta}}{1 + b s^{\beta}}$	Bagley and Torvik (1983)
$0 < \alpha < 1, \quad 0 < \beta < 1$	
$sG(s) = G^{\infty} \left[1 + \sum_k \alpha_k \frac{s^2 + 2\zeta_k \omega_k s}{s^2 + 2\zeta_k \omega_k s + \omega_k^2} \right]$	Golla and Hughes (1985)
	and McTavish and Hughes (1993)
$G(s) = 1 + \sum_{k=1}^{n} \frac{\Delta_k s}{s + \beta_k}$	Lesieutre and Mingori (1990)
$G(s) = c \frac{1 - e^{-st_0}}{st_0}$	Adhikari (1998)
$G(s) = c \frac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}$	Adhikari (1998)

Parametrization of Models (2)

- $g(t) = g_\infty\,\mu\exp(-\mu t)$ so that $G(\omega) = \frac{g_\infty\,\mu}{\mathrm{i}\omega+\mu}$
- $c(\mathbf{r}-\boldsymbol{\xi}) = \frac{\alpha}{2}\exp(-\alpha|\mathbf{r}-\boldsymbol{\xi}|)$ and
- $C(\mathbf{r})$, g_∞, μ and α are all positive

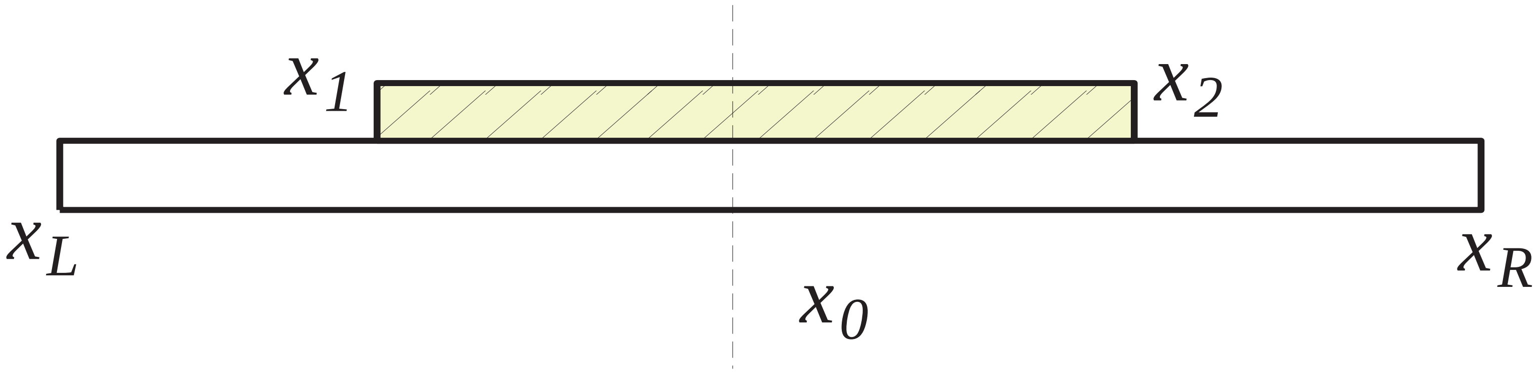
The damping force:

$$\int_{\mathcal{D}}\int_{-\infty}^{t} C(\mathbf{r})\,g_\infty\,\mu\exp(-\mu\{t-\tau\}) \\ \frac{\alpha}{2}\exp(-\alpha|\mathbf{r}-\boldsymbol{\xi}|)\dot{u}(\boldsymbol{\xi},\tau)\;d\boldsymbol{\xi}\,d\tau$$

Special Cases

- if $\alpha \to \infty, \mu \to \infty$ one obtains the standard viscous model in (4)
- if $\alpha \to \infty$ and μ is finite one obtains the local non-viscous model in (5)
- if α is finite but $\mu \to \infty$ one obtains the non-local viscous damping model in (6)
- if both α and μ are finite one obtains the non-local viscoelastic damping model in (7)

Damped Euler-Bernoulli Beam



Homogeneous Euler-Bernoulli beam with non-viscous damping

Objectives:

- To obtain eigenvalues and eigenvectors of the system

Equation of Motion (1)

Part within the damping patch:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} + \int_{x_1}^{x_2}\int_{-\infty}^{t}\frac{\alpha}{2}\exp\left(-\alpha\left|x-\xi\right|\right)$$

$$g_\infty \mu \exp\left(-\mu(t-\tau)\right)\left.\frac{\partial w(\xi,t)}{\partial t}\right|_{t=\tau} d\xi d\tau = f(x,t) \tag{8}$$

when $x \in [x_1, x_2]$

Equation of Motion (2)

Part outside the non-viscous damping patch:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}+C_0\frac{\partial w(x,t)}{\partial t}=f(x,t) \quad (9)$$

when $x \in (x_L, x_1) \cup (x_2, x_R)$.

- Appropriate boundary conditions must be satisfied at $x = x_L$ and at $x = x_R$
- relevant continuity conditions at the internal points x_1 and x_2 must be satisfied

Outline of the Solution Method

- Transform the equations into Laplace domain
- differentiate with respect to the spatial variable to eliminate the spatial correlation terms (possible due to the exponential assumption)
- express the BCs corresponding to the higher order derivatives in terms of the known BCs
- repeat the process for all three segments
- merge the solutions from the three segments by matching the displacements and their derivatives at the interfaces

Eigensolutions of the Beam

The eigenvalues λ_j are the roots of

$$\det\left[\mathbf{M}(s)\exp\left(\bar{\boldsymbol{\Phi}}(s)(x_L - x_1)\right)\mathbf{T}\left(x_1, s\right) + \mathbf{N}(s)\exp\left(\bar{\boldsymbol{\Phi}}(s)(x_R - x_2)\right)\mathbf{T}\left(x_2, s\right)\right] = 0$$

The corresponding mode shapes are

$$\boldsymbol{\psi}_j(x) = \begin{cases} \exp\left(\bar{\boldsymbol{\Phi}}(\lambda_j)(x - x_1)\right)\mathbf{T}(x_1, \lambda_j)\mathbf{u}_0(\lambda_j), & x_L \le x \le x_1 \\ \mathbf{T}(x, \lambda_j)\mathbf{u}_0(\lambda_j), & x_1 \le x \le x_2 \\ \exp\left(\bar{\boldsymbol{\Phi}}(\lambda_j)(x - x_2)\right)\mathbf{T}(x_2, \lambda_j)\mathbf{u}_0(\lambda_j), & x_2 \le x \le x_R \end{cases}$$

Boundary Conditions

The matrices $\mathbf{M}(s)$ and $\mathbf{N}(s)$ depend on the boundary conditions:

- Clamped-clamped (C-C):

$$\mathbf{M}(s) = \begin{bmatrix} \mathbf{I}_{2\times 2} & \mathbf{O}_{2\times 2} \\ \mathbf{O}_{2\times 2} & \mathbf{O}_{2\times 2} \end{bmatrix}, \quad \mathbf{N}(s) = \begin{bmatrix} \mathbf{O}_{2\times 2} & \mathbf{O}_{2\times 2} \\ \mathbf{I}_{2\times 2} & \mathbf{O}_{2\times 2} \end{bmatrix}$$

- Free-Free (F-F):

$$\mathbf{M}(s) = \begin{bmatrix} \mathbf{O}_{2\times 2} & \mathbf{I}_{2\times 2} \\ \mathbf{O}_{2\times 2} & \mathbf{O}_{2\times 2} \end{bmatrix}, \quad \mathbf{N}(s) = \begin{bmatrix} \mathbf{O}_{2\times 2} & \mathbf{O}_{2\times 2} \\ \mathbf{O}_{2\times 2} & \mathbf{I}_{2\times 2} \end{bmatrix}$$

Example 1: The System

Damped beam with step variation in the system properties and pinned boundary conditions (Friswell and Lees, 2001)

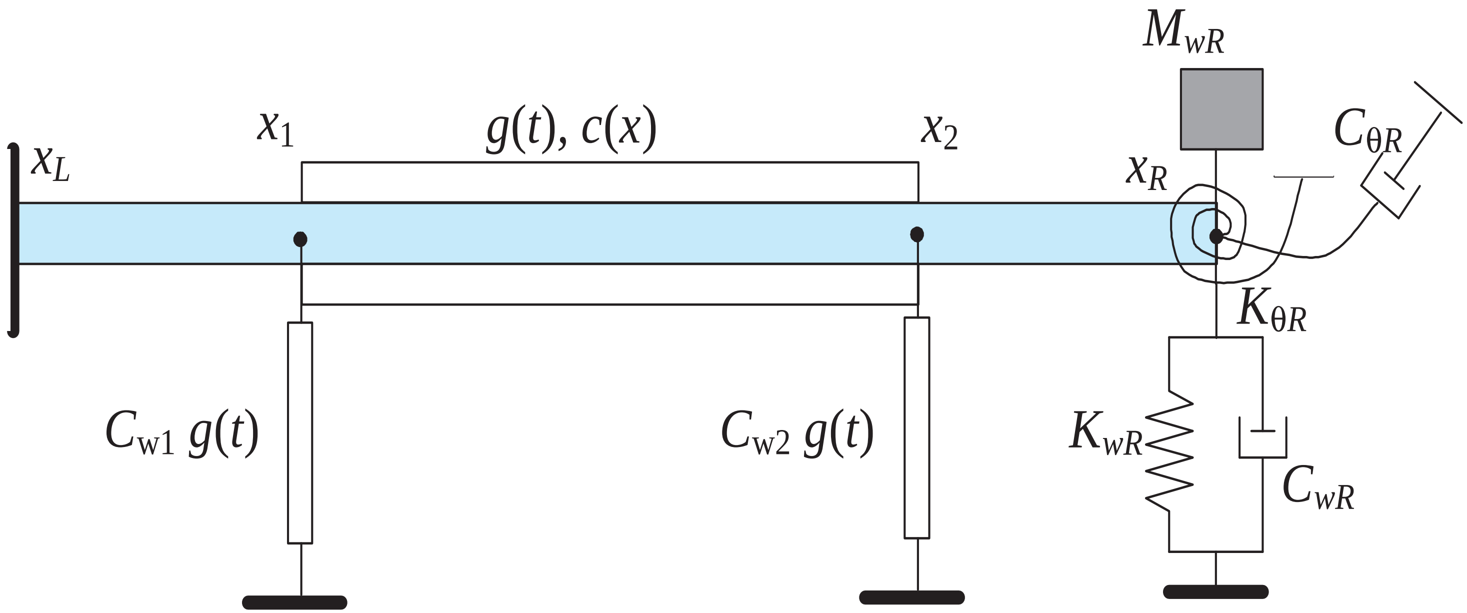
Parameters	Part 1	Part 2
L_i	1m	2m
ρA_i	10 kg/m	20 kg/m
c_i	0 Ns/m^2	10 Ns/m^2
EI_i	100 Nm^2	100 Nm^2

Example 1: Results

Proposed method	Friswell and Lees (2001)
-2.2552 ± 1.2711i	-2.2552 ± 1.2711i
-1.7936 ± 10.903i	-1.7936 ± 10.903i
-1.5741 ± 24.863i	-1.5741 ± 24.863i
-1.7876 ± 43.165i	-1.7876 ± 43.165i
-1.8781 ± 68.118i	-1.8781 ± 68.118i
-1.6984 ± 99.327i	-1.6984 ± 99.327i
-1.6775 ± 133.66i	-1.6775 ± 133.66i
-1.8549 ± 174.00i	-1.8549 ± 174.00i

The first eight eigenvalues of the beam

Example 2: The System (1)



Euler-Bernoulli beam with complex boundary conditions and middle supports

Example 2: The System (2)

The numerical values (in SI units) of the system parameters are as follows:

- Case 1: Local viscous damping:
  L=1, EI=1, m=16, M_{wR}=4, K_{wR}=8, C_{wR}=4,
  g_∞=1.6,$g(t)$=δ(t), c(x) $= \delta(x)$,
  $C_{\theta R} = K_{\theta R} = C_{w1} = C_{w2}$=0
- Case 2: Non-local non-viscous damping:
  L=1, EI=1, m=16, M_{wR}=4, K_{wR}=8, C_{wR}=4, g_∞=16,
  $g(t) = \mu \exp\left(-\mu t\right)$,
  $c(x) = \alpha \exp\left(-\alpha \left|x\right|\right)$,$K_{\theta R}$ = 8, $C_{\theta R} = C_{w1} = C_{w2}$=4

Example 2: Results

j	λ_j	
	Proposed method	Yang and Wu (1997)
1	-0.2705 $\pm$ 1.1451i	-0.2705 $\pm$ 1.1451i
2	-0.1357 $\pm$ 4.4930i	-0.1357 $\pm$ 4.4930i
3	-0.0896 $\pm$ 13.2586i	-0.0896 $\pm$ 13.2586i
4	-0.0727 $\pm$ 26.8877i	-0.0727 $\pm$ 26.8877i
5	-0.0647 $\pm$ 45.4297i	-0.0647 $\pm$ 45.4297i
6	-0.0602 $\pm$ 68.8941i	-0.0602 $\pm$ 68.8941i
7	-0.0575 $\pm$ 97.2862i	-0.0575 $\pm$ 97.2862i
8	-0.0557 $\pm$ 130.6088i	-0.0557 $\pm$ 130.6088i
9	-0.0546 $\pm$ 168.8632i	-0.0546 $\pm$ 168.8632i
10	-0.0537 $\pm$ 212.0505i	-0.0537 $\pm$ 212.0505i

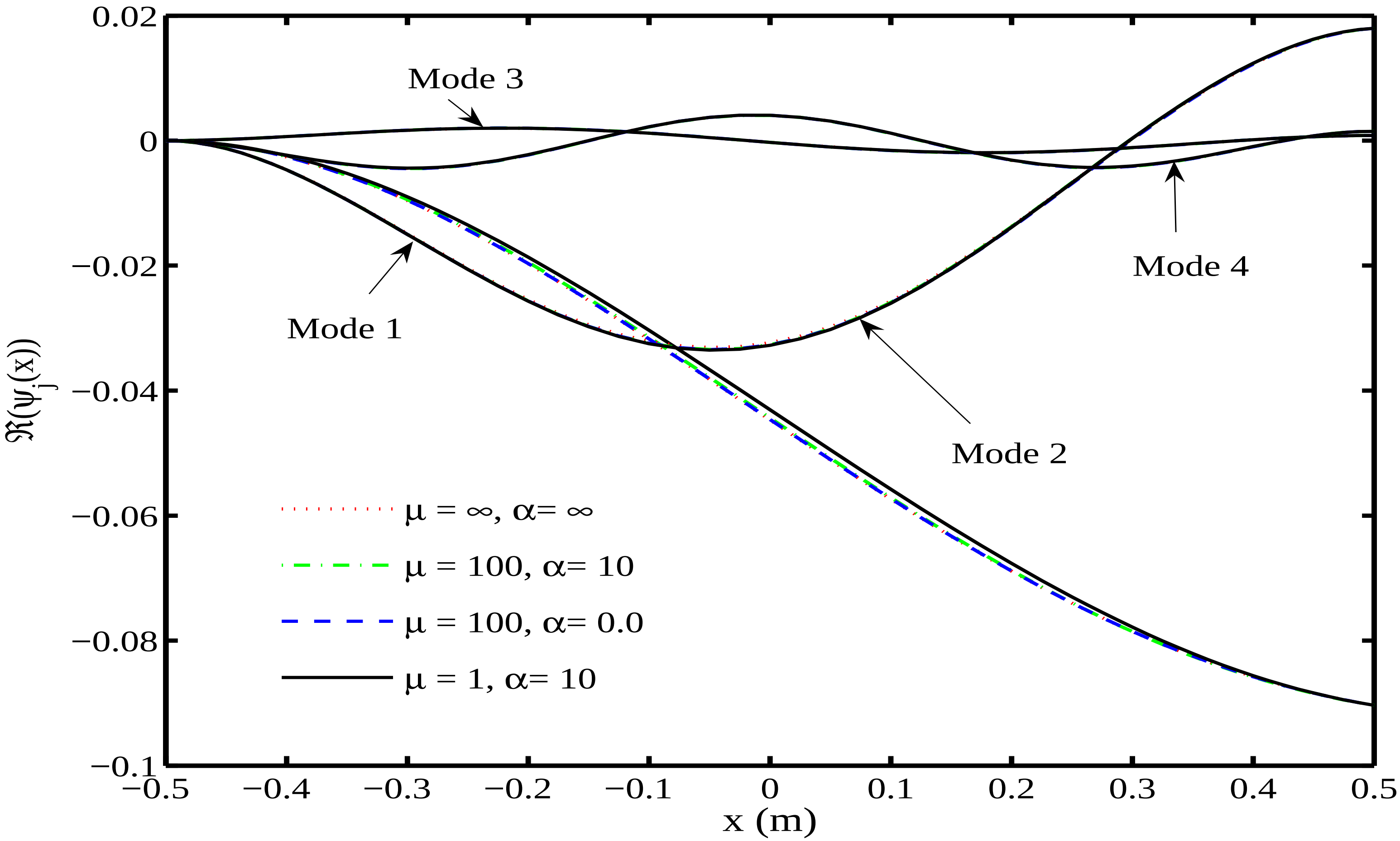
First ten eigenvalues of the beam for Case 1

Example 2: Results

j	λ_j			
	$\mu = \infty, \quad \alpha = \infty$	$\mu = 100, \quad \alpha = 10$	$\mu = 100, \quad \alpha = 0.1$	$\mu = 1, \quad \alpha = 10$
1	-0.67921 $\pm$ 1.1780i	-0.65439 $\pm$ 1.1966i	-0.56874 $\pm$ 1.2444i	-0.52895 $\pm$ 1.4404i
2	-0.98559 $\pm$ 6.4492i	-0.92155 $\pm$ 6.4835i	-0.61797 $\pm$ 6.4677i	-0.49096 $\pm$ 6.5182i
3	-1.3833 $\pm$ 16.611i	-1.2245 $\pm$ 16.723i	-1.0592 $\pm$ 16.703i	-0.61363 $\pm$ 16.656i
4	-1.4263 $\pm$ 31.584i	-1.2455 $\pm$ 31.754i	-1.1706 $\pm$ 31.727i	-0.73244 $\pm$ 31.620i
5	-1.0714 $\pm$ 51.442i	-0.90384 $\pm$ 51.521i	-0.83720 $\pm$ 51.483i	-0.74997 $\pm$ 51.456i
6	-1.0353 $\pm$ 76.208i	-0.79867 $\pm$ 76.276i	-0.74770 $\pm$ 76.237i	-0.71290 $\pm$ 76.207i
7	-1.3874 $\pm$ 105.87i	-0.92919 $\pm$ 106.11i	-0.90512 $\pm$ 106.08i	-0.71677 $\pm$ 105.88i
8	-1.4200 $\pm$ 140.46i	-0.91144 $\pm$ 140.68i	-0.89997 $\pm$ 140.67i	-0.75157 $\pm$ 140.47i
9	-1.0745 $\pm$ 179.97i	-0.78527 $\pm$ 180.03i	-0.77680 $\pm$ 180.01i	-0.75400 $\pm$ 179.98i
10	-1.0541 $\pm$ 224.42i	-0.74574 $\pm$ 224.45i	-0.74022 $\pm$ 224.44i	-0.73084 $\pm$ 224.42i

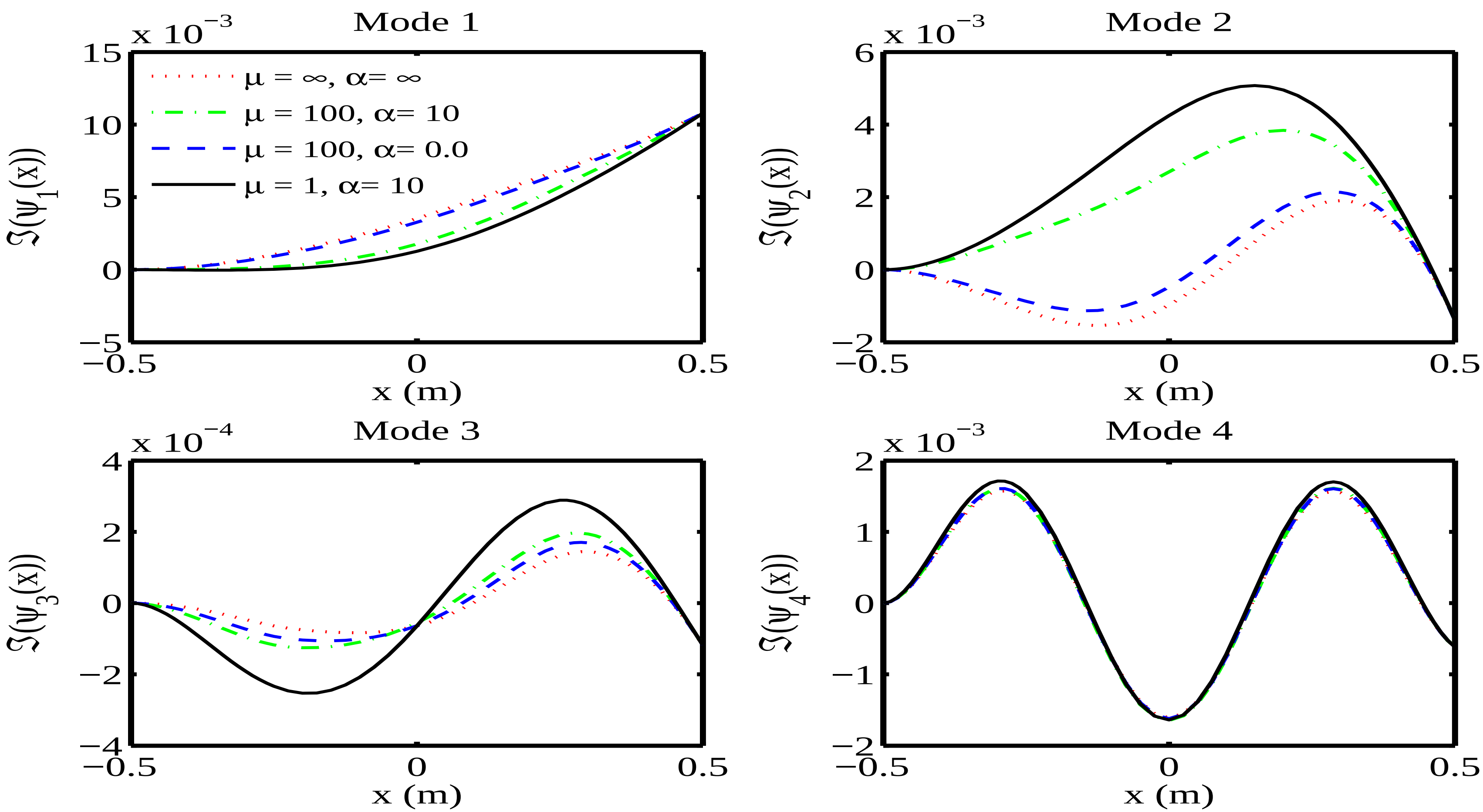
First ten eigenvalues of the beam for Case 2

Example 2: Results



Real parts of the first four modes for Case 2

Example 2: Results



Imaginary parts of the first four modes for Case 2

Conclusions (1)

- A method to obtain the natural frequencies and mode-shapes of Euler-Bernoulli beams with general linear damping models has been proposed
- it is assumed that the damping force at a given point in the beam depends on the past history of velocities at different points via convolution integrals over exponentially decaying kernel functions

Conclusions (2)

- conventional viscous and viscoelastic damping models can be obtained as special cases of this general linear damping model
- the choice of damping models effects the imaginary parts of the complex modes
- future work will discuss computational issues, forced vibration problems and experimental identification of non-viscous damping models

Open Problems

- To what extent different damping models with 'correct' sets of parameters influence the dynamics?
- which aspects of dynamic behavior are wrongly predicted by an incorrect damping model?
- how to choose a damping model (not the parameters!) for a given system?

References

Adhikari, S. (1998), *Energy Dissipation in Vibrating Structures*, Cambridge University Engineering Department, Cambridge, UK, first Year Report.

Bagley, R. L. and Torvik, P. J. (1983), "Fractional calculus–a different approach to the analysis of viscoelastically damped structures", *AIAA Journal*, **21** (5), pp. 741–748.

Biot, M. A. (1955), "Variational principles in irreversible thermodynamics with application to viscoelasticity", *Physical Review*, **97** (6), pp. 1463–1469.

Biot, M. A. (1958), "Linear thermodynamics and the mechanics of solids", in "Proceedings of the Third U. S. National Congress on Applied Mechanics", ASME, New York, (pp. 1–18).

Friswell, M. I. and Lees, A. W. (2001), "The modes of non-homogeneous damped beams", *Journal of Sound and Vibration*, **242** (2), pp. 355–361.

Golla, D. F. and Hughes, P. C. (1985), "Dynamics of viscoelastic structures - a time domain finite element formulation", *Transactions of ASME, Journal of Applied Mechanics*, **52**, pp. 897–906.

Lesieutre, G. A. and Mingori, D. L. (1990), "Finite element modeling of frequency-dependent material properties using augmented thermodynamic fields", *AIAA Journal of Guidance, Control and Dynamics*, **13**, pp. 1040–1050.

McTavish, D. J. and Hughes, P. C. (1993), "Modeling of linear viscoelastic space structures", *Transactions of ASME, Journal of Vibration and Acoustics*, **115**, pp. 103–110.

Yang, B. and Wu, X. (1997), "Transient response of one- dimensional distributed systems: a closed form eignfunction expansion realization", *Journal of Sound and Vibration*, **208** (5), pp. 763–776.