Joint Distribution of Eigenvalues of Linear Stochastic Systems

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Outline of the Presentation

- Random eigenvalue problem
- Existing methods
  - Exact methods
  - Perturbation methods
- Asymptotic analysis of multidimensional integrals
- Joint moments and pdf of the natural frequencies
- Numerical examples & results
- Conclusions

Random Eigenvalue Problem

The random eigenvalue problem of undamped or proportionally damped linear systems:

$$\mathbf{K}(\mathbf{x})\boldsymbol{\phi}_j = \omega_j^2 \mathbf{M}(\mathbf{x})\boldsymbol{\phi}_j \tag{1}$$

ω_j natural frequencies; $\boldsymbol{\phi}_j$ eigenvectors;
$\mathbf{M}(\mathbf{x}) \in \mathbb{R}^{N\times N}$ mass matrix and $\mathbf{K}(\mathbf{x}) \in \mathbb{R}^{N\times N}$ stiffness matrix.
$\mathbf{x} \in \mathbb{R}^m$ is random parameter vector with pdf

$$p_{\mathbf{x}}(\mathbf{x}) = e^{-L(\mathbf{x})}$$

$-L(\mathbf{x})$ is the log-likelihood function.

The Objectives

- The aim is to obtain the joint probability density function of the natural frequencies and the eigenvectors
- in this work we look at the joint statistics of the eigenvalues
- while several papers are available on the distribution of individual eigenvalues, only first-order perturbation results are available for the joint pdf of the eigenvalues

Exact Joint pdf

Without any loss of generality the original eigenvalue problem can be expressed by

$$\mathbf{H}(\mathbf{x})\boldsymbol{\psi}_j = \omega_j^2\boldsymbol{\psi}_j \tag{2}$$

where

$$\mathbf{H}(\mathbf{x}) = \mathbf{M}^{-1/2}(\mathbf{x})\mathbf{K}(\mathbf{x})\mathbf{M}^{-1/2}(\mathbf{x}) \in \mathbb{R}^{N\times N}$$

and $\boldsymbol{\psi}_j = \mathbf{M}^{1/2}\boldsymbol{\phi}_j$

Exact Joint pdf

The joint probability (following Muirhead, 1982) density function of the natural frequencies of an N-dimensional linear positive definite dynamic system is given by

$$p_{\mathbf{\Omega}}\left(\omega_1, \omega_2, \cdots, \omega_N\right) = \frac{\pi^{N^2/2}}{\Gamma(N/2)} \prod_{i<j\leq N} \left(\omega_j^2 - \omega_i^2\right) \int_{O(N)} p_{\mathbf{H}}\left(\mathbf{\Psi}\mathbf{\Omega}^2\mathbf{\Psi}^T\right)\left(d\mathbf{\Psi}\right) \quad (3)$$

where $\mathbf{H} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$ & $p_{\mathbf{H}}(\mathbf{H})$ is the pdf of $\mathbf{H}$.

Limitations of the Exact Method

- the multidimensional integral over the orthogonal group $O(N)$ is difficult to carry out in practice and exact closed-form results can be derived only for few special cases
- the derivation of an expression of the joint pdf of the system matrix $p_{\mathbf{H}}(\mathbf{H})$ is non-trivial even if the joint pdf of the random system parameters $\mathbf{x}$ is known

Limitations of the Exact Method

- even one can overcome the previous two problems, the joint pdf of the natural frequencies given by Eq. (3) is ‘too much information’ to be useful for practical problems because
  - it is not easy to ‘visualize’ the joint pdf in the space of N natural frequencies, and
  - the derivation of the marginal density functions of the natural frequencies from Eq. (3) is not straightforward, especially when N is large.

Eigenvalues of GOE Matrices

Suppose the system matrix $\mathbf{H}$ is from a Gaussian orthogonal ensemble (GOE). The pdf of $\mathbf{H}$:

$$p_{\mathbf{H}}(\mathbf{H}) = \exp\left(-\theta_2 \mathrm{Trace}\left(\mathbf{H}^2\right) + \theta_1 \mathrm{Trace}\left(\mathbf{H}\right) + \theta_0\right)$$

The joint pdf of the natural frequencies:

$$p_{\mathbf{\Omega}}\left(\omega_1, \omega_2, \cdots, \omega_N\right) = \exp\left[-\left(\sum_{j=1}^{N} \theta_2 \omega_j^4 - \theta_1 \omega_j^2 - \theta_0\right)\right] \prod_{i<j} \left|\omega_j^2 - \omega_i^2\right|$$

Perturbation Method

Taylor series expansion of $\omega_j(\mathbf{x})$ about the mean $\mathbf{x} = \boldsymbol{\mu}$

$$\omega_j(\mathbf{x}) \approx \omega_j(\boldsymbol{\mu}) + \mathbf{d}_{\omega_j}^T(\boldsymbol{\mu})\,(\mathbf{x} - \boldsymbol{\mu}) + \frac{1}{2}\,(\mathbf{x} - \boldsymbol{\mu})^T\,\mathbf{D}_{\omega_j}(\boldsymbol{\mu})\,(\mathbf{x} - \boldsymbol{\mu})$$

Here $\mathbf{d}_{\omega_j}(\boldsymbol{\mu}) \in \mathbb{R}^m$ and $\mathbf{D}_{\omega_j}(\boldsymbol{\mu}) \in \mathbb{R}^{m\times m}$ are respectively the gradient vector and the Hessian matrix of $\omega_j(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\mu}$.

Joint Statistics

Joint statistics of the natural frequencies can be obtained provided it is assumed that the $\mathbf{x}$ is Gaussian. Assuming $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, first few cumulants can be obtained as

$$\kappa_{jk}^{(1,0)} = \mathrm{E}\left[\omega_j\right] = \overline{\omega}_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{D}_{\omega_j}\boldsymbol{\Sigma}\right),$$

$$\kappa_{jk}^{(0,1)} = \mathrm{E}\left[\omega_k\right] = \overline{\omega}_k + \frac{1}{2}\mathrm{Trace}\left(\mathbf{D}_{\omega_k}\boldsymbol{\Sigma}\right),$$

$$\kappa_{jk}^{(1,1)} = \mathrm{Cov}\left(\omega_j, \omega_k\right) = \frac{1}{2}\mathrm{Trace}\left(\left(\mathbf{D}_{\omega_j}\boldsymbol{\Sigma}\right)\left(\mathbf{D}_{\omega_k}\boldsymbol{\Sigma}\right)\right) + \mathbf{d}_{\omega_j}^T\boldsymbol{\Sigma}\mathbf{d}_{\omega_k}$$

Multidimensional Integrals

We want to evaluate an m-dimensional integral over the unbounded domain $\mathbb{R}^m$:

$$\mathcal{J} = \int_{\mathbb{R}^m} e^{-f(\mathbf{x})}\, d\mathbf{x}$$

- Assume $f(\mathbf{x})$ is smooth and at least twice differentiable
- The maximum contribution to this integral comes from the neighborhood where $f(\mathbf{x})$ reaches its global minimum, say $\boldsymbol{\theta} \in \mathbb{R}^m$

Multidimensional Integrals

Therefore, at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 0, \forall k \quad \text{or} \quad \mathbf{d}_f(\boldsymbol{\theta}) = \mathbf{0}$$

Expand $f(\mathbf{x})$ in a Taylor series about $\boldsymbol{\theta}$:

$$\mathcal{J} = \int_{\mathbb{R}^m} e^{-\left\{f(\boldsymbol{\theta}) + \frac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^T \mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta}) + \varepsilon(\mathbf{x},\boldsymbol{\theta})\right\}} d\mathbf{x}$$

$$= e^{-f(\boldsymbol{\theta})} \int_{\mathbb{R}^m} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^T \mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta}) - \varepsilon(\mathbf{x},\boldsymbol{\theta})} d\mathbf{x}$$

Multidimensional Integrals

The error $\varepsilon(\mathbf{x}, \boldsymbol{\theta})$ depends on higher derivatives of $f(\mathbf{x})$ at $\mathbf{x} = \boldsymbol{\theta}$. If they are small compared to $f(\boldsymbol{\theta})$ their contribution will negligible to the value of the integral. So we assume that $f(\boldsymbol{\theta})$ is large so that

$$\left| \frac{1}{f(\boldsymbol{\theta})} \mathcal{D}^{(j)}(f(\boldsymbol{\theta})) \right| \rightarrow 0 \quad \text{for} \quad j > 2$$

where $\mathcal{D}^{(j)}(f(\boldsymbol{\theta}))$ is jth order derivative of $f(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\theta}$. Under such assumptions $\varepsilon(\mathbf{x}, \boldsymbol{\theta}) \rightarrow 0$.

Multidimensional Integrals

- Use the coordinate transformation:

$$\boldsymbol{\xi} = (\mathbf{x} - \boldsymbol{\theta})\, \mathbf{D}_f^{-1/2}(\boldsymbol{\theta})$$

- The Jacobian: $\|\mathbf{J}\| = \|\mathbf{D}_f(\boldsymbol{\theta})\|^{-1/2}$
- The integral becomes:

$$\mathcal{J} \approx e^{-f(\boldsymbol{\theta})} \int_{\mathbb{R}^m} \|\mathbf{D}_f(\boldsymbol{\theta})\|^{-1/2} e^{-\frac{1}{2}\left(\boldsymbol{\xi}^T \boldsymbol{\xi}\right)} d\boldsymbol{\xi}$$

or

$$\boxed{\mathcal{J} \approx (2\pi)^{m/2} e^{-f(\boldsymbol{\theta})} \|\mathbf{D}_f(\boldsymbol{\theta})\|^{-1/2}}$$

Moments of Single Eigenvalues

An arbitrary rth order moment of the natural frequencies can be obtained from

$$\mu_j^{(r)} = \mathrm{E}\left[\omega_j^r(\mathbf{x})\right] = \int_{\mathbb{R}^m} \omega_j^r(\mathbf{x}) p_{\mathbf{x}}(\mathbf{x})\, d\mathbf{x}$$

$$= \int_{\mathbb{R}^m} e^{-(L(\mathbf{x})-r\ln\omega_j(\mathbf{x}))}\, d\mathbf{x}, \quad r = 1, 2, 3 \cdots$$

- Previous result can be used by choosing $f(\mathbf{x}) = L(\mathbf{x}) - r\ln\omega_j(\mathbf{x})$

Moments of Single Eigenvalues

After some simplifications

$$\mu_j^{(r)} \approx (2\pi)^{m/2}\omega_j^r(\boldsymbol{\theta})e^{-L(\boldsymbol{\theta})}$$
$$\left\|\mathbf{D}_L(\boldsymbol{\theta}) + \frac{1}{r}\mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^T - \frac{r}{\omega_j(\boldsymbol{\theta})}\mathbf{D}_{\omega_j}(\boldsymbol{\theta})\right\|^{-1/2}$$
$$r = 1, 2, 3, \cdots$$

$\boldsymbol{\theta}$ is obtained from:

$$\mathbf{d}_{\omega_j}(\boldsymbol{\theta})r = \omega_j(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})$$

Maximum Entropy pdf

Constraints for $u \in [0, \infty]$:

$$\int_0^\infty p_{\omega_j}(u)du = 1$$

$$\int_0^\infty u^r p_{\omega_j}(u)du = \mu_j^{(r)}, \quad r = 1, 2, 3, \cdots, n$$

Maximizing Shannon's measure of entropy $\mathcal{S} = -\int_0^\infty p_{\omega_j}(u) \ln p_{\omega_j}(u)du$, the pdf of ω_j is

$$p_{\omega_j}(u) = e^{-\left\{\rho_0 + \sum_{i=1}^n \rho_i u^i\right\}} = e^{-\rho_0} e^{-\sum_{i=1}^n \rho_i u^i}, \quad u \geq 0$$

Maximum Entropy pdf

Taking first two moments, the resulting pdf is a truncated Gaussian density function

$$p_{\omega_j}(u) = \frac{1}{\sqrt{2\pi}\sigma_j \, \Phi\left(\widehat{\omega}_j/\sigma_j\right)} \exp\left\{-\frac{\left(u-\widehat{\omega}_j\right)^2}{2\sigma_j^2}\right\}$$

where $\sigma_j^2 = \mu_j^{(2)} - \widehat{\omega}_j^2$

- Ensures that the probability of any natural frequencies becoming negative is zero

Joint Moments of Two Eigenvalues

Arbitrary $r-s$-th order joint moment of two natural frequencies

$$\mu_{jl}^{(rs)} = \mathrm{E}\left[\omega_j^r(\mathbf{x})\omega_l^s(\mathbf{x})\right]$$
$$= \int_{\mathbb{R}^m} \exp\left\{-\left(L(\mathbf{x}) - r\ln\omega_j(\mathbf{x}) - s\ln\omega_l(\mathbf{x})\right)\right\} d\mathbf{x},$$
$$r = 1, 2, 3 \cdots$$

- Choose $f(\mathbf{x}) = L(\mathbf{x}) - r\ln\omega_j(\mathbf{x}) - s\ln\omega_l(\mathbf{x})$

Joint Moments of Two Eigenvalues

After some simplifications

$$\mu_{jl}^{(rs)} \approx (2\pi)^{m/2} \omega_j^r(\boldsymbol{\theta}) \omega_l^s(\boldsymbol{\theta}) \exp\left\{-L\left(\boldsymbol{\theta}\right)\right\} \left\|\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\|^{-1/2}$$

where $\boldsymbol{\theta}$ is obtained from:

$$\mathbf{d}_L(\boldsymbol{\theta}) = \frac{r}{\omega_j(\boldsymbol{\theta})} \mathbf{d}_{\omega_j}(\boldsymbol{\theta}) + \frac{s}{\omega_l(\boldsymbol{\theta})} \mathbf{d}_{\omega_l}(\boldsymbol{\theta})$$

and $\mathbf{D}_f\left(\boldsymbol{\theta}\right) = \mathbf{D}_L(\boldsymbol{\theta}) + \frac{r}{\omega_j^2(\boldsymbol{\theta})} \mathbf{d}_{\omega_j}(\boldsymbol{\theta}) \mathbf{d}_{\omega_j}(\boldsymbol{\theta})^T - \frac{r}{\omega_j(\boldsymbol{\theta})} \mathbf{D}_{\omega_j}(\boldsymbol{\theta}) + \frac{s}{\omega_l^2(\boldsymbol{\theta})} \mathbf{d}_{\omega_l}(\boldsymbol{\theta}) \mathbf{d}_{\omega_l}(\boldsymbol{\theta})^T - \frac{s}{\omega_l(\boldsymbol{\theta})} \mathbf{D}_{\omega_l}(\boldsymbol{\theta})$

Joint Moments of Multiple Eigenvalues

We want to obtain

$$\mu_{j_1 j_2 \cdots j_n}^{(r_1 r_2 \cdots r_n)} = \int_{\mathbb{R}^m} \left\{ \omega_{j_1}^{r_1}(\mathbf{x}) \omega_{j_2}^{r_2}(\mathbf{x}) \cdots \omega_{j_n}^{r_n}(\mathbf{x}) \right\} p_{\mathbf{x}}(\mathbf{x})\, d\mathbf{x}$$

It can be shown that

$$\mu_{j_1 j_2 \cdots j_n}^{(r_1 r_2 \cdots r_n)} \approx (2\pi)^{m/2} \left\{ \omega_{j_1}^{r_1}(\boldsymbol{\theta})\, \omega_{j_2}^{r_2}(\boldsymbol{\theta}) \cdots \omega_{j_n}^{r_n}(\boldsymbol{\theta}) \right\} \exp\left\{-L(\boldsymbol{\theta})\right\} \left\|\mathbf{D}_f(\boldsymbol{\theta})\right\|^{-1/2}$$

Joint Moments of Multiple Eigenvalues

Here $\boldsymbol{\theta}$ is obtained from

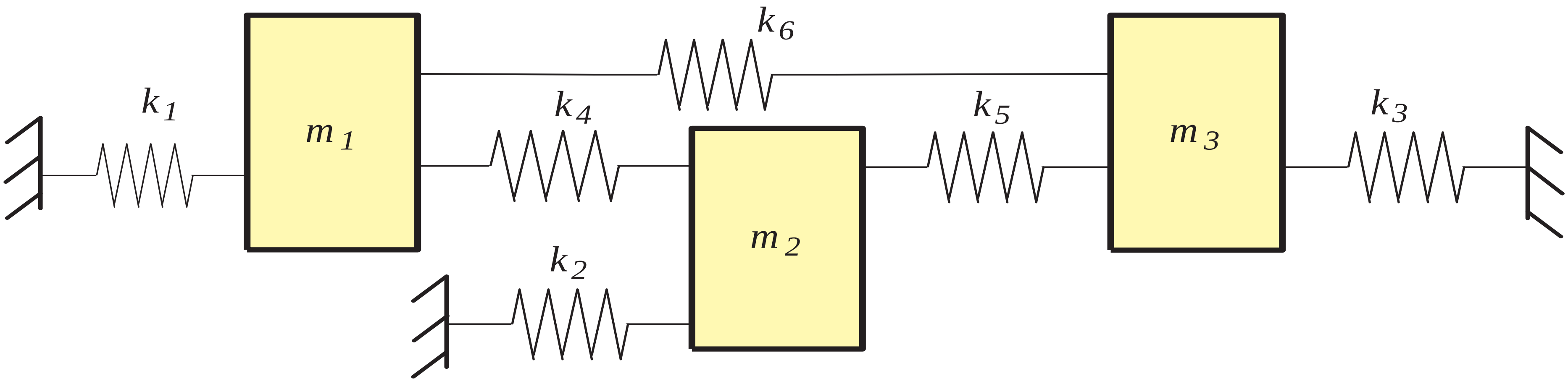
$$\mathbf{d}_L(\boldsymbol{\theta}) = \frac{r_1}{\omega_{j_1}(\boldsymbol{\theta})}\mathbf{d}_{\omega_{j_1}}(\boldsymbol{\theta}) + \frac{r_2}{\omega_{j_1}(\boldsymbol{\theta})}\mathbf{d}_{\omega_{j_2}}(\boldsymbol{\theta}) + \cdots \frac{r_n}{\omega_{j_n}(\boldsymbol{\theta})}\mathbf{d}_{\omega_{j_n}}(\boldsymbol{\theta})$$

and the Hessian matrix is given by

$$\mathbf{D}_f(\boldsymbol{\theta}) = \mathbf{D}_L(\boldsymbol{\theta}) + \sum_{\substack{j = j_1, j_2, \cdots \\ r = r_1, r_2, \cdots}}^{j_n, r_n} \frac{r}{\omega_j^2(\boldsymbol{\theta})}\mathbf{d}_{\omega_j}(\boldsymbol{\theta})\mathbf{d}_{\omega_j}(\boldsymbol{\theta})^T - \frac{r}{\omega_j(\boldsymbol{\theta})}\mathbf{D}_{\omega_j}(\boldsymbol{\theta})$$

Example System

Undamped three degree-of-freedom random system:



$\overline{m}_i = 1.0$ kg for $i = 1, 2, 3$; $\overline{k}_i = 1.0$ N/m for $i = 1, \cdots, 5$ and $k_6 = 3.0$ N/m

Example System

$m_i = \overline{m}_i \left(1 + \epsilon_m x_i\right)$, $i = 1, 2, 3$

$k_i = \overline{k}_i \left(1 + \epsilon_k x_{i+3}\right)$, $i = 1, \cdots, 6$

Vector of random variables: $\mathbf{x} = \{x_1, \cdots, x_9\}^T \in \mathbb{R}^9$

- $\mathbf{x}$ is standard Gaussian, $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$
- Strength parameters $\epsilon_m = 0.15$ and $\epsilon_k = 0.20$

Computational Methods

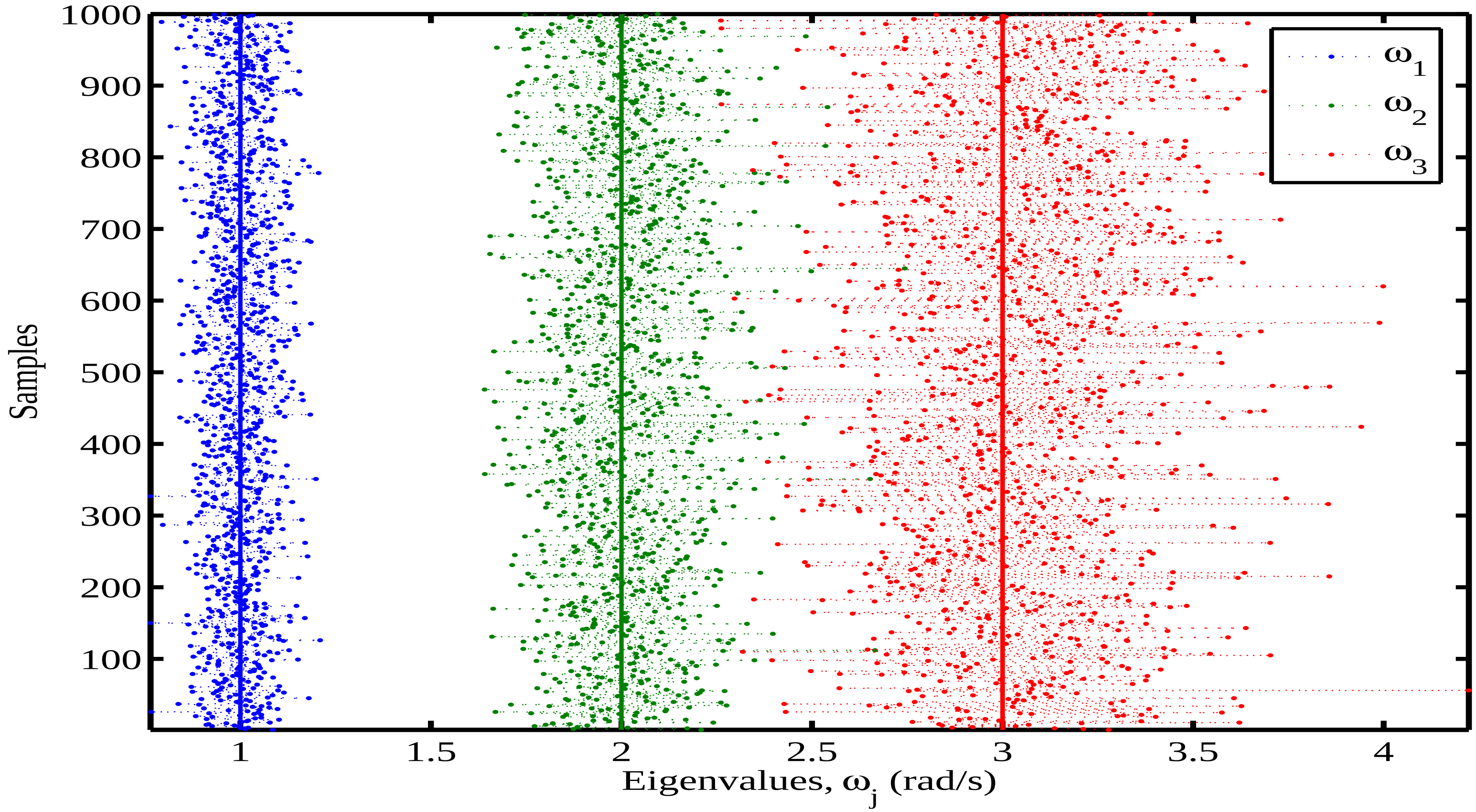
Following four methods are compared

1. *First-order perturbation*
2. *Second-order perturbation*
3. *Asymptotic method*
4. *Monte Carlo Simulation (15K samples)* - can be considered as benchmark.

The percentage error:

$$\text{Error} = \frac{(\bullet) - (\bullet)_{\text{MCS}}}{(\bullet)_{\text{MCS}}} \times 100$$

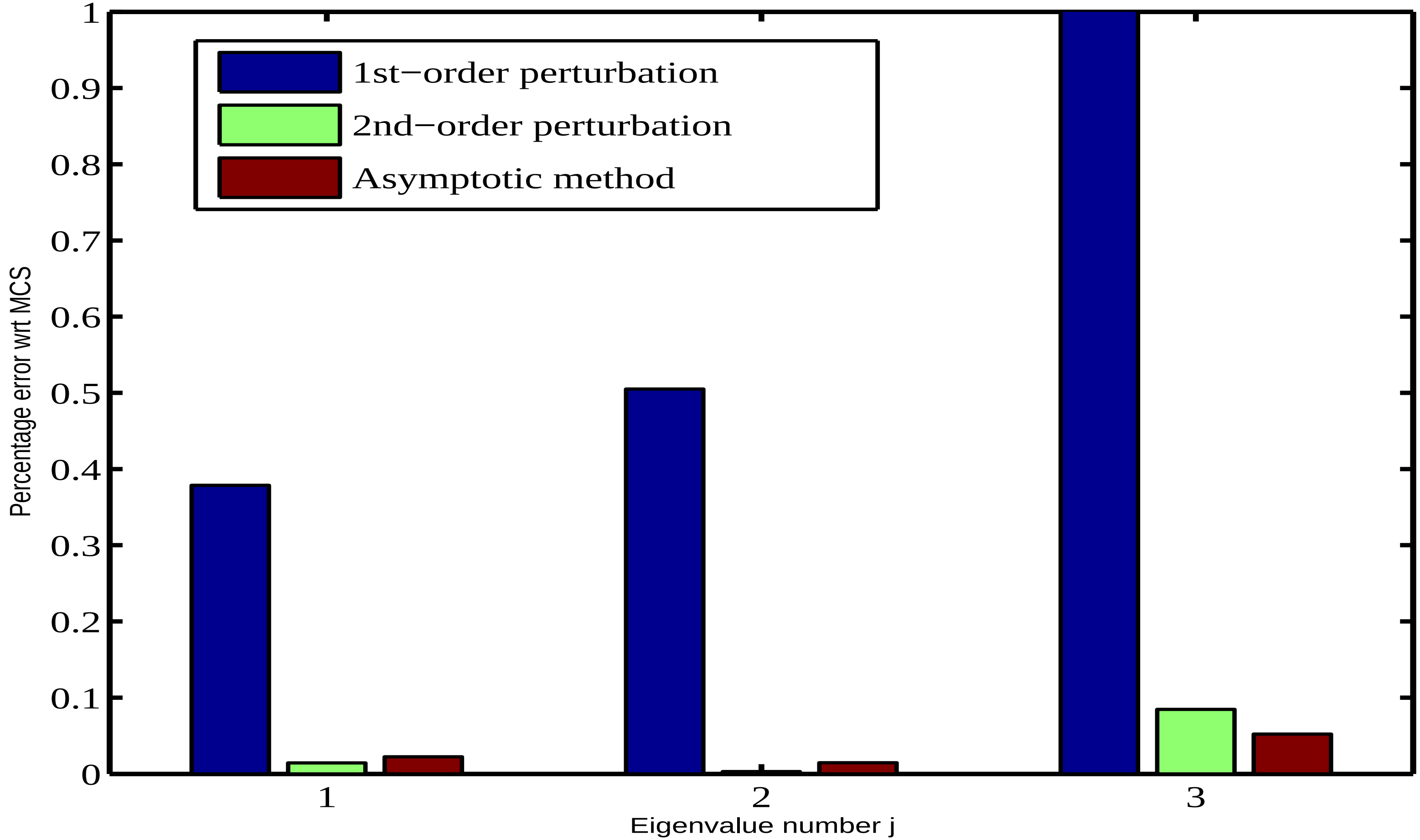
Scatter of the Eigenvalues



Statistical scatter of the natural frequencies

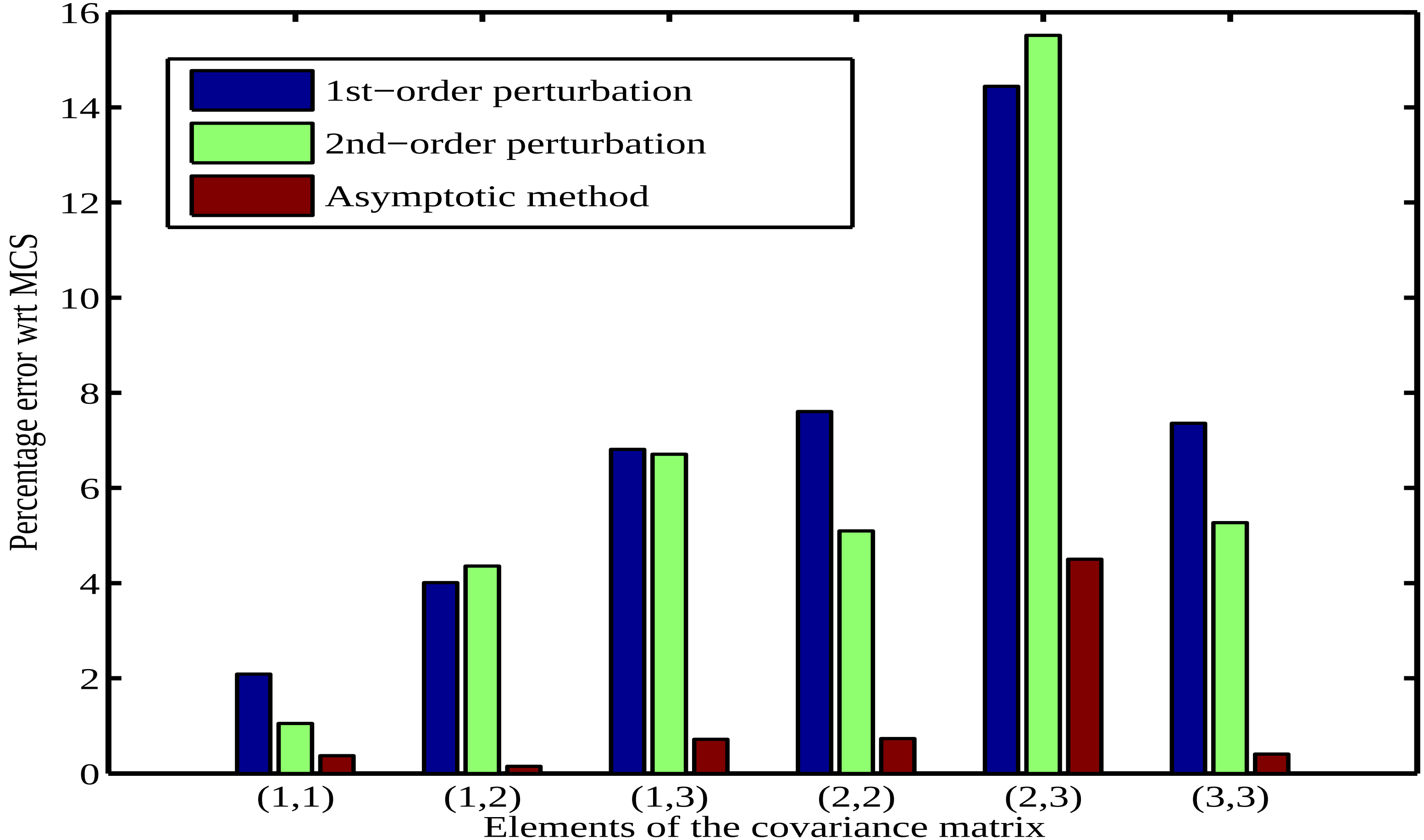
$$\overline{\omega}_1 = 1, \quad \overline{\omega}_2 = 2, \quad \text{and} \quad \overline{\omega}_3 = 3$$

Error in the Mean Values



Error in the mean values

Error in Covariance Matrix



Error in the elements of the covariance matrix

Mean and Covariance

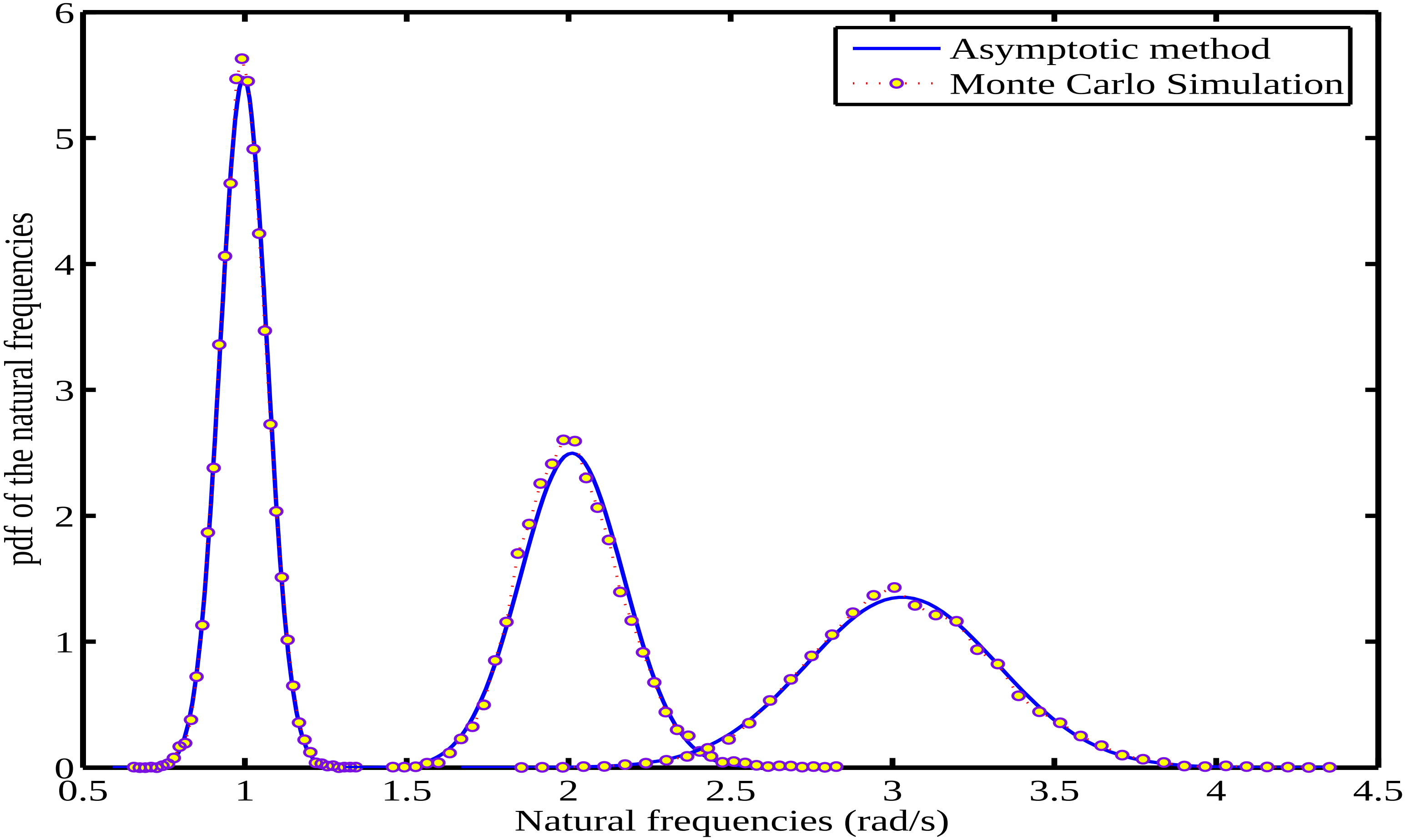
Using the asymptotic method, the mean and covariance matrix of the natural frequencies are obtained as

$$\boldsymbol{\mu}_{\boldsymbol{\Omega}} = \{0.9962,\ 2.0102,\ 3.0312\}^T$$

and $$\boldsymbol{\Sigma}_{\boldsymbol{\Omega}} = \begin{bmatrix} 0.5319 & 0.5643 & 0.7228 \\ 0.5643 & 2.5705 & 0.9821 \\ 0.7228 & 0.9821 & 8.7292 \end{bmatrix} \times 10^{-2}$$

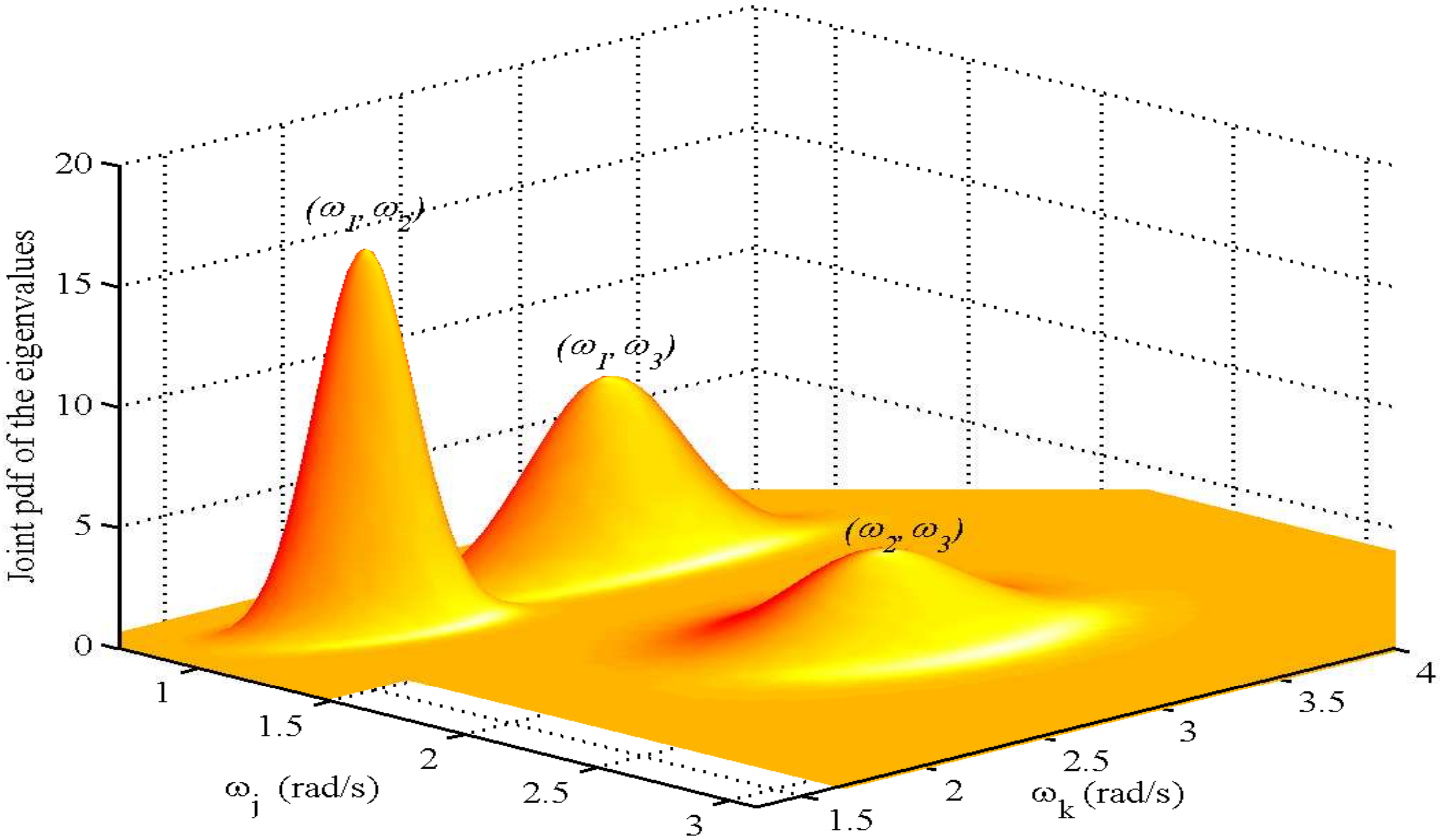
Individual pdf and joint pdf of the natural frequencies are computed using these values.

Individual pdf



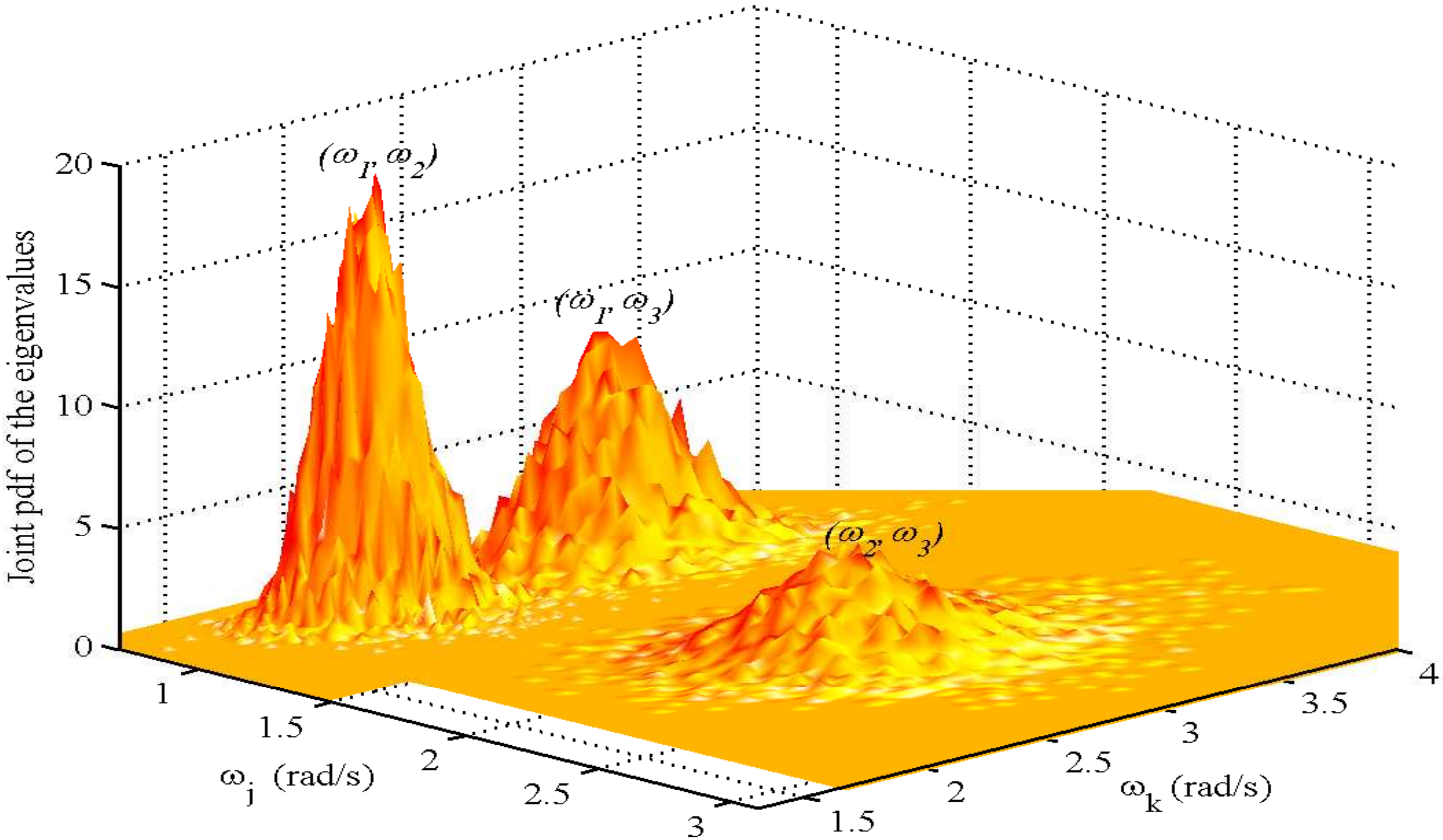
Individual pdf of the natural frequencies

Analytical Joint pdf



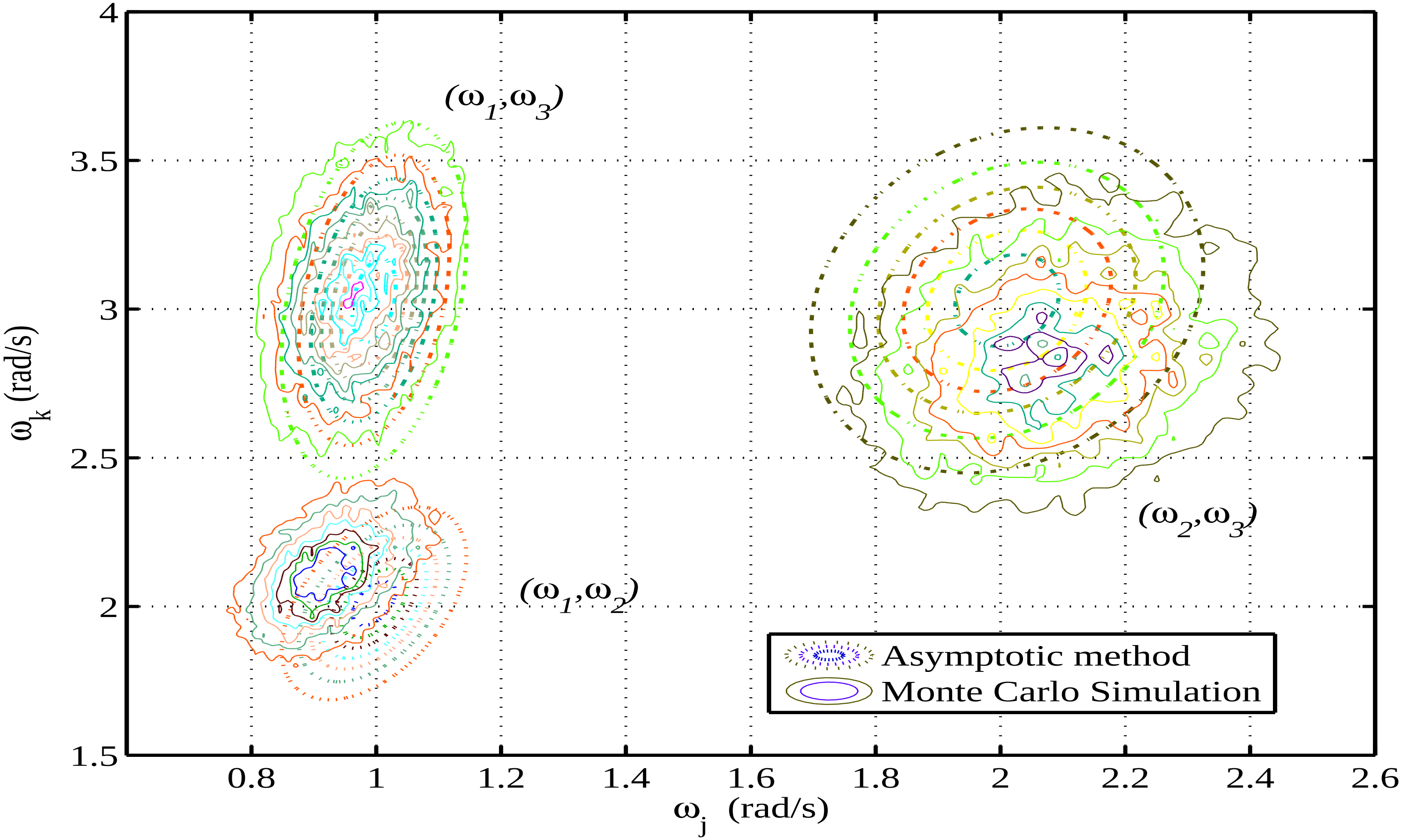
Joint pdf using asymptotic method

Joint pdf from MCS



Joint pdf from Monte Carlo Simulation

Contours of the joint pdf



Contours of the joint pdf

Conclusions

- Statistics of the natural frequencies of linear stochastic dynamic systems has been considered
- usual assumption of small randomness is not employed in this study.
- a general expression of the joint pdf of the natural frequencies of linear stochastic systems has been given

Conclusions

- a closed-form expression is obtained for the general order joint moments of the eigenvalues
- it was observed that the natural frequencies are *not* jointly Gaussian even they are so individually
- future studies will consider joint statistics of the eigenvalues and eigenvectors and dynamic response analysis using eigensolution distributions

References

Muirhead, R. J. (1982), *Aspects of Multivariate Statistical Theory*, John Wiely and Sons, New York, USA.