Dynamics of harmonically excited irregular cellular metamaterials

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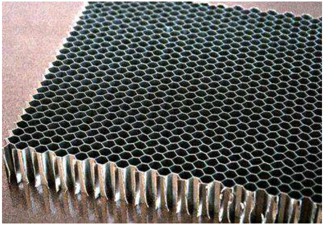
Outline

Lattice based metamaterials

- Metamaterials are artificial materials designed to outperform naturally occurring materials in various fronts.
- We are interested in mechanical metamaterials - here the main concern is in mechanical response of a material due to applied forces
- Lattice based metamaterials are abundant in man-made and natural systems at various length scales
- Among various lattice geometries (triangle, square, rectangle, pentagon, hexagon), hexagonal lattice is most common
- This talk is about in-plane elastic properties of 2D hexagonal lattice materials - commonly known as “honeycombs”

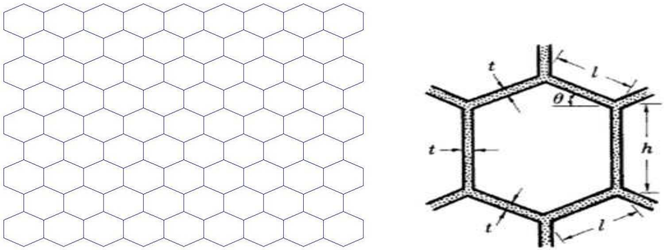


Mechanics of lattice materials

- Honeycombs have been modelled as a continuous solid with an equivalent elastic moduli throughout its domain.
- This approach eliminates the need of detail numerical (finite element) modelling in complex structural systems like sandwich structures.
- Extensive amount of research has been carried out to predict the equivalent elastic properties of regular honeycombs consisting of perfectly periodic hexagonal cells (Gibson and Ashby, 1999).
- Analysis of two dimensional honeycombs dealing with in-plane elastic properties are commonly based on the assumption of static forces
- In this work, we are interested in in-plane elastic properties when the applied forces are dynamic in nature
- Dynamic forcing is relevant, for example, in helicopter/wind turbine blades, aircraft wings where light weight and high stiffness is necessary

Equivalent elastic properties of regular honeycombs

- Unit cell approach - Gibson and Ashby (1999)



(a) Regular hexagon ($\theta = 30^{\circ}$) (b) Unit cell

- We are interested in homogenised equivalent in-plane elastic properties
- This way, we can avoid a detailed structural analysis considering all the beams and treat it as a material

Equivalent elastic properties of regular honeycombs

- The cell walls are treated as beams of thickness t, depth b and Young's modulus E_s. l and h are the lengths of inclined cell walls having inclination angle θ and the vertical cell walls respectively.
- The equivalent elastic properties are:

$$E_1 = E_s \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l}+\sin\theta)\sin^2\theta} \tag{1}$$

$$E_2 = E_s \left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l}+\sin\theta)}{\cos^3\theta} \tag{2}$$

$$\nu_{12} = \frac{\cos^2\theta}{(\frac{h}{l}+\sin\theta)\sin\theta} \tag{3}$$

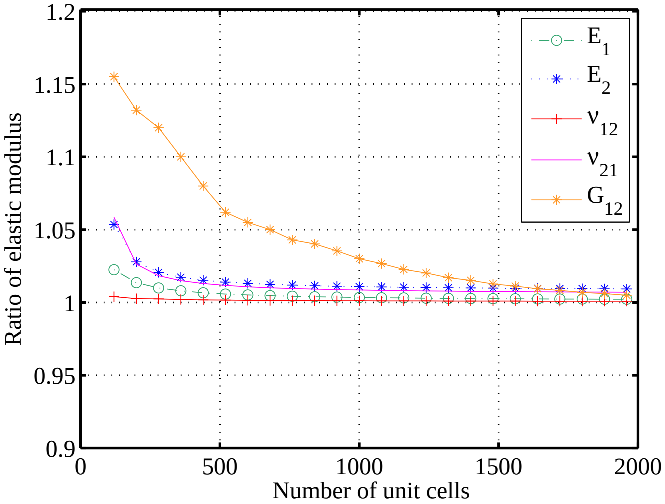
$$\nu_{21} = \frac{(\frac{h}{l}+\sin\theta)\sin\theta}{\cos^2\theta} \tag{4}$$

$$G_{12} = E_s \left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l}+\sin\theta)}{\left(\frac{h}{l}\right)^2(1+2\frac{h}{l})\cos\theta} \tag{5}$$

Finite element modelling and verification

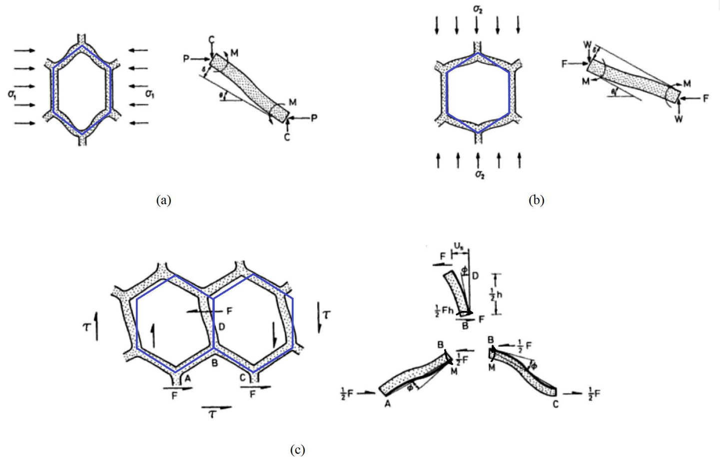
- A finite element code has been developed to obtain the in-plane elastic moduli numerically for honeycombs.
- Each cell wall has been modelled as an Euler-Bernoulli beam element having three degrees of freedom at each node.
- For E_1 and ν_{12}: two opposite edges parallel to direction-2 of the entire honeycomb structure are considered. Along one of these two edges, uniform stress parallel to direction-1 is applied while the opposite edge is restrained against translation in direction-1. Remaining two edges (parallel to direction-1) are kept free.
- For E_2 and ν_{21}: two opposite edges parallel to direction-1 of the entire honeycomb structure are considered. Along one of these two edges, uniform stress parallel to direction-2 is applied while the opposite edge is restrained against translation in direction-2. Remaining two edges (parallel to direction-2) are kept free.
- For G_{12}: uniform shear stress is applied along one edge keeping the opposite edge restrained against translation in direction-1 and 2, while the remaining two edges are kept free.

Finite element modelling and verification



$\theta = 30^\circ$, $h/l = 1.5$. FE results converge to analytical predictions after 1681 cells.

The deformation of a unit cell



(a) Deformed shape and free body diagram under the application of stress in direction - 1 (b) Deformed shape and free body diagram under the application of stress in direction - 2 (c) Deformed shape and free body diagram under the application of shear stress (The undeformed shapes of the hexagonal cell are indicated using blue colour for each of the loading conditions.

The element stiffness matrix of a beam

- The equation governing the transverse deflection $V(x)$ of the beam can be expressed as

$$EI\frac{d^4V(x)}{dx^4} = f(x) \tag{6}$$

  It is assumed that the behaviour of the beam follows the Euler-Bernoulli hypotheses

- Using the finite element method, the element stiffness matrix of a beam can be expressed as

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L^2 \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{7}$$

- The area moment of inertia $I = \dfrac{\bar{b}t^3}{12}$

Equivalent elastic properties

- Young's modulus E_1:

$$E_1 = \frac{\sigma_1}{\epsilon_{11}} = \frac{A_{33} l \cos\theta}{(h + l\sin\theta)\bar{b}\sin^2\theta} = \frac{A_{33}}{\bar{b}} \frac{\cos\theta}{(\frac{h}{l} + \sin\theta)\sin^2\theta} \quad (8)$$

- Young's modulus E_2:

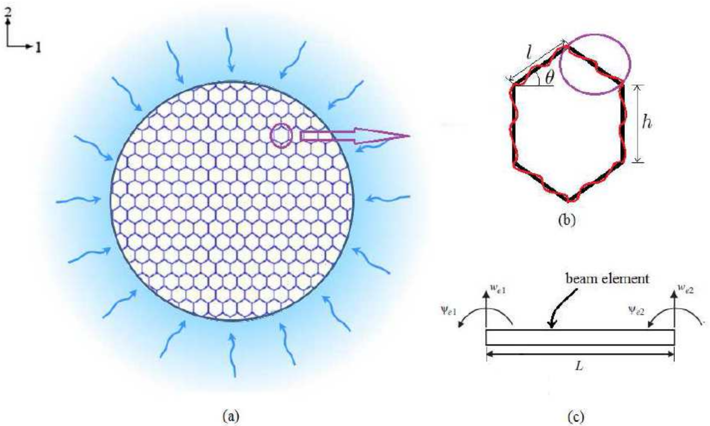
$$E_2 = \frac{\sigma_2}{\epsilon_{22}} = \frac{A_{33}(h + l\sin\theta)}{l\bar{b}\cos^3\theta} = \frac{A_{33}}{\bar{b}} \frac{(\frac{h}{l} + \sin\theta)}{\cos^3\theta} \quad (9)$$

- Shear modulus G_{12}:

$$G_{12} = \frac{\tau}{\gamma} = \frac{1}{\frac{2l\bar{b}\cos\theta}{(h+l\sin\theta)}\left(\frac{h^2}{4lA^s_{43}} + \frac{2}{\left(A^v_{33} - \frac{A^v_{34}A^v_{43}}{A^v_{44}}\right)}\right)} \quad (10)$$

$(\bullet)^v$ = vertical element; $(\bullet)^s$ = slant element

Dynamic equivalent proerties



(a) Typical representation of a hexagonal lattice structure in a dynamic environment (e.g., the honeycomb as part of a host structure experiencing wave propagation). (b) One hexagonal unit cell under dynamic environment (c) A dynamic element for the bending vibration of a damped beam with length L. It has two nodes and four degrees of freedom.

Dynamic stiffness matrix

- Individual elements of the lattice have been considered as damped Euler-Bernoulli beams with the equation of motion

$$EI\frac{\partial^4 V(x,t)}{\partial x^4} + c_1\frac{\partial^5 V(x,t)}{\partial x^4 \partial t} + m\frac{\partial^2 V(x,t)}{\partial t^2} + c_2\frac{\partial V(x,t)}{\partial t} = 0 \qquad (11)$$

- Using the dynamic finite element method, the element stiffness matrix of a beam can be expressed as

$$\mathbf{A} = \frac{EIb}{(cC-1)}\begin{bmatrix} -b^2(cS+Cs) & -sbS & b^2(S+s) & -b(C-c) \\ -sbS & -Cs+cS & b(C-c) & -S+s \\ b^2(S+s) & b(C-c) & -b^2(cS+Cs) & sbS \\ -b(C-c) & -S+s & sbS & -Cs+cS \end{bmatrix} \qquad (12)$$

where

$$C = \cosh(bL), \quad c = \cos(bL), \quad S = \sinh(bL) \quad \text{and} \quad s = \sin(bL) \qquad (13)$$

- $$b^4 = \frac{m\omega^2(1 - \mathrm{i}\zeta_m/\omega)}{EI(1+\mathrm{i}\omega\zeta_k)}; \zeta_k = c_1/(EI), \zeta_m = c_2/m \qquad (14)$$

Equivalent dynamic elastic properties

- Young's modulus E_1:

$$E_1 = \frac{Et^3(cS + Cs)b^3 l \cos\theta}{12(h + l\sin\theta)\sin^2\theta(-1 + cC)} \tag{15}$$

- Young's modulus E_2:

$$E_2 = \frac{Et^3(cS + Cs)b^3(h + l\sin\theta)}{12 l \cos^3\theta(-1 + cC)} \tag{16}$$

- Shear modulus G_{12}:

$$G_{12} = \frac{Et^3(h + l\sin\theta)}{48 l \cos\theta \left(\frac{h^2(c^s C^s - 1)}{8 l s^s S^s b^2} + \frac{(c^v C^v - 1)(c^v S^v - C^v s^v)}{b^3 (C^{v2} s^{v2} - c^{v2} S^{v2} - s^{v2} S^{v2})} \right)} \tag{17}$$

$C^s = \cosh(bl)$, $c^s = \cos(bl)$, $S^s = \sinh(bl)$, $s^s = \sin(bl)$, $C^v = \cosh\left(\frac{bh}{2}\right)$, $c^v = \cos\left(\frac{bh}{2}\right)$, $S^v = \sinh\left(\frac{bh}{2}\right)$ and $s^v = \sin\left(\frac{bh}{2}\right)$

Frequency dependent Young's modulus: E_1

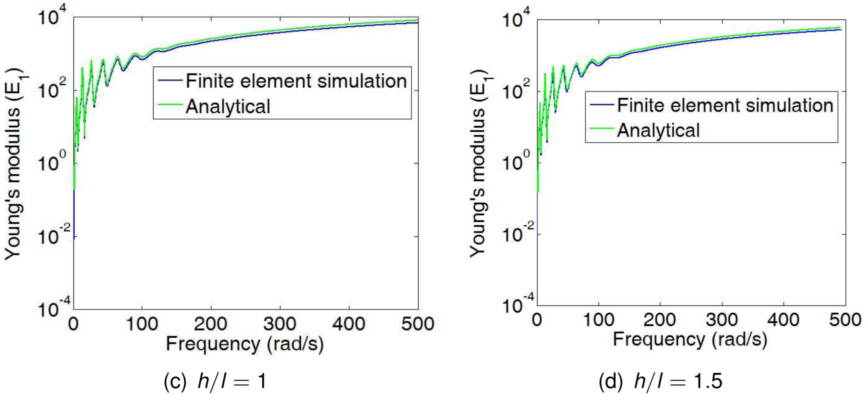
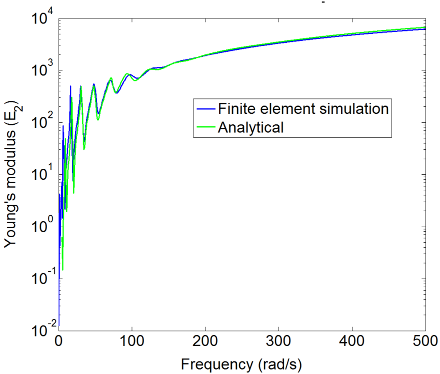
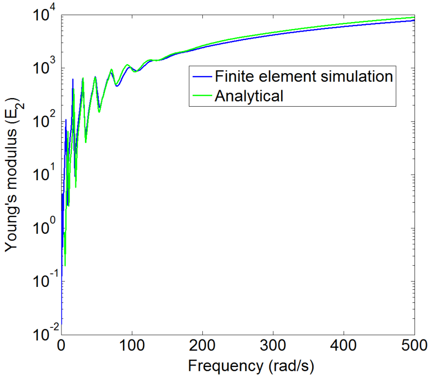


Figure: Frequency dependent Young's modulus (E_1) of regular hexagonal lattices with $\theta = 30^\circ$ and $\zeta_m = 0.05$ and $\zeta_k = 0.002$

Frequency dependent Young's modulus: E_2



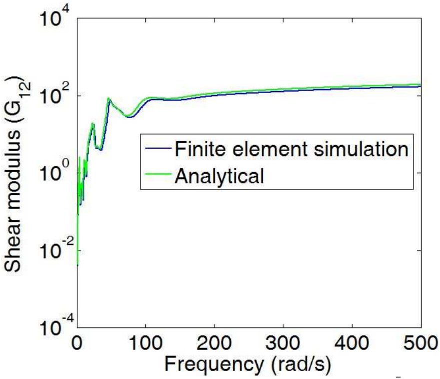
(a) $h/l = 1$



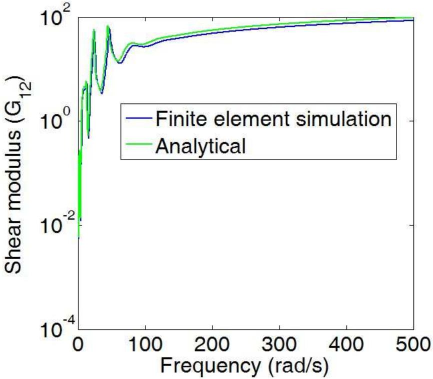
(b) $h/l = 1.5$

Figure: Frequency dependent Young's modulus (E_2) of regular hexagonal lattices with $\theta = 30^\circ$ and $\zeta_m = 0.05$ and $\zeta_k = 0.002$

Frequency dependent shear modulus: G_{12}



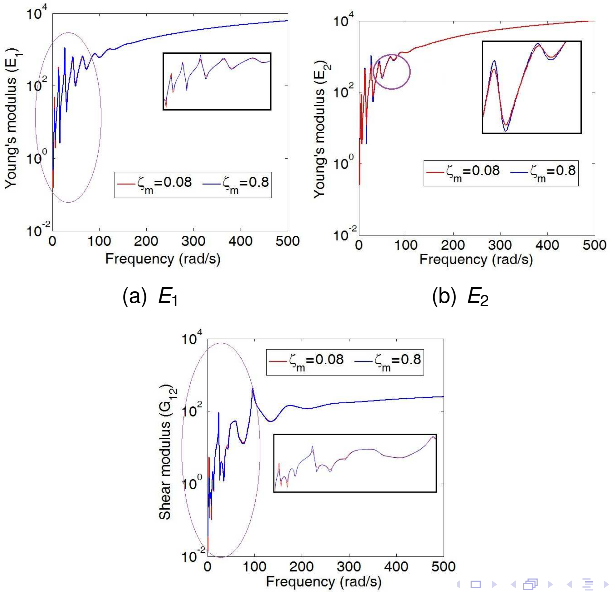
(a) $h/l = 1$



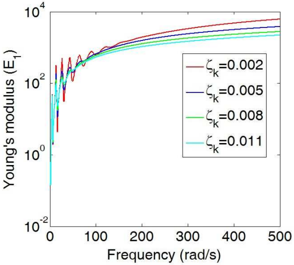
(b) $h/l = 1.5$

Figure: Frequency dependent shear modulus (G_{12}) of regular hexagonal lattices with $\theta = 30^\circ$ and $\zeta_m = 0.05$ and $\zeta_k = 0.002$

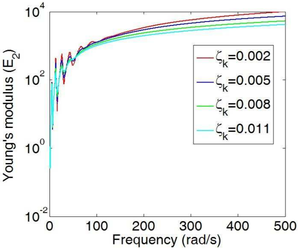
Effect of variation in mass proportional damping factor



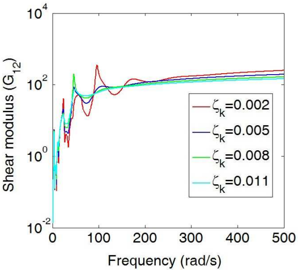
Effect of variation in stiffness proportional damping factor



(a) E_1



(b) E_2



Conclusions

- Equivalent dynamic homogenised elastic properties of damped cellular metamaterials have been considered.
- It is shown that the material is anisotropic with complex-valued equivalent elastic moduli.
- The two Young's moduli and shear modulus are dependent on frequency values. Two in-plane Poisson's ratios depend only on structural geometry of the lattice structure.
- Using the principle of basic structural dynamics on a unit cell with a dynamic stiffness technique, closed-form expressions have been obtained for E_1, E_2, ν_{12}, ν_{21} and G_{12}.
- The new results reduce to the classical formulae of Gibson and Ashby for the special case when frequency goes to zero (static).
- Future research will consider different types of unit cell geometries.

Closed-form expressions: Dynamic Homogenisation

$$E_1 = \frac{Et^3(cS + Cs)b^3 l \cos\theta}{12(h + l\sin\theta)\sin^2\theta(-1 + cC)} \tag{18}$$

$$E_2 = \frac{Et^3(cS + Cs)b^3(h + l\sin\theta)}{12 l \cos^3\theta(-1 + cC)} \tag{19}$$

$$\nu_{12} = \frac{\cos^2\theta}{(\frac{h}{l} + \sin\theta)\sin\theta} \tag{20}$$

$$\nu_{21} = \frac{(\frac{h}{l} + \sin\theta)\sin\theta}{\cos^2\theta} \tag{21}$$

$$G_{12} = \frac{Et^3(h + l\sin\theta)}{48 l \cos\theta \left(\frac{h^2(c^s C^s - 1)}{8 l s^s S^s b^2} + \frac{(c^v C^v - 1)(c^v S^v - C^v s^v)}{b^3 (C^{v2} s^{v2} - c^{v2} S^{v2} - s^{v2} S^{v2})} \right)} \tag{22}$$

In the limit, frequency $\omega \to 0$, they reduce to the classical 'static' homogenised values.

Some of our papers on this topic

1. Mukhopadhyay, T. and Adhikari, S., "Effective in-plane elastic properties of quasi-random spatially irregular hexagonal lattices", International Journal of Engineering Science, (in press).
2. Mukhopadhyay, T., Mahata, A., Asle Zaeem, M. and Adhikari, S., "Effective elastic properties of two dimensional multiplanar hexagonal nano-structures", 2D Materials, 4[2] (2017), pp. 025006:1-15.
3. Mukhopadhyay, T. and Adhikari, S., "Stochastic mechanics of metamaterials", Composite Structures, 162[2] (2017), pp. 85-97.
4. Mukhopadhyay, T. and Adhikari, S., "Free vibration of sandwich panels with randomly irregular honeycomb core", ASCE Journal of Engineering Mechanics,141[6] (2016), pp. 06016008:1-5.
5. Mukhopadhyay, T. and Adhikari, S., "Equivalent in-plane elastic properties of irregular honeycombs: An analytical approach", International Journal of Solids and Structures, 91[8] (2016), pp. 169-184.
6. Mukhopadhyay, T. and Adhikari, S., "Effective in-plane elastic properties of auxetic honeycombs with spatial irregularity", Mechanics of Materials, 95[2] (2016), pp. 204-222.