Matrix-Eigenvalue Problems in Stochastic Structural Dynamics

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Outline of the Presentation

- Random eigenvalue problem
- Perturbation Methods
- Asymptotic analysis of multidimensional integrals
- Moments and pdf of the eigenvalues
- Numerical Example & results
- Conclusions & open problems

Random Eigenvalue Problem

The random eigenvalue problem of undamped or proportionally damped linear systems:

$$\mathbf{K}(\mathbf{x})\boldsymbol{\phi}_j = \lambda_j \mathbf{M}(\mathbf{x})\boldsymbol{\phi}_j$$

λ_j eigenvalues; $\boldsymbol{\phi}_j$ eigenvectors; $\mathbf{M}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ mass matrix and $\mathbf{K}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ stiffness matrix. $\mathbf{x} \in \mathbb{R}^m$ is random parameter vector with pdf

$$p_{\mathbf{x}}(\mathbf{x}) = e^{-L(\mathbf{x})}$$

$-L(\mathbf{x})$ is the log-likelihood function.

The Broad Issues

- To obtain the joint probability density function of the eigenvalues and the eigenvectors
- If a matrix $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$ has pdf $f(\mathbf{A})$ then the joint pdf of the eigenvalues (R. J. Muirhead, Theorem 3.2.17, pp 104)

$$\frac{\pi^{N^2/2}}{\Gamma_N((N/2))} \Pi_{i\leq j}^{N} (\lambda_i - \lambda_j) \int_{O(N)} f(\mathbf{H}\mathbf{\Lambda}\mathbf{H}^T) d\mathbf{H}$$

- It is hard to get marginal distribution of the eigenvalues - too much information!!

Perturbation Method

Taylor series expansion of $\lambda_j(\mathbf{x})$ about $\mathbf{x} = \boldsymbol{\alpha}$

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha}) + \mathbf{d}_{\lambda_j}^T(\boldsymbol{\alpha})\left(\mathbf{x} - \boldsymbol{\alpha}\right) + \frac{1}{2}\left(\mathbf{x} - \boldsymbol{\alpha}\right)^T \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\left(\mathbf{x} - \boldsymbol{\alpha}\right)$$

- In mean-centered approach $\boldsymbol{\alpha}$ is the mean of $\mathbf{x}$
- Alternatively, $\boldsymbol{\alpha}$ can be obtained such that the any moment of each eigenvalue is calculated most accurately

α-centered perturbation

The rth moment of $\lambda_j(\mathbf{x})$:

$$\lambda_j^{(r)} = \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x}) e^{-L(\mathbf{x})}\, d\mathbf{x} = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-h_j(\mathbf{x})}\, d\mathbf{x} \tag{1}$$

where $$h_j(\mathbf{x}) = L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x}) \tag{2}$$

Expand the function $h(\mathbf{x})$ in a Taylor series about a point where $h_j(\mathbf{x})$ attends its global minimum.

α-centered perturbation

Therefore, the optimal point can be obtained as

$$\frac{\partial h_j(\mathbf{x})}{\partial x_k} = 0, \quad \forall k \tag{3}$$

Combining for all k we have

$$\mathbf{d}_{\lambda_j}(\boldsymbol{\alpha}) = \lambda_j(\boldsymbol{\alpha})\mathbf{d}_L(\boldsymbol{\alpha})/r \tag{4}$$

Multidimensional Integrals

We want to evaluate an m-dimensional integral over the unbounded domain $\mathbb{R}^m$:

$$\mathcal{J} = \int_{\mathbb{R}^m} e^{-f(\mathbf{x})}\, d\mathbf{x}$$

- Assume $f(\mathbf{x})$ is smooth and at least twice differentiable
- The maximum contribution to this integral comes from the neighborhood where $f(\mathbf{x})$ reaches its global minimum, say $\boldsymbol{\theta} \in \mathbb{R}^m$

Multidimensional Integrals

Therefore, at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 0, \forall k \quad \text{or} \quad \mathbf{d}_f(\boldsymbol{\theta}) = \mathbf{0}$$

Expand $f(\mathbf{x})$ in a Taylor series about $\boldsymbol{\theta}$:

$$\mathcal{J} = \int_{\mathbb{R}^m} e^{-\left\{f(\boldsymbol{\theta}) + \frac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^T \mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta}) + \varepsilon(\mathbf{x},\boldsymbol{\theta})\right\}} d\mathbf{x}$$

$$= e^{-f(\boldsymbol{\theta})} \int_{\mathbb{R}^m} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^T \mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta}) - \varepsilon(\mathbf{x},\boldsymbol{\theta})} d\mathbf{x}$$

Multidimensional Integrals

- Use the coordinate transformation:

$$\boldsymbol{\xi} = (\mathbf{x} - \boldsymbol{\theta})\, \mathbf{D}_f^{-1/2}(\boldsymbol{\theta})$$

- The Jacobian: $\|\mathbf{J}\| = \|\mathbf{D}_f(\boldsymbol{\theta})\|^{-1/2}$
- The integral becomes:

$$\mathcal{J} \approx e^{-f(\boldsymbol{\theta})} \int_{\mathbb{R}^m} \|\mathbf{D}_f(\boldsymbol{\theta})\|^{-1/2}\, e^{-\frac{1}{2}\left(\boldsymbol{\xi}^T \boldsymbol{\xi}\right)}\, d\boldsymbol{\xi}$$

or

$$\mathcal{J} \approx (2\pi)^{m/2} e^{-f(\boldsymbol{\theta})} \|\mathbf{D}_f(\boldsymbol{\theta})\|^{-1/2}$$

Moments of Single Eigenvalues

An arbitrary rth order moment of the eigenvalues can be obtained from

$$\mu_j^{(r)} = \mathrm{E}\left[\lambda_j^r(\mathbf{x})\right] = \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x}) p_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$
$$= \int_{\mathbb{R}^m} e^{-(L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x}))}\, d\mathbf{x}, \quad r = 1, 2, 3 \cdots$$

- Previous result can be used by choosing $f(\mathbf{x}) = L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x})$

Moments of Single Eigenvalues

After some simplifications

$$\mu_j^{(r)} \approx (2\pi)^{m/2} \lambda_j^r(\boldsymbol{\theta}) e^{-L(\boldsymbol{\theta})}$$

$$\left\| \mathbf{D}_L(\boldsymbol{\theta}) + \frac{1}{r}\mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^T - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \right\|^{-1/2}$$

$$r = 1, 2, 3, \cdots$$

$\boldsymbol{\theta}$ is obtained from:

$$\mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) r = \lambda_j(\boldsymbol{\theta}) \mathbf{d}_L(\boldsymbol{\theta})$$

Maximum Entropy pdf

Constraints for $u \in [0, \infty]$:

$$\int_0^\infty p_{\lambda_j}(u) du = 1$$

$$\int_0^\infty u^r p_{\lambda_j}(u) du = \mu_j^{(r)}, \quad r = 1, 2, 3, \cdots, n$$

Maximizing Shannon's measure of entropy $\mathcal{S} = -\int_0^\infty p_{\lambda_j}(u) \ln p_{\lambda_j}(u) du$, the pdf of λ_j is

$$p_{\lambda_j}(u) = e^{-\left\{\rho_0 + \sum_{i=1}^n \rho_i u^i\right\}} = e^{-\rho_0} e^{-\sum_{i=1}^n \rho_i u^i}, \quad u \geq 0$$

Maximum Entropy pdf

Taking first two moments, the resulting pdf is a truncated Gaussian density function

$$p_{\lambda_j}(u) = \frac{1}{\sqrt{2\pi}\sigma_j \, \Phi\left(\widehat{\lambda}_j/\sigma_j\right)} \exp\left\{ -\frac{\left(u - \widehat{\lambda}_j\right)^2}{2\sigma_j^2} \right\}$$

where $\sigma_j^2 = \mu_j^{(2)} - \widehat{\lambda}_j^2$

- Ensures that the probability of any eigenvalues becoming negative is zero

Central χ^2 Approximation

Pdf of jth eigenvalue

$$p_{\lambda_j}(u) \approx \frac{1}{\gamma_j} p_{\chi^2_{\nu_j}} \left(\frac{u - \eta_j}{\gamma_j} \right) = \frac{(u - \eta_j)^{\nu_j/2-1} e^{-(u-\eta_j)/2\gamma_j}}{(2\gamma_j)^{\nu_j/2} \Gamma(\nu_j/2)}$$

The constants η_j, γ_j, and ν_j are such that the first three moments of λ_j are the same.

Joint Moments of Two Eigenvalues

Arbitrary $r-s$-th order joint moment of two eigenvalues

$$\mu_{jl}^{(rs)} = \mathrm{E}\left[\lambda_j^r(\mathbf{x})\lambda_l^s(\mathbf{x})\right]$$
$$= \int_{\mathbb{R}^m} \exp\left\{-\left(L(\mathbf{x}) - r\ln\lambda_j(\mathbf{x}) - s\ln\lambda_l(\mathbf{x})\right)\right\}\, d\mathbf{x},$$

- Choose $f(\mathbf{x}) = L(\mathbf{x}) - r\ln\lambda_j(\mathbf{x}) - s\ln\lambda_l(\mathbf{x})$

Joint Moments of Two Eigenvalues

After some simplifications

$$\mu_{jl}^{(rs)} \approx (2\pi)^{m/2} \lambda_j^r(\boldsymbol{\theta}) \lambda_l^s(\boldsymbol{\theta}) \exp\left\{-L\left(\boldsymbol{\theta}\right)\right\} \left\|\mathbf{D}_f\left(\boldsymbol{\theta}\right)\right\|^{-1/2}$$

where $\boldsymbol{\theta}$ is obtained from:

$$\mathbf{d}_L(\boldsymbol{\theta}) = \frac{r}{\lambda_j(\boldsymbol{\theta})} \mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) + \frac{s}{\lambda_l(\boldsymbol{\theta})} \mathbf{d}_{\lambda_l}(\boldsymbol{\theta})$$

$$\mathbf{D}_f\left(\boldsymbol{\theta}\right) = \mathbf{D}_L(\boldsymbol{\theta}) + \frac{r}{\lambda_j^2(\boldsymbol{\theta})} \mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) \mathbf{d}_{\lambda_j}(\boldsymbol{\theta})^T - \frac{r}{\lambda_j(\boldsymbol{\theta})} \mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) + \frac{s}{\lambda_l^2(\boldsymbol{\theta})} \mathbf{d}_{\lambda_l}(\boldsymbol{\theta}) \mathbf{d}_{\lambda_l}(\boldsymbol{\theta})^T - \frac{s}{\lambda_l(\boldsymbol{\theta})} \mathbf{D}_{\lambda_l}(\boldsymbol{\theta})$$

Joint Moments of Multiple Eigenvalues

We want to obtain

$$\mu_{j_1 j_2 \cdots j_n}^{(r_1 r_2 \cdots r_n)} = \int_{\mathbb{R}^m} \left\{ \lambda_{j_1}^{r_1}(\mathbf{x}) \lambda_{j_2}^{r_2}(\mathbf{x}) \cdots \lambda_{j_n}^{r_n}(\mathbf{x}) \right\} p_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

It can be shown that

$$\mu_{j_1 j_2 \cdots j_n}^{(r_1 r_2 \cdots r_n)} \approx (2\pi)^{m/2} \left\{ \lambda_{j_1}^{r_1}(\boldsymbol{\theta})\, \lambda_{j_2}^{r_2}(\boldsymbol{\theta}) \cdots \lambda_{j_n}^{r_n}(\boldsymbol{\theta}) \right\} \exp\left\{ -L(\boldsymbol{\theta}) \right\} \left\| \mathbf{D}_f(\boldsymbol{\theta}) \right\|^{-1/2}$$

Joint Moments of Multiple Eigenvalues

Here $\boldsymbol{\theta}$ is obtained from

$$\mathbf{d}_L(\boldsymbol{\theta}) = \frac{r_1}{\lambda_{j_1}(\boldsymbol{\theta})}\mathbf{d}_{\lambda_{j_1}}(\boldsymbol{\theta}) + \frac{r_2}{\lambda_{j_1}(\boldsymbol{\theta})}\mathbf{d}_{\lambda_{j_2}}(\boldsymbol{\theta}) + \cdots \frac{r_n}{\lambda_{j_n}(\boldsymbol{\theta})}\mathbf{d}_{\lambda_{j_n}}(\boldsymbol{\theta}$$

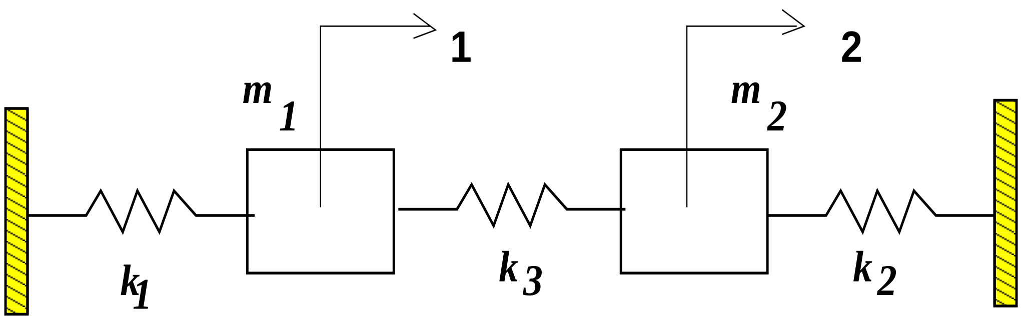
and the Hessian matrix is given by

$$\mathbf{D}_f(\boldsymbol{\theta}) = \mathbf{D}_L(\boldsymbol{\theta}) + \sum_{\substack{j = j_1, j_2, \cdots \\ r = r_1, r_2, \cdots}}^{j_n, r_n} \frac{r}{\lambda_j^2(\boldsymbol{\theta})}\mathbf{d}_{\lambda_j}(\boldsymbol{\theta})\mathbf{d}_{\lambda_j}(\boldsymbol{\theta})^T$$

$$- \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta})$$

Numerical example

Undamped two degree-of-system system: $m_1 = 1$ Kg, $m_2 = 1.5$ Kg, $\bar{k}_1 = 1000$ N/m, $\bar{k}_2 = 1100$ N/m and $k_3 = 100$ N/m.



Only the stiffness parameters k_1 and k_2 are uncertain: $k_i = \bar{k}_i(1 + \epsilon_i x_i)$, $i = 1, 2$. $\mathbf{x} = \{x_1, x_2\}^T \in \mathbb{R}^2$ and the 'strength parameters' $\epsilon_1 = \epsilon_2 = 0.25$.

Numerical example

Following six methods are compared

1. *Mean-centered first-order perturbation*
2. *Mean-centered second-order perturbation*
3. α*-centered first-order perturbation*
4. α*-centered second-order perturbation*
5. *Asymptotic method*
6. *Monte Carlo Simulation (10K samples)* - can be considered as benchmark.

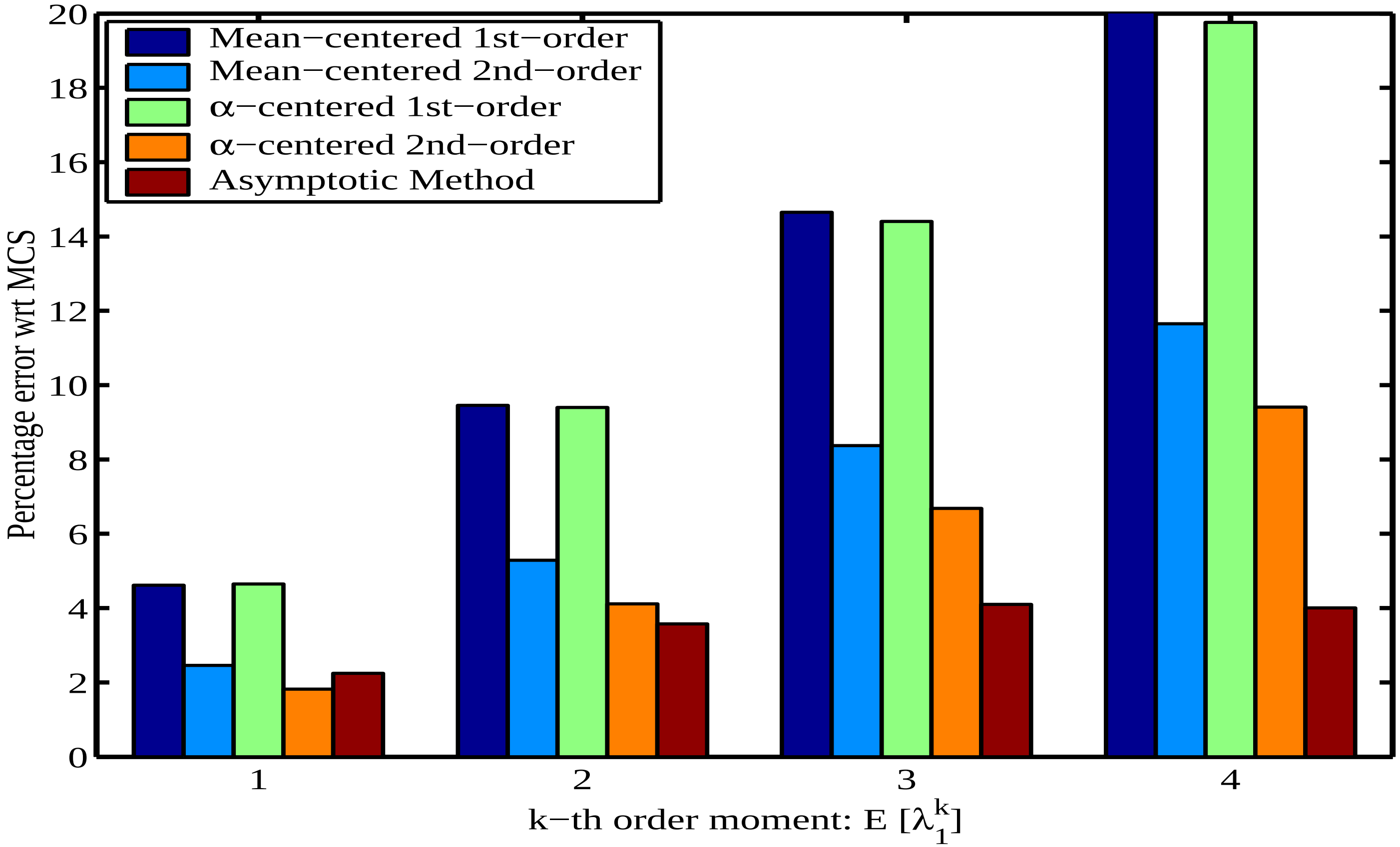
Numerical example

The percentage error:

$$\text{Error}_{i\text{th method}} = \frac{\{\mu'_k\}_{i\text{th method}} - \{\mu'_k\}_{\text{MCS}}}{\{\mu'_k\}_{\text{MCS}}} \times 100$$

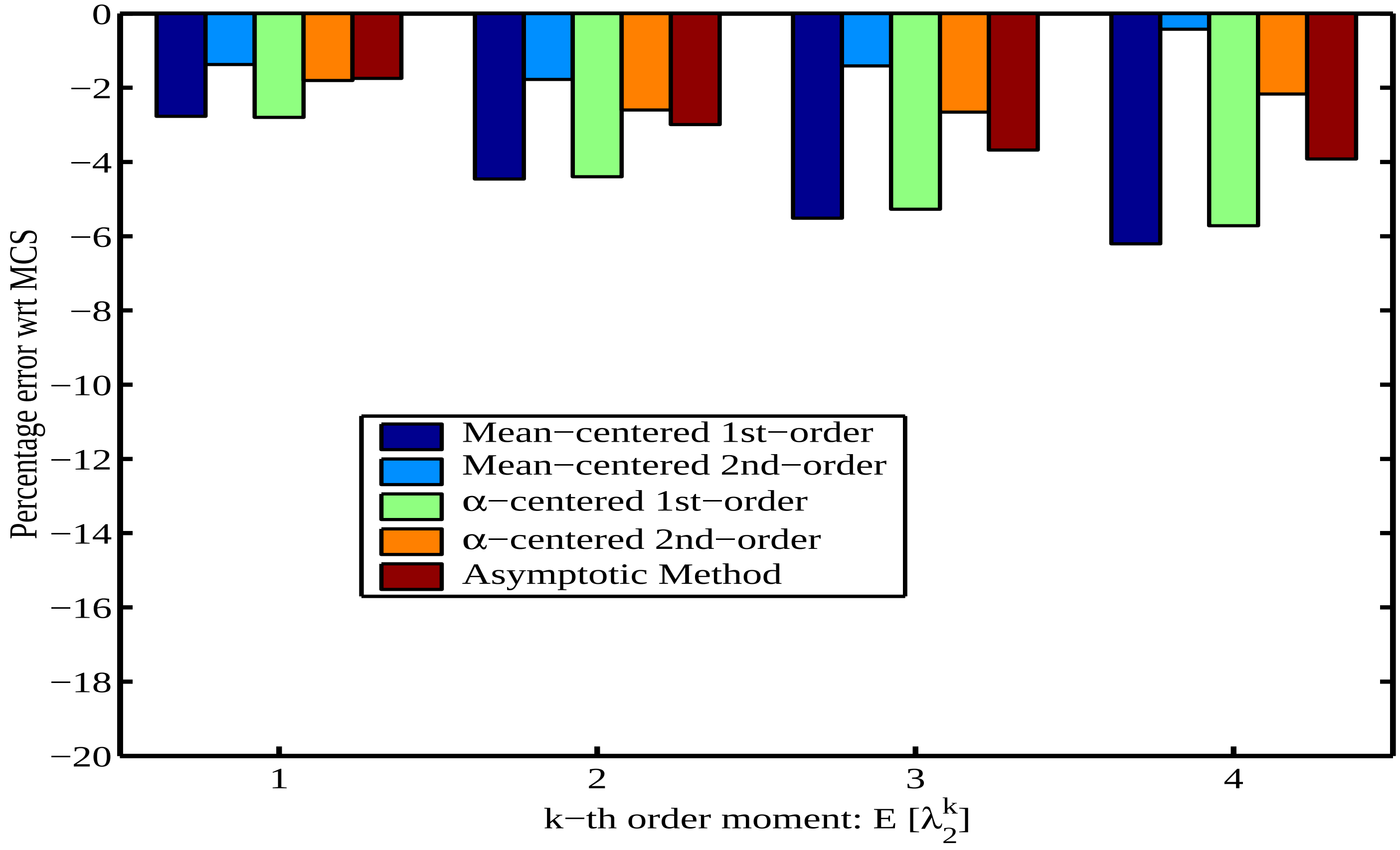
$i = 1, \cdots 5$.

Numerical example



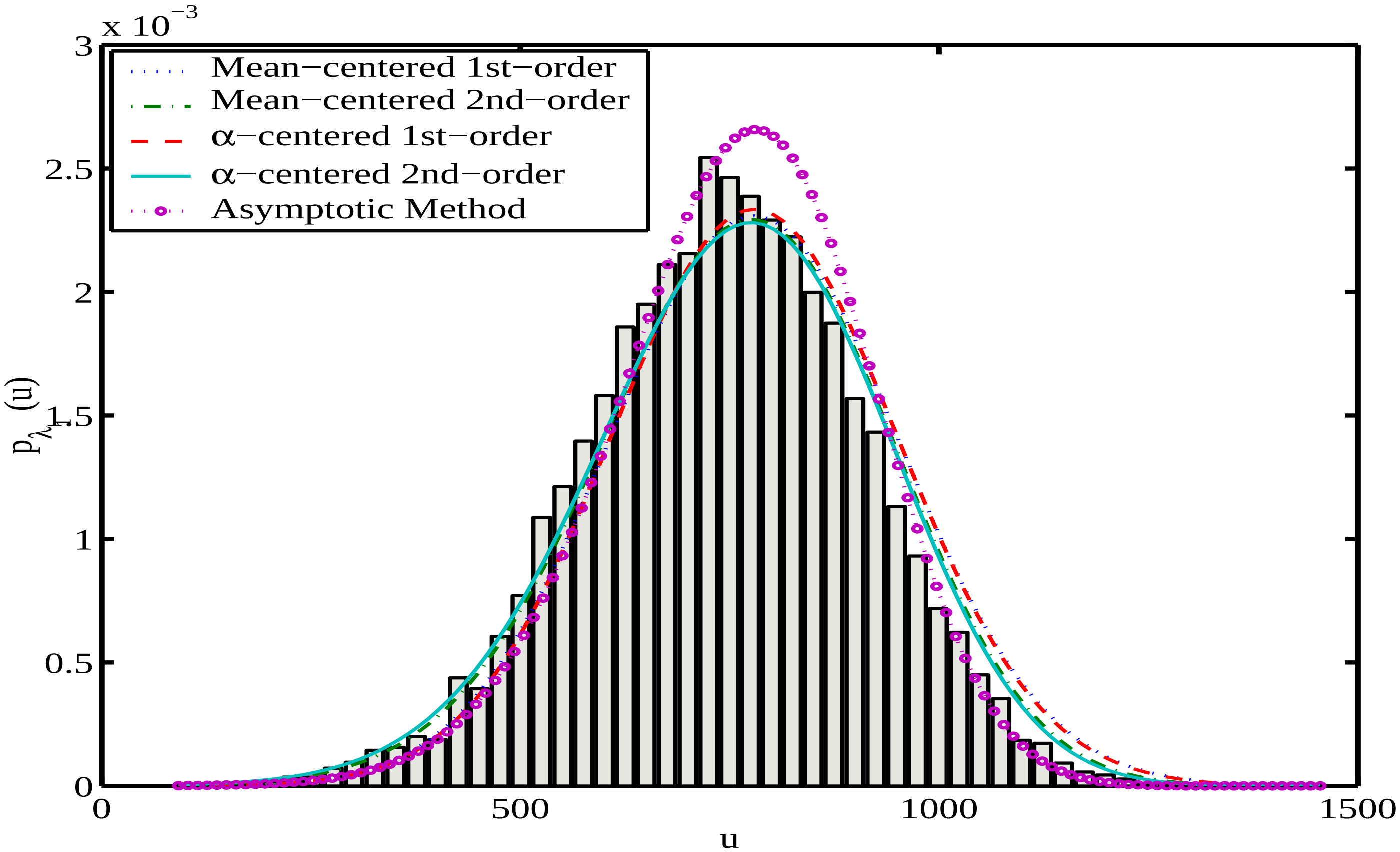
Percentage error for the first four raw moments of the first eigenvalue

Numerical example



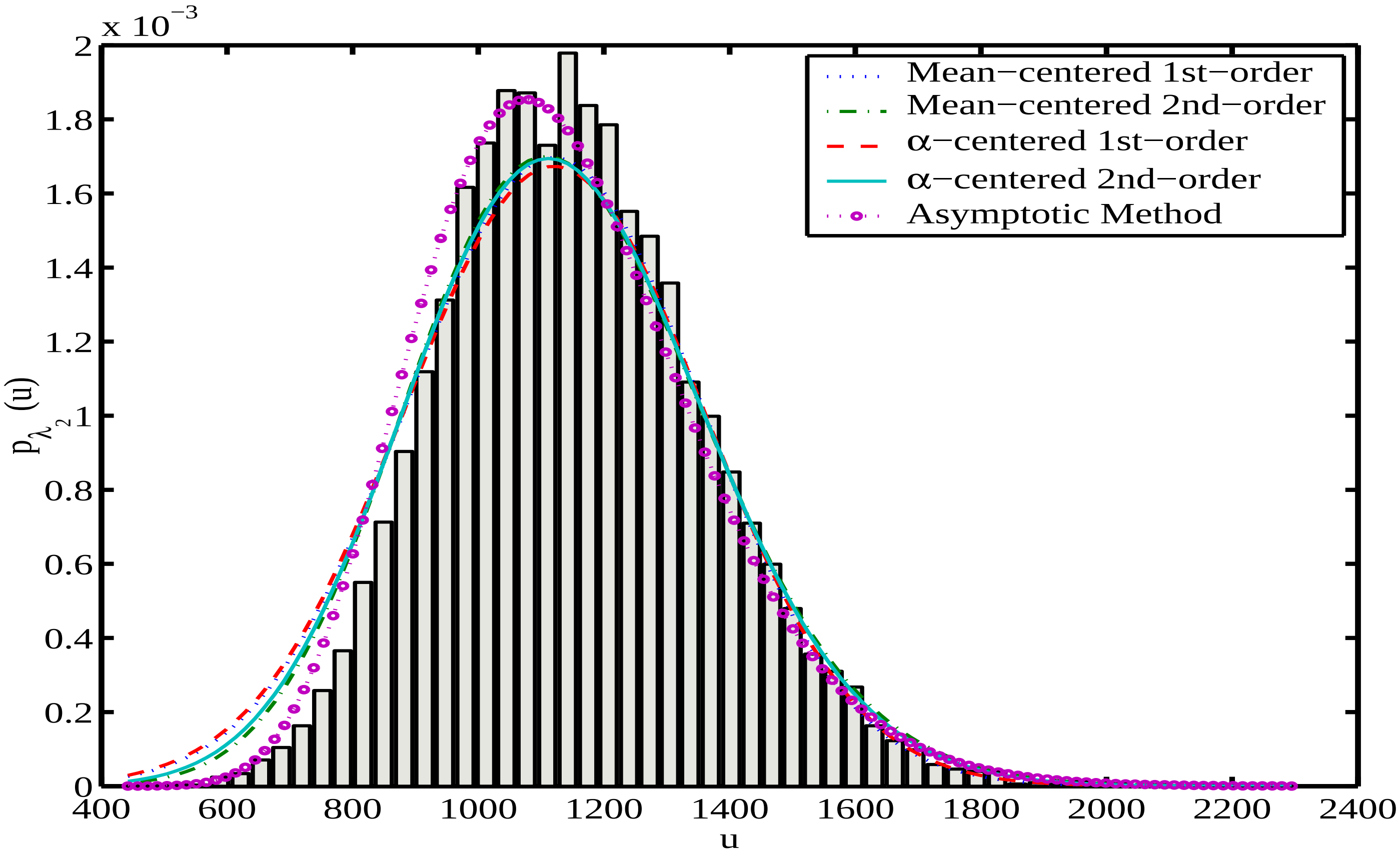
Percentage error for the first four raw moments of the second eigenvalue

Numerical example



Probability density function of the first eigenvalue

Numerical example



Probability density function of the second eigenvalue

Conclusions

- The statistics of the eigenvalues of linear stochastic dynamic systems has been considered
- A closed form expression is obtained for general order joint moments of the eigenvalues
- Pdf of the eigenvalues are obtained:
  - using maximum entropy method
  - in terms of central χ^2 density

Open Problems

- Joint statistics of the eigenvectors
- Joint statistics of the eigenvalues and eigenvectors
- Systems with non-proportional damping