Reliability Analysis in High Dimensions

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Outline of the presentation

- Introduction to structural reliability analysis
- Limitation of current methods in high dimension
- Asymptotic distribution of quadratic forms
- Strict asymptotic formulation
- Weak asymptotic formulation
- Numerical result
- Open problems & discussions

Reliability analysis: basics

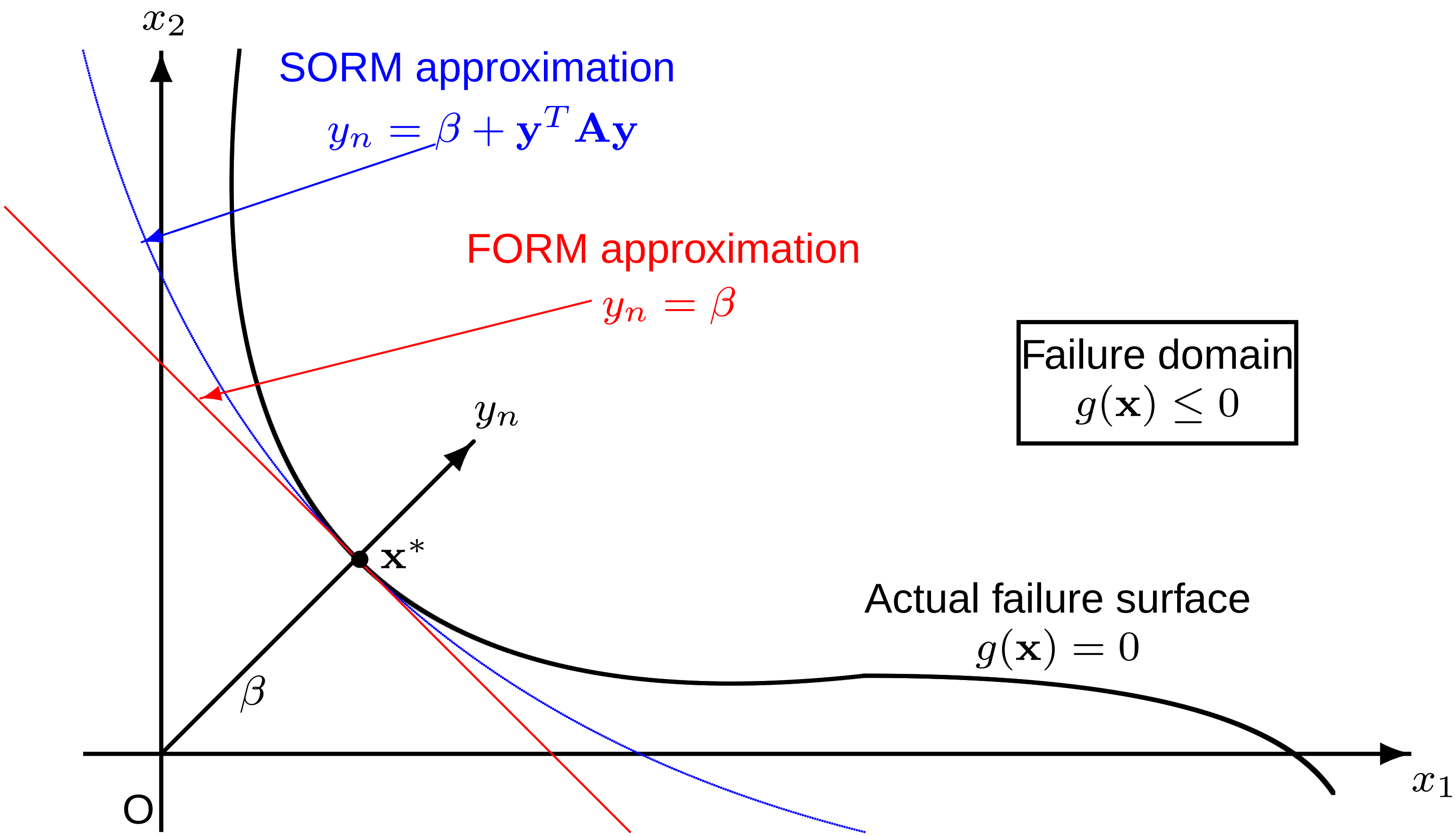
Probability of failure

$$P_f = (2\pi)^{-n/2} \int_{g(\mathbf{x}) \leq 0} e^{-\mathbf{x}^T \mathbf{x}/2} d\mathbf{x}$$

$\mathbf{x} \in \mathbb{R}^n$: Gaussian parameter vector
$g(\mathbf{x})$: failure surface
Maximum contribution comes from the neighborhood where $\mathbf{x}^T \mathbf{x}/2$ is minimum subject to $g(\mathbf{x}) \leq 0$. The design point $\mathbf{x}^*$:

$$\mathbf{x}^* : \min\{(\mathbf{x}^T \mathbf{x})/2\} \quad \text{subject to} \quad g(\mathbf{x}) = 0.$$

Graphical explanation



$$\frac{\mathbf{X}^*}{\beta} = -\frac{\boldsymbol{\nabla} g}{|\boldsymbol{\nabla} g|} = \boldsymbol{\alpha}^*$$

FORM/SORM approximations

$$P_f \approx \text{Prob}\left[y_n \geq \beta + \mathbf{y}^T\mathbf{A}\mathbf{y}\right] = \text{Prob}\left[y_n \geq \beta + U\right] \tag{1}$$

where

$$U : \mathbb{R}^{n-1} \mapsto \mathbb{R} = \mathbf{y}^T\mathbf{A}\mathbf{y},$$

is a quadratic form in Gaussian random variable. The eigenvalues of $\mathbf{A}$, say a_j, can be related to the principal curvatures of the surface κ_j as $a_j = \kappa_j/2$. Considering $\mathbf{A} = \mathbf{O}$ in Eq. (1), we have the FORM:

$$P_f \approx \Phi(-\beta)$$

SORM approximations

Breitung's asymptotic formula (1984):

$$P_f \to \Phi(-\beta)\,\|\mathbf{I}_{n-1} + 2\beta\mathbf{A}\|^{-1/2} \quad \text{when} \quad \beta \to \infty$$

Hohenbichler and Rackwitz's improved formula (1988):

$$P_f \approx \Phi(-\beta)\left\|\mathbf{I}_{n-1} + 2\frac{\varphi(\beta)}{\Phi(-\beta)}\mathbf{A}\right\|^{-1/2}$$

The curse of dimensionality

- If n, i.e. the dimension is large, the computation time to obtain P_f using any tools will be high (no magic is possible!)

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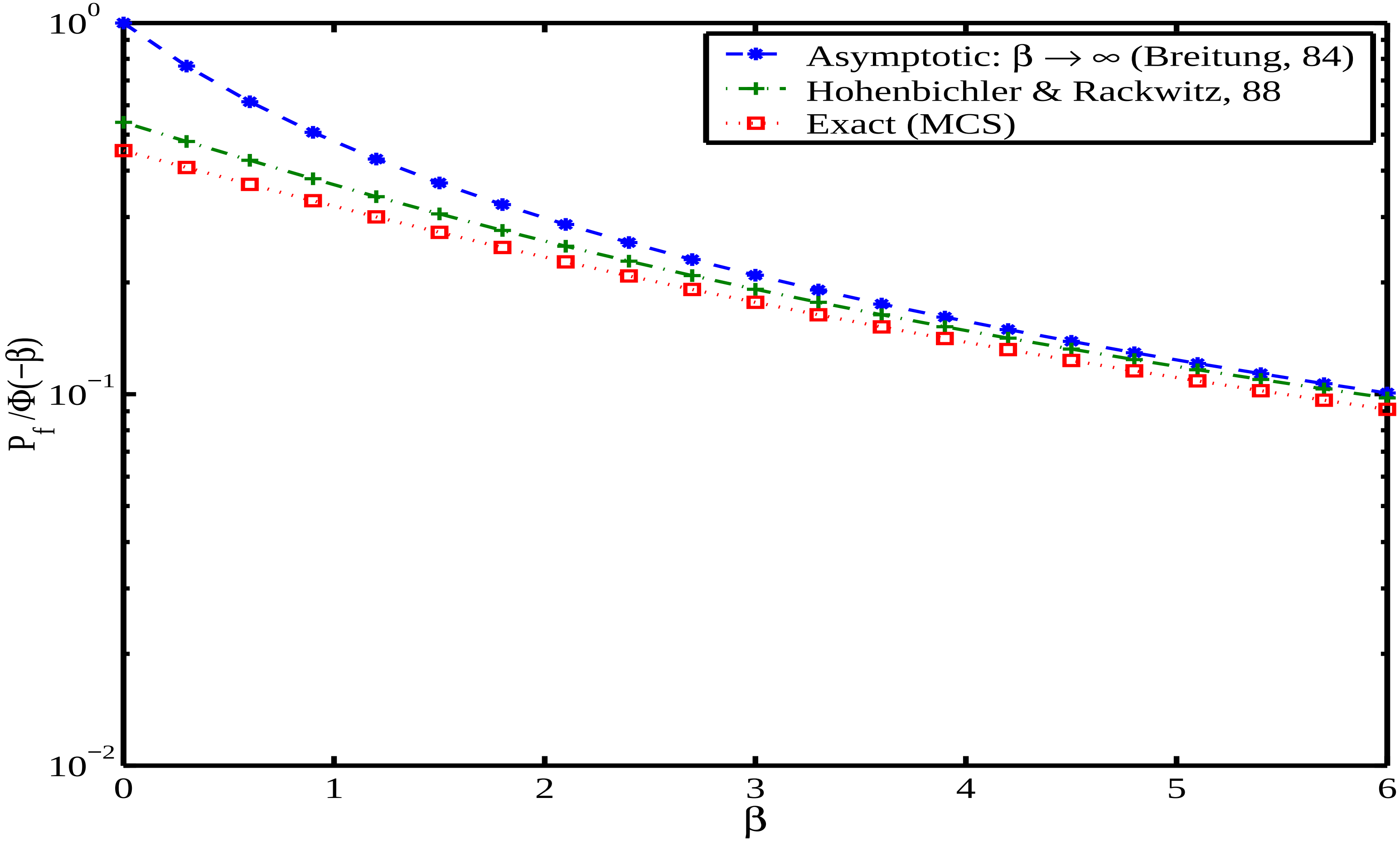
- If n, i.e. the dimension is large, the computation time to obtain P_f using any tools will be high (no magic is possible!)
- Question 1: What is a 'high dimension'?
- Question 2: Suppose we have followed the 'normal route' and did all the calculations (i.e., x^*, β and $\mathbf{A}$). Can we still trust the results from classical FORM/SORM in high dimension?

Numerical example

Consider a problem for which the failure surface is exactly parabolic: $g = -y_n + \beta + \mathbf{y}^T \mathbf{A} \mathbf{y}$

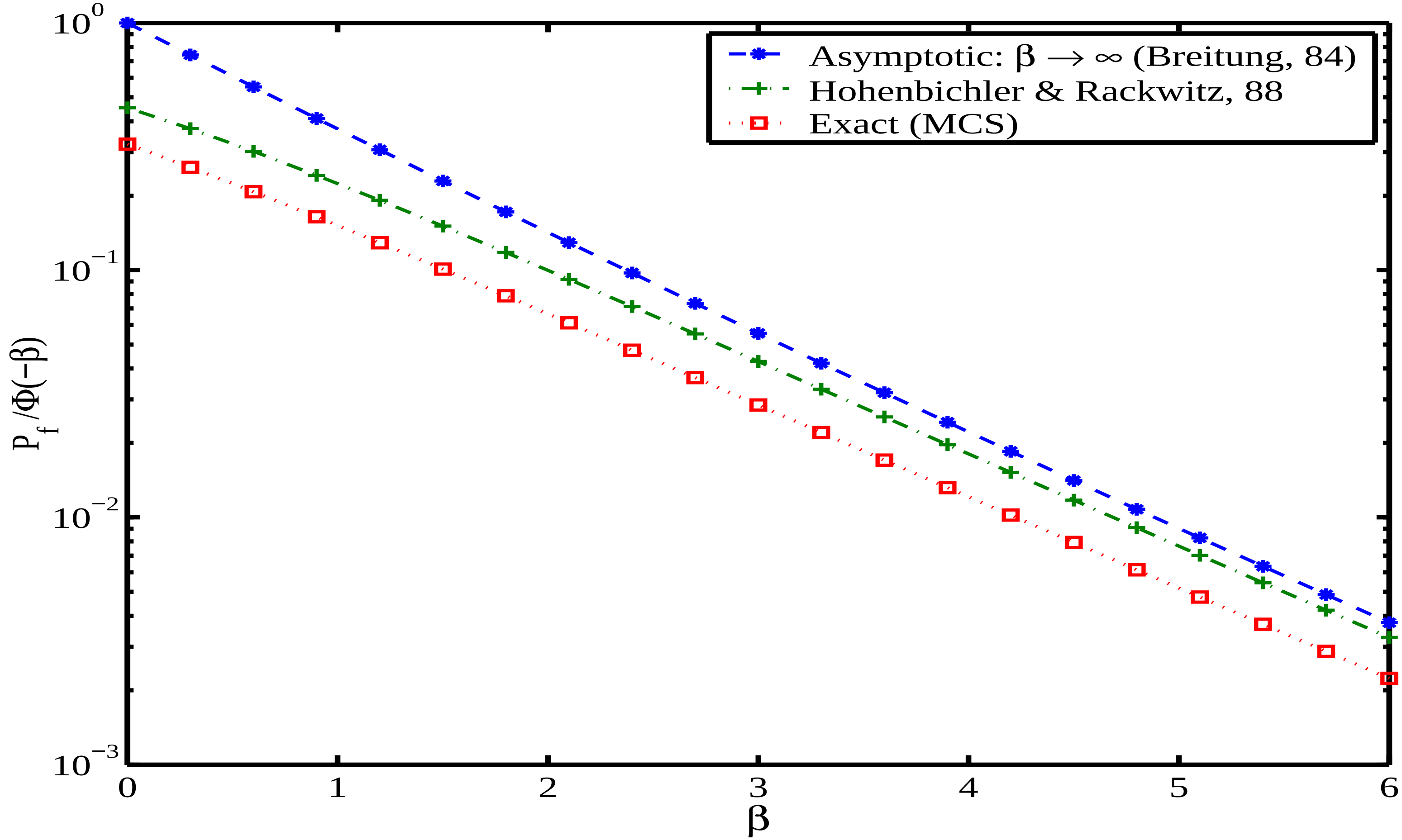
- We choose n and the value of $\mathrm{Trace}\left(\mathbf{A}\right)$
- When $\mathrm{Trace}\left(\mathbf{A}\right) = 0$ the failure surface is effectively linear. Therefore, the more the value of $\mathrm{Trace}\left(\mathbf{A}\right)$, the more non-linear the failure surface becomes.
- It is assumed that the eigenvalues of $\mathbf{A}$ are uniform random numbers.

P_f for small n



Failure probability for $n - 1 = 3$, $\mathrm{Trace}\,(\mathbf{A}) = 1$

P_f for large n



Failure probability for $n-1=100$, $\mathrm{Trace}\left(\mathbf{A}\right)=1$

Asymptotic distribution of quadratic forms

Moment generating function:

$$M_U(s) = \|\mathbf{I}_{n-1} - 2s\mathbf{A}\|^{-1/2} = \prod_{k=1}^{n-1} (1 - 2sa_k)^{-1/2}$$

Now construct a sequence of new random variables $q = U/\sqrt{n}$. The moment generating function of q:

$$M_q(s) = M_U(s/\sqrt{n}) = \prod_{k=1}^{n-1} \left(1 - 2sa_k/\sqrt{n}\right)^{-1/2}$$

Asymptotic distribution

Truncating the Taylor series expansion:

$$\ln\left(M_q(s)\right) \approx \text{Trace}\left(\mathbf{A}\right) s/\sqrt{n} + \left(2\ \text{Trace}\left(\mathbf{A}^2\right)\right) s^2/2n$$

We assume n is large such that the following conditions hold

$$\frac{2}{n}\text{Trace}\left(\mathbf{A}^2\right) < \infty$$

and $$\frac{2^r}{n^{r/2}\ r}\text{Trace}\left(\mathbf{A}^r\right) \rightarrow 0, \forall r \geq 3$$

Asymptotic distribution

Therefore, the moment generating function of $U = q\sqrt{n}$ can be approximated by:

$$M_U(s) \approx e^{\mathrm{Trace}(\mathbf{A})s + \left(2\,\mathrm{Trace}\left(\mathbf{A}^2\right)\right)s^2/2}$$

From the uniqueness of the Laplace Transform pair it follows that U asymptotically approaches a Gaussian random variable with mean $\mathrm{Trace}\,(\mathbf{A})$ and variance $2\mathrm{Trace}\,(\mathbf{A}^2)$, that is

$$U \simeq \mathbb{N}_1\left(\mathrm{Trace}\,(\mathbf{A})\,, 2\,\mathrm{Trace}\,(\mathbf{A}^2)\right) \quad \text{when} \quad n \to \infty$$

Minimum number of random variables

The error in neglecting higher order terms:

$$\frac{1}{r}\left(\frac{2s}{\sqrt{n}}\right)^r \mathrm{Trace}\left(\mathbf{A}^r\right), \text{ for } r \geq 3.$$

Using $s = \beta$ and assuming there exist a small real number ϵ (the error) we have

$$\frac{1}{r}\frac{(2\beta)^r}{n^{r/2}}\mathrm{Trace}\left(\mathbf{A}^r\right) < \epsilon \text{ or } n > \frac{4\beta^2}{\sqrt[r]{r^2\epsilon^2}}\left(\sqrt[r]{\mathrm{Trace}\left(\mathbf{A}^r\right)}\right)^2$$

Strict asymptotic formulation

We rewrite (1):

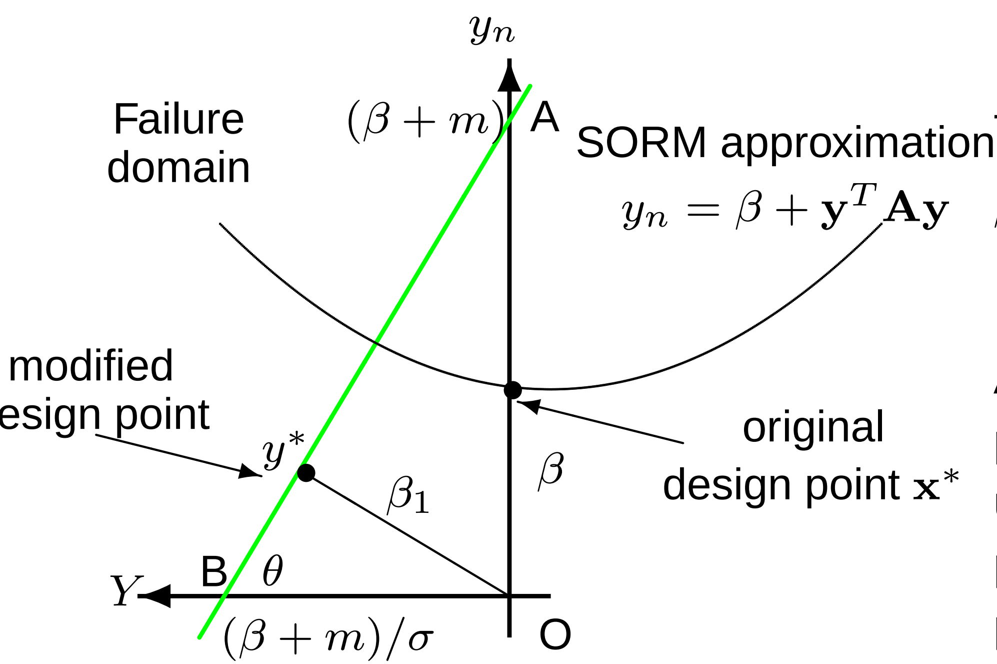
$$P_f \approx \mathrm{Prob}\left[y_n \geq \beta + U\right] = \mathrm{Prob}\left[y_n - U \geq \beta\right]$$

Since U is asymptotically Gaussian, the variable $z = y_n - U$ is also Gaussian with mean $\left(-\mathrm{Trace}\left(\mathbf{A}\right)\right)$ and variance $\left(1 + 2\,\mathrm{Trace}\left(\mathbf{A}^2\right)\right)$. Thus,

$$P_{f_{\mathrm{Strict}}} \rightarrow \Phi\left(-\beta_1\right),\ \beta_1 = \frac{\beta + \mathrm{Trace}\left(\mathbf{A}\right)}{\sqrt{1 + 2\,\mathrm{Trace}\left(\mathbf{A}^2\right)}}, n \rightarrow \infty$$

Graphical explanation

$$m = \text{Trace}\,(\mathbf{A}),\ \sigma^2 = 2\text{Trace}\,\left(\mathbf{A}^2\right)$$



Failure surface: $y_n - U \geq \beta$. Using the standardizing transformation $Y = (U - m)/\sigma$, modified failure surface $\frac{y_n}{\beta+m} + \frac{Y}{-\frac{\beta+m}{\sigma}} \geq 1$.

From $\triangle$AOB, $\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\sigma}{\sqrt{1+\sigma^2}}$.

Therefore, from $\triangle$OBy^*:

$$\beta_1 = \frac{\beta+m}{\sigma}\sin\theta = \frac{\beta+m}{\sqrt{1+\sigma^2}} = \frac{\beta+\text{Trace}(\mathbf{A})}{\sqrt{1+2\ \text{Trace}\left(\mathbf{A}^2\right)}}.$$

If n is small, m, σ will be small. When $m, \sigma \to 0$, AB rotates clockwise and eventually becomes parallel to the Y-axis with a shift of $+\beta$. In this situation $y^* \to x^*$ in the y_n-axis and $\beta_1 \to \beta$ as expected. This explains why classical F/SORM approximations based on the original design point $\mathbf{x}^*$ do not work well when a large number of random variables are considered.

Weak asymptotic formulation

$$P_f \approx \text{Prob}\left[y_n \geq \beta + U\right]$$

$$= \int_{\mathbb{R}} \left\{ \int_{\beta+u}^{\infty} \varphi(y_n) dy_n \right\} p_U(u) du = \text{E}\left[\Phi(-\beta - U)\right]$$

Noticing that $u \in \mathbb{R}^+$ as $\mathbf{A}$ is positive definite we rewrite

$$P_f \approx \int_{\mathbb{R}^+} e^{\ln[\Phi(-\beta-u)]+\ln[p_U(u)]}\, du$$

Weak asymptotic formulation

For the maxima of the integrand (say at point u^*)

$$\frac{\partial}{\partial u}\left\{\ln\left[\Phi(-\beta-u)\right]+\ln\left[p_U(u)\right]\right\}=0$$

Recalling that

$$p_U(u)=(2\pi)^{-1/2}\sigma^{-1}e^{-(u-m)^2/(2\sigma^2)}$$

we have

$$\frac{\varphi(\beta+u)}{\Phi(-(\beta+u))}=\frac{m-u}{\sigma^2}$$

Weak asymptotic formulation

Because this relationship holds at the optimal point u^*, define a constant η as

$$\eta = \frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))} = \frac{m - u^*}{\sigma^2}$$

Taking a first-order Taylor series expansion of $\ln\left[\Phi(-\beta - u)\right]$ about $u = u^*$:

$$\Phi(-\beta - u) \approx e^{\ln[\Phi(-(\beta+u^*))] - \frac{\varphi(\beta+u^*)}{\Phi(-(\beta+u^*))}(u-u^*)}$$

Weak asymptotic formulation

Using η we have

$$\Phi(-\beta - u) \approx \Phi(-\beta_2) e^{\eta u^*} e^{-\eta u} \quad (1)$$

where the modified reliability index

$$\beta_2 = \beta + u^*$$

Taking the expectation of (1) and using the expression of the moment generating function:

$$P_f \approx \mathrm{E}\left[\Phi(-\beta - U)\right] = \Phi(-\beta_2) e^{\eta u^*} \left\| \mathbf{I}_{n-1} + 2\,\eta\,\mathbf{A} \right\|^{-1/2}$$

Weak asymptotic formulation

Considering the asymptotic expansion of the ratio

$$\eta = \frac{\varphi(\beta + u^*)}{\Phi(-(\beta + u^*))} \approx (\beta + u^*) = \beta_2 \approx \frac{m - u^*}{\sigma^2}$$

We obtain

$$u^* \approx \frac{m - \beta\sigma^2}{1 + \sigma^2}, \; \beta_2 = \beta + u^* \approx \frac{\beta + m}{1 + \sigma^2} = \frac{\beta + \mathrm{Trace}\,(\mathbf{A})}{1 + 2\,\mathrm{Trace}\,(\mathbf{A}^2)}$$

Since $\eta \approx \beta_2$, u^* can be expressed in terms of β_2 as

$$u^* \approx -\left(\beta_2\sigma^2 - m\right) = -\left(2\beta_2 \mathrm{Trace}\left(\mathbf{A}^2\right) - \mathrm{Trace}\,(\mathbf{A})\right)$$

Weak asymptotic formulation

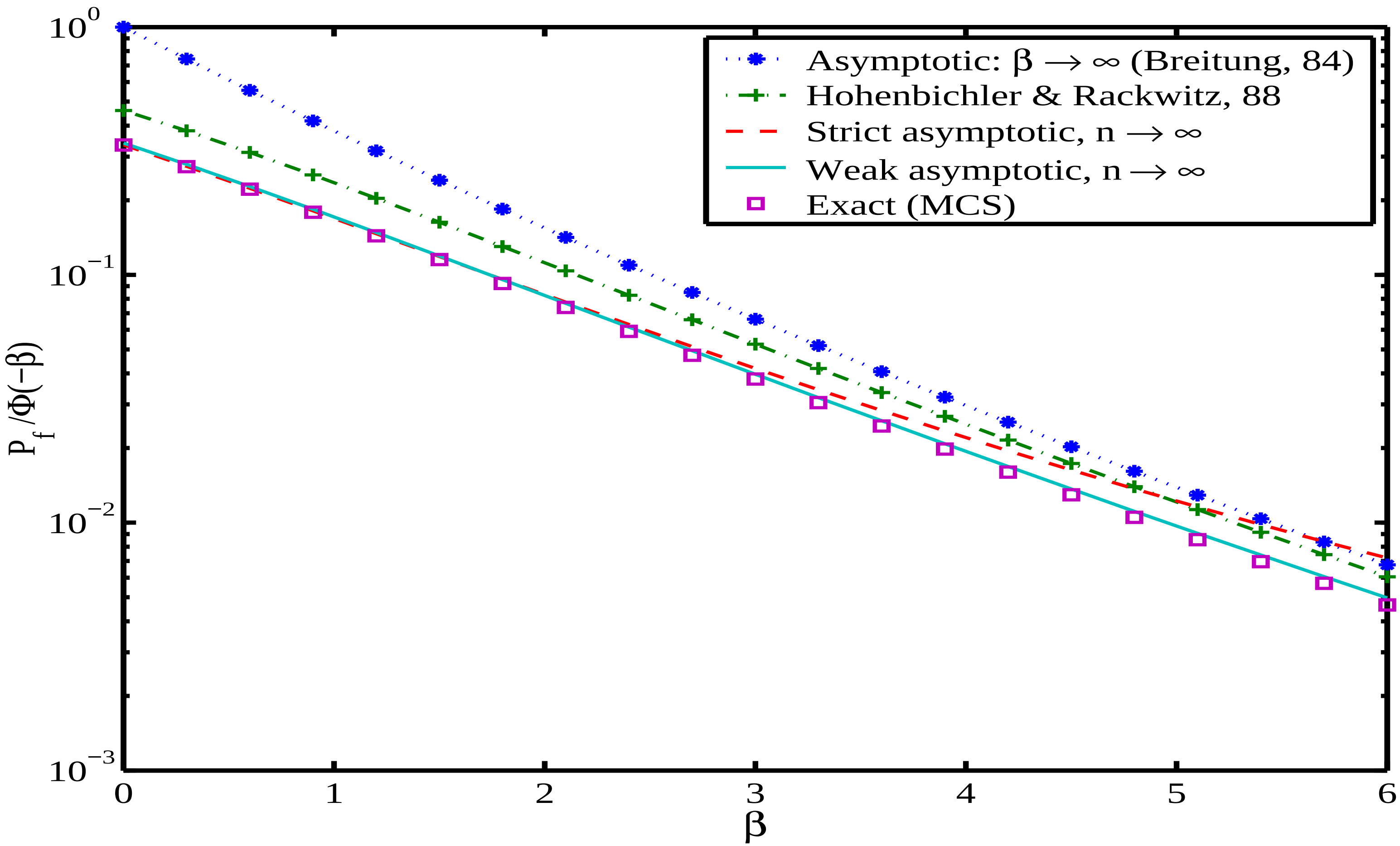
Using the expression of η and u^*, the failure probability using weak asymptotic formulation:

$$P_{f_{\text{Weak}}} \to \frac{\Phi\left(-\beta_2\right) e^{-\left(2\beta_2^2 \text{Trace}\left(\mathbf{A}^2\right) - \beta_2 \text{Trace}\left(\mathbf{A}\right)\right)}}{\sqrt{\|\mathbf{I}_{n-1} + 2\beta_2 \mathbf{A}\|}},$$

$$\text{where } \beta_2 = \frac{\beta + \text{Trace}\left(\mathbf{A}\right)}{1 + 2\,\text{Trace}\left(\mathbf{A}^2\right)} \text{ when } n \to \infty$$

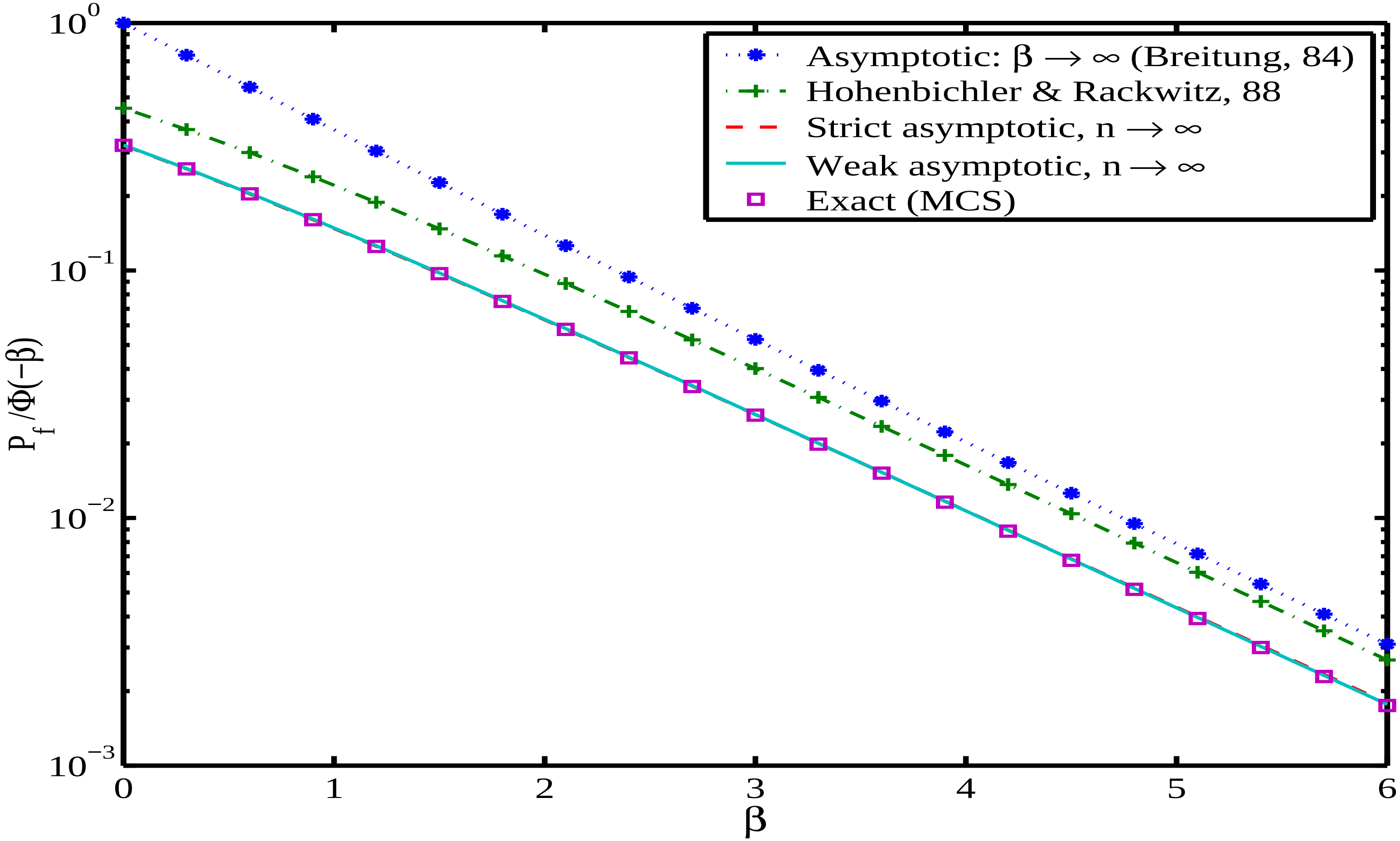
For the small n case, $\text{Trace}\left(\mathbf{A}\right), \text{Trace}\left(\mathbf{A}^2\right) \to 0$ and it can be seen that $P_{f_{\text{Weak}}}$ approaches to Breitung's formula.

P_f from asymptotic analysis



Failure probability for $n - 1 = 35$, $\mathrm{Trace}\,(\mathbf{A}) = 1$ $[n_{min} = 176]$

P_f from asymptotic analysis



Failure probability for $n-1=200$, $\mathrm{Trace}\,(\mathbf{A})=1$

Summary & conclusions

- Geometric analysis shows that the classical design point should be modified in high dimension. This also explains why classical FORM/SORM work poorly in high dimension.

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- Geometric analysis shows that the classical design point should be modified in high dimension. This also explains why classical FORM/SORM work poorly in high dimension.

- $$P_{f_{\text{Strict}}} \to \Phi\left(-\beta_1\right), \; \beta_1 = \frac{\beta + \text{Trace}\left(\mathbf{A}\right)}{\sqrt{1 + 2\,\text{Trace}\left(\mathbf{A}^2\right)}}, n \to \infty$$

  The strict asymptotic formula can viewed as the 'correction' needed to the existing FORM formula in high dimension.

Summary & conclusions

$$P_{f_{\text{Weak}}} \to \frac{\Phi\left(-\beta_2\right) e^{-\left(2\beta_2^2 \text{Trace}\left(\mathbf{A}^2\right) - \beta_2 \text{Trace}\left(\mathbf{A}\right)\right)}}{\sqrt{\left\|\mathbf{I}_{n-1} + 2\beta_2 \mathbf{A}\right\|}},$$

where $\beta_2 = \dfrac{\beta + \text{Trace}\left(\mathbf{A}\right)}{1 + 2\,\text{Trace}\left(\mathbf{A}^2\right)}$ when $n \to \infty$

The weak asymptotic formula can viewed as the correction needed to the existing SORM formula in high dimension.

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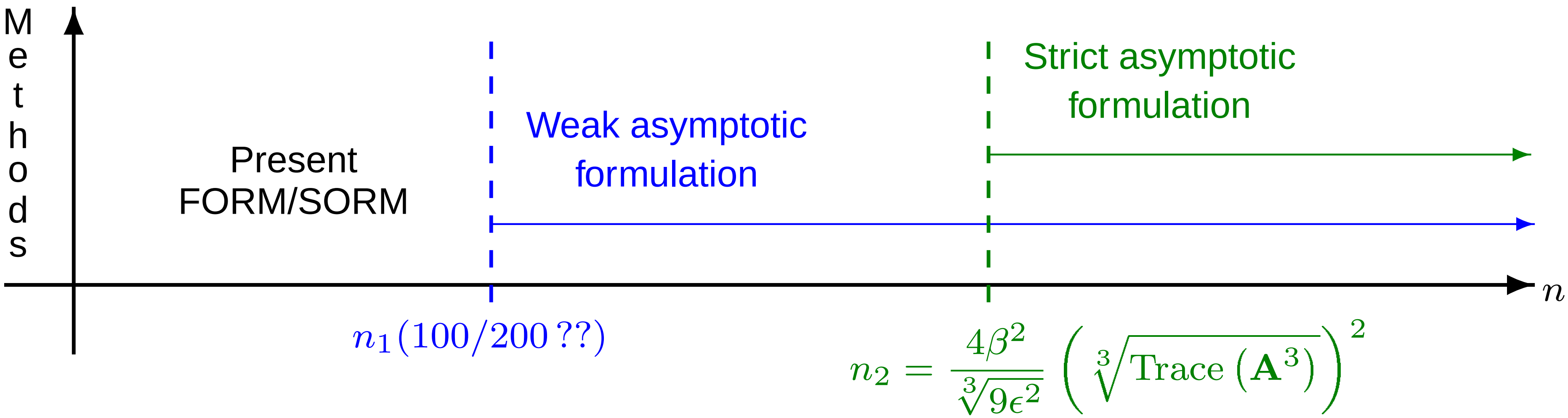
- Any geometric interpretation for the weak formulation?
- Why these asymptotic results degrade as β becomes high?
- Any expression of $n_{\mathrm{m}in}$ for the weak formulation?

Open Questions

The broad picture:



$\beta \downarrow, n \downarrow$ ✓

$\beta \uparrow, n \downarrow$ ✓ (Asymptotic: $\beta \rightarrow \infty$)

$\beta \downarrow, n \uparrow$ ✓ (Asymptotic: $n \rightarrow \infty$)

$\beta \uparrow, n \uparrow$ × (Joint asymptotic: $n, \beta \rightarrow \infty$?)

References

Breitung, K. 1984. Asymptotic approximations for multinormal integrals. *Journal of Engineering Mechanics, ASCE*, **110**(3), 357–367.

Hohenbichler, M., and Rackwitz, R. 1988. Improvement of second-order reliability estimates by importance sampling. *Journal of Engineering Mechanics, ASCE*, **14**(12), 2195–2199.