The ‘damping effect’ in the dynamic response of stochastic systems

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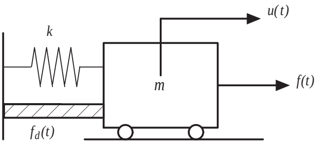
Outline of the talk

Stochastic SDOF systems



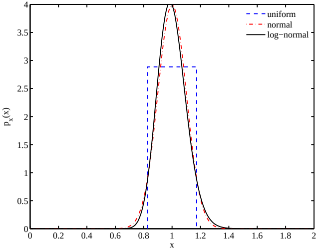
Consider a normalised single degrees of freedom system (SDOF):

$$\ddot{u}(t) + 2\zeta\omega_n \dot{u}(t) + \omega_n^2 u(t) = f(t)/m \tag{1}$$

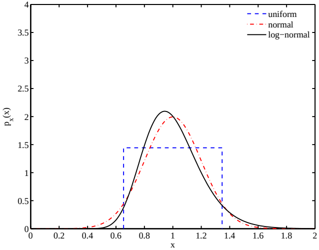
Here $\omega_n = \sqrt{k/m}$ is the natural frequency and $\xi = c/2\sqrt{km}$ is the damping ratio.

- We are interested in understanding the motion when the natural frequency of the system is perturbed in a stochastic manner.
- Stochastic perturbation can represent statistical scatter of measured values or a lack of knowledge regarding the natural frequency.

Frequency variability



(a) Pdf: $\sigma_a = 0.1$

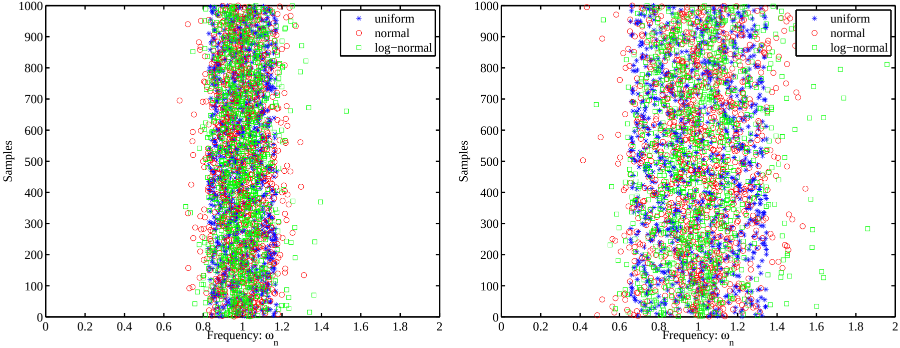


(b) Pdf: $\sigma_a = 0.2$

Figure : We assume that the mean of r is 1 and the standard deviation is σ_a.

- Suppose the natural frequency is expressed as $\omega_n^2 = \omega_{n_0}^2 r$, where ω_{n_0} is deterministic frequency and r is a random variable with a given probability distribution function.

Frequency samples

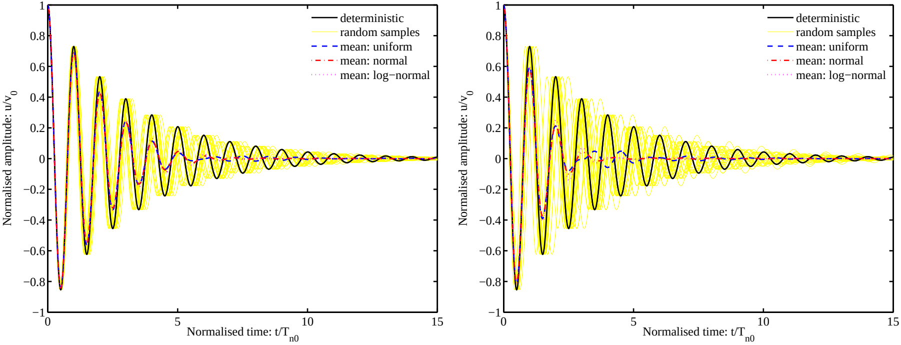


(a) Frequencies: $\sigma_a = 0.1$

(b) Frequencies: $\sigma_a = 0.2$

Figure : 1000 sample realisations of the frequencies for the three distributions

Response in the time domain

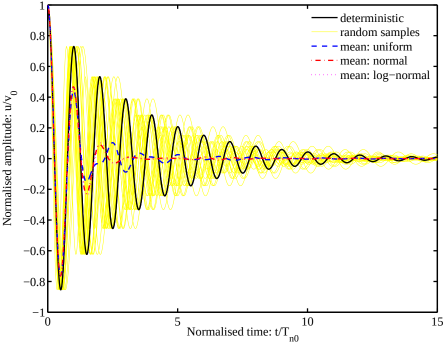


(a) Response: $\sigma_a = 0.05$

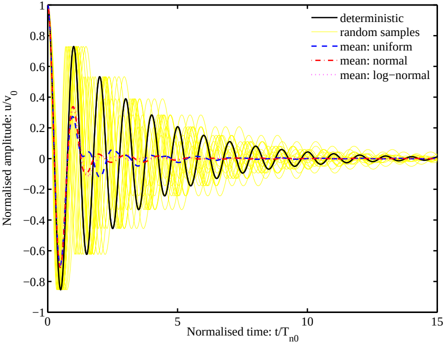
(b) Response: $\sigma_a = 0.1$

Figure : Response due to initial velocity v_0 with 5% damping

Response in the time domain



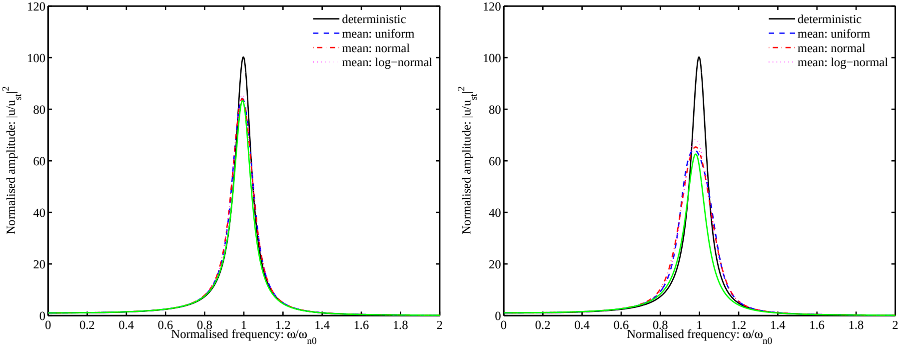
(a) Response: $\sigma_a = 0.15$



(b) Response: $\sigma_a = 0.2$

Figure : Response due to initial velocity v_0 with 5% damping

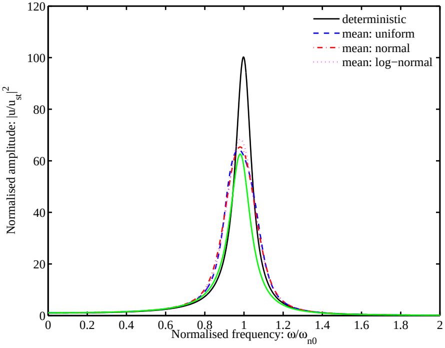
Frequency response function



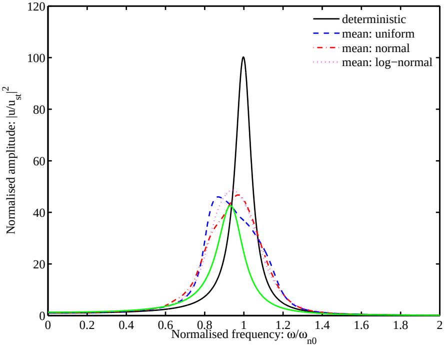
(a) Response: $\sigma_a = 0.05$ (b) Response: $\sigma_a = 0.1$

Figure : Normalised frequency response function with 5% damping $|u/u_{st}|^2$, where $u_{st} = f/k$

Frequency response function



(a) Response: $\sigma_a = 0.15$



(b) Response: $\sigma_a = 0.2$

Figure : Normalised frequency response function with 5% damping $|u/u_{st}|^2$, where $u_{st} = f/k$

Key observations

- The mean response is significantly more damped compared to deterministic response - 'The ghost damping'!
- The higher the randomness, the higher the "effective damping".
- The qualitative features are almost independent of the distribution the random natural frequency.
- We often use averaging to obtain more reliable experimental results - is it always true?

Assuming uniform random variable, we aim to explain some of these observations.

Equivalent damping

- Assume that the random natural frequencies are $\omega_n^2 = \omega_{n_0}^2(1 + \epsilon x)$, where x has zero mean and unit standard deviation.
- The normalised harmonic response in the frequency domain

$$\frac{u(\mathrm{i}\omega)}{f/k} = \frac{k/m}{[-\omega^2 + \omega_{n_0}^2(1 + \epsilon x)] + 2\mathrm{i}\xi\omega\omega_{n_0}\sqrt{1 + \epsilon x}} \quad (2)$$

- Considering $\omega_{n_0} = \sqrt{k/m}$ and frequency ratio $r = \omega/\omega_{n_0}$ we have

$$\frac{u}{f/k} = \frac{1}{[(1 + \epsilon x) - r^2] + 2\mathrm{i}\xi r\sqrt{1 + \epsilon x}} \quad (3)$$

Equivalent damping

- The squared-amplitude of the normalised dynamic response can be obtained as

$$\left(\frac{|u|}{f/k}\right)^2 = \frac{1}{[(1+\epsilon x) - r^2]^2 + 4\xi^2 r^2 1 + \epsilon x} \quad (4)$$

- We are interested in the response at the resonance. Therefore considering $\omega = \omega_{n_0}$ (that is $r = 1$) , we have

$$\hat{U} = \left(\frac{|u|}{f/k}\right)^2 |_{r=1} = \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} \quad (5)$$

- Since x is zero mean unit standard deviation uniform random variable, its pdf is given by $p_x(x) = 1/2\sqrt{3}, -\sqrt{3} \leq x \leq \sqrt{3}$

Equivalent damping

- The mean of $\hat{U}$ is therefore

$$
\begin{aligned}
\mathrm{E}\left[\hat{U}\right] &= \int \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} p_x(x)\mathrm{d}x \\
&= \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} \mathrm{d}x \\
&= \frac{1}{4\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}} - \frac{\xi}{\sqrt{1-\xi^2}}\right) \\
&+ \frac{1}{4\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}} + \frac{\xi}{\sqrt{1-\xi^2}}\right) \qquad (6)
\end{aligned}
$$

Equivalent damping

- Note that for any δ we have

$$\frac{1}{2}\left\{\tan^{-1}(a+\delta)+\tan^{-1}(a-\delta)\right\}=\tan^{-1}(a)+O(\delta^2) \tag{7}$$

- Neglecting terms of the order $O(\xi^2)$ we have

$$\mathrm{E}\left[\hat{U}\right]\approx\frac{1}{2\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}}\tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}}\right)\approx\frac{\tan^{-1}(\sqrt{3}\epsilon/2\xi)}{2\sqrt{3}\epsilon\xi} \tag{8}$$

Equivalent damping

- For small damping, the maximum amplitude at $\omega = \omega_{n_0}$ is $1/4\xi_e^2$ where ξ_e is the equivalent damping for the mean response
- Therefore, the equivalent damping for the mean response is given by

$$(2\xi_e)^2 = \frac{2\sqrt{3}\epsilon\xi}{\tan^{-1}(\sqrt{3}\epsilon/2\xi)} \tag{9}$$

- From this we can obtain the equivalent damping factor as

$$\xi_e = \frac{3^{1/4}\sqrt{\epsilon\xi}}{\sqrt{2\tan^{-1}(\sqrt{3}\epsilon/2\xi)}} \tag{10}$$

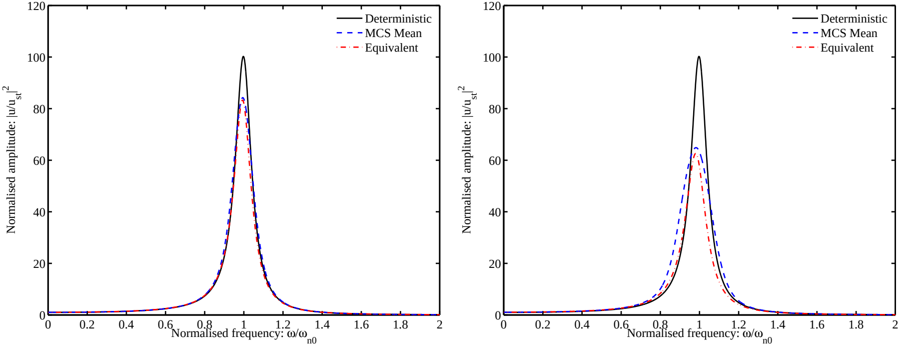
The 'Ghost damping'

- For small damping and randomness, ignoring higher-order terms in ϵ and ξ we obtain

$$\xi_e = \frac{3^{1/4}\sqrt{\epsilon\xi}}{\sqrt{2\tan^{-1}(\sqrt{3}\epsilon/2\xi)}} \approx \frac{3^{1/4}}{\sqrt{\pi}}\sqrt{\epsilon}\sqrt{\xi} \tag{11}$$

- *The equivalent damping factor of the mean system is proportional to the square root of the damping factor of the underlying baseline system*

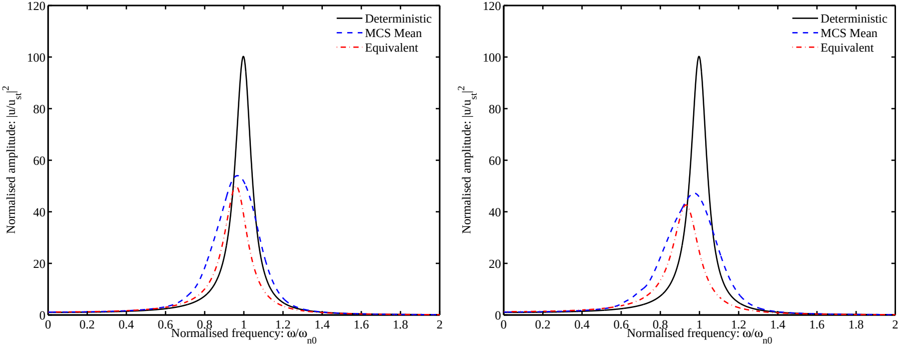
Equivalent frequency response function



(a) Response: $\sigma_a = 0.05$ (b) Response: $\sigma_a = 0.1$

Figure : Normalised frequency response function with equivalent damping ($\xi_e = 0.05$ in the ensembles). For the two cases $\xi_e = 0.0551$ and $\xi_e = 0.0643$ and respectively.

Equivalent frequency response function



(a) Response: $\sigma_a = 0.15$

(b) Response: $\sigma_a = 0.2$

Figure : Normalised frequency response function with equivalent damping ($\xi_e = 0.05$ in the ensembles). For the two cases $\xi_e = 0.0735$ and $\xi_e = 0.0819$ respectively.

What are implications - could this 'Ghost Damping' haunt us?

- The mean response of a damped stochastic system is more damped than the underlying baseline system
- The 'damping effect' depends on the standard deviation of the resonance frequency - the 'random' the system is, the more 'damped' the mean response becomes
- Any engineering design decision based on the mean result will NOT be a conservative one! This is because the 'sample response' can be significantly higher than the mean response.

Conclusions

- The mean response of a damped stochastic system is more damped than the underlying baseline system.
- For small baseline damping factors, the equivalent damping for the mean response is given by

$$\xi_e \approx \frac{3^{1/4}\sqrt{\epsilon\xi}}{\sqrt{2\tan^{-1}(\sqrt{3}\epsilon/2\xi)}} \approx 3^{1/4}\sqrt{\frac{\epsilon\xi}{\pi}}$$

  where ϵ is the standard deviation of the squared natural frequency and ξ is the damping factor of the baseline system.
- Higher standard deviation of the resonance frequency will ‘dampen’ the mean response.
- Any computational approach, based on perturbation around the baseline response, may have difficulty in predicting the statistical average when the damping of the baseline system is low.