Ultra Sensitive sensor using coupled cantilevers

Ezra Clarke and Sondipon Adhikari

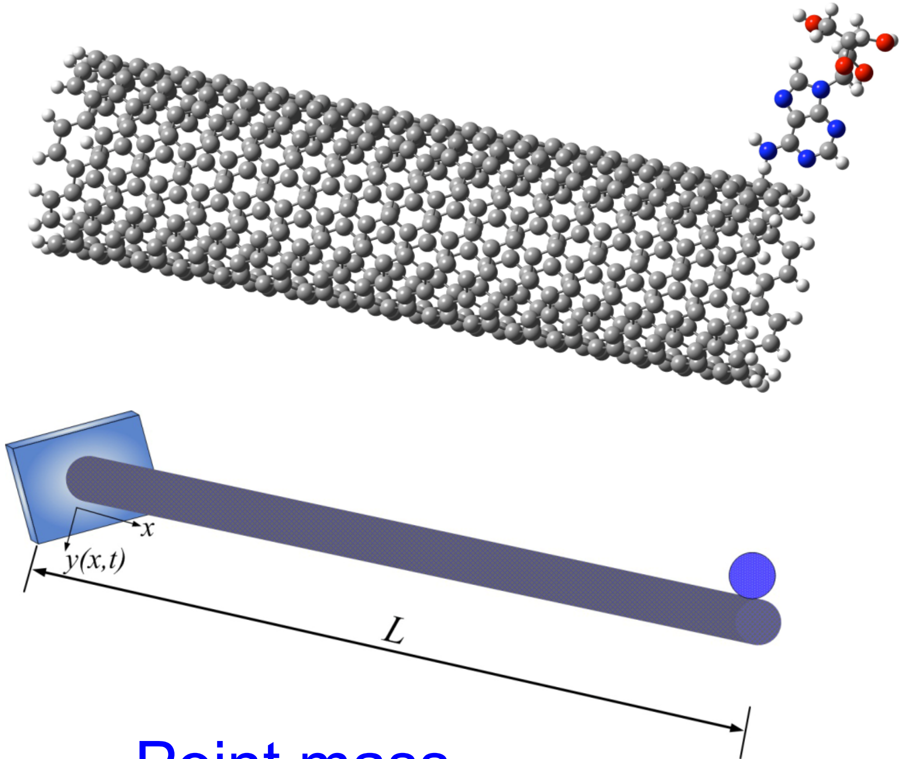
Zienkiewicz Centre for Computational Engineering

College of Engineering, Swansea University, UK

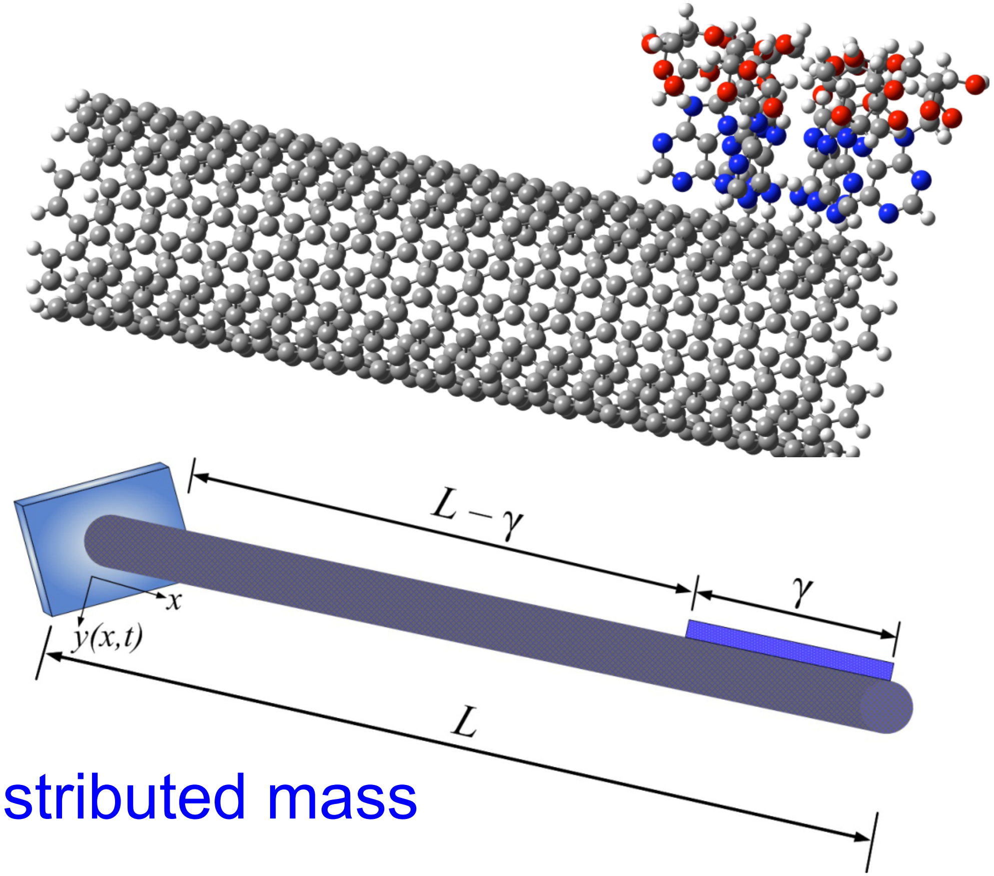
Outline of the Talk

- Background & Motivation
- Coupled cantilevers
- Two-degree-of-freedom model
  - System without added mass
  - System with added mass
- Frequency sensitivity vs eigenmode sensitivity
- Sensor equations
- Validation: Finite element simulations
- Conclusions

Single cantilever based mass sensor



Point mass



Distributed mass

Chowdhury, R., Adhikari, S. and Mitchell, J., "Vibrating carbon nanotube based bio-sensors", Physica E: Low-dimensional Systems and Nanostructures, 42[2] (2009), pp. 104-109.

Adhikari, S. and Chowdhury, R., "The calibration of carbon nanotube based bio-nano sensors", *Journal of Applied Physics*, 107[12] (2010), pp. 124322:1-8

Vibration based mass sensor: CNT

Natural frequency with the added mass:

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m_{eq}}} = \frac{\beta}{2\pi}\frac{c_k}{\sqrt{1+c_m\Delta M}}$$

where

$$\beta = \sqrt{\frac{EI}{\rho A L^4}}$$

the stiffness calibration constant

$$c_k = \sqrt{\frac{I_3}{I_1}}$$

and the mass calibration constant

$$c_m = \frac{I_2}{I_1}$$

Identification of the added mass

$$f_n = \frac{f_{0_n}}{\sqrt{1+c_m\Delta M}} \qquad (22)$$

The frequency-shift can be expressed using Eq. (22) as

$$\Delta f = f_{0_n} - f_n = f_{0_n} - \frac{f_{0_n}}{\sqrt{1+c_m\Delta M}} \qquad (23)$$

From this we obtain

$$\frac{\Delta f}{f_{0_n}} = 1 - \frac{1}{\sqrt{1+c_m\Delta M}} \qquad (24)$$

Rearranging gives the expression

$$\Delta M = \frac{1}{c_m\left(1-\frac{\Delta f}{f_{0_n}}\right)^2} - \frac{1}{c_m} \qquad (25)$$

Vibration based mass sensor: CNT

Mass of a nano object can be detected from the frequency shift Δf

$$M = \frac{\rho A L}{c_m} \frac{(c_k^2 \beta^2)}{(c_k \beta - 2\pi \Delta f)^2} - \frac{\rho A L}{c_m}$$

$$I_1 = \int_0^1 Y_j^2(\xi) \mathrm{d}\xi = 1.0$$

$$I_2 = \frac{1}{\gamma} \int_{\xi=1-\gamma}^1 Y_j^2(\xi) \mathrm{d}\xi; \quad 0 \le \gamma \le 1$$

$$I_3 = \int_0^1 Y_j''^2(\xi) \mathrm{d}\xi = 12.3624$$

$$c_k = \sqrt{\frac{I_3}{I_1}} = 3.5160 \quad \text{and} \quad c_m = \frac{I_2}{I_1}$$

Adhikari, S. and Chowdhury, R., "The calibration of carbon nanotube ba bio-nano sensors", *Journal of Applied Physics*, **107**[12] (2010), pp. 124322:1-8

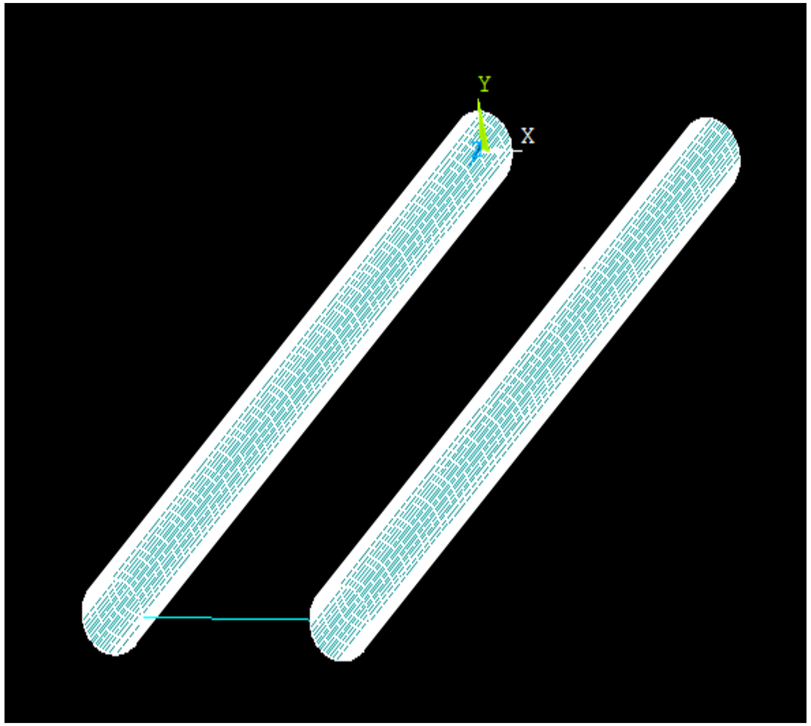
TABLE I. The stiffness (c_k) and mass (c_m) calibration constants for CNT based bio-nano sensor. The value of γ indicates the length of the mass as a fraction of the length of the CNT.

Mass size	Cantilevered CNT		Bridged CNT	
	c_k	c_m	c_k	c_m
Point mass ($\gamma \to 0$)	3.5160152	4.0	22.373285	2.522208547
$\gamma = 0.1$		3.474732666		2.486573805
$\gamma = 0.2$		3.000820053		2.383894805
$\gamma = 0.3$		2.579653837		2.226110255
$\gamma = 0.4$		2.212267400		2.030797235
$\gamma = 0.5$		1.898480438		1.818142650
$\gamma = 0.6$		1.636330135		1.607531183
$\gamma = 0.7$		1.421839146		1.414412512
$\gamma = 0.8$		1.249156270		1.248100151

Proposed approach

- Use two coupled cantilevers – not one!
- Use eigenmodes and not eigenfrequencies

- Motivation: Under certain situations the eigenmodes may prove to be more sensitive to the changes in the mass than the classical approach to consider resonant frequency



Two-degree-of-freedom model

Equivalent two-DOF model

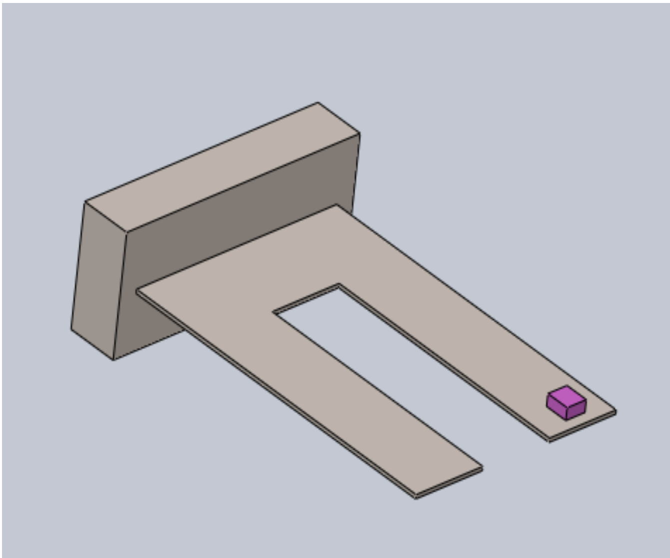


Fig1.(a)

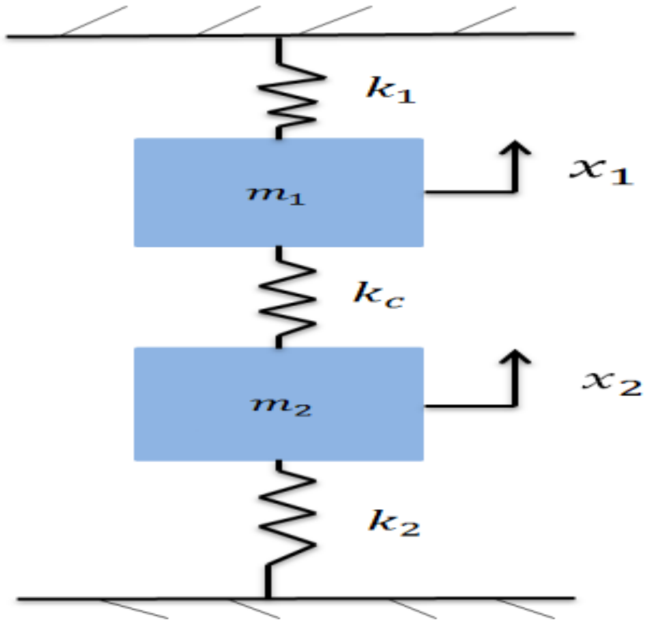


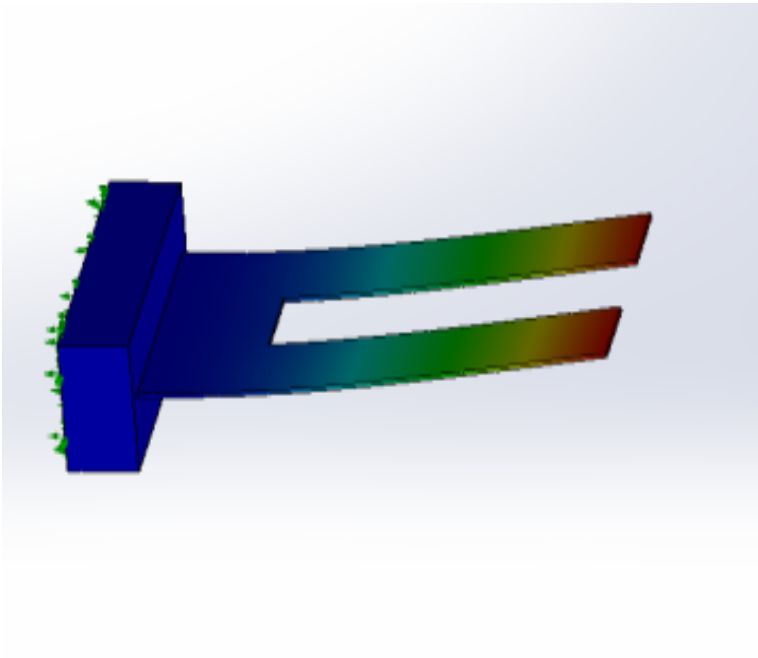
Fig1.(b)

The tip dynamics can be modeled by a 2-DOF spring-mass system

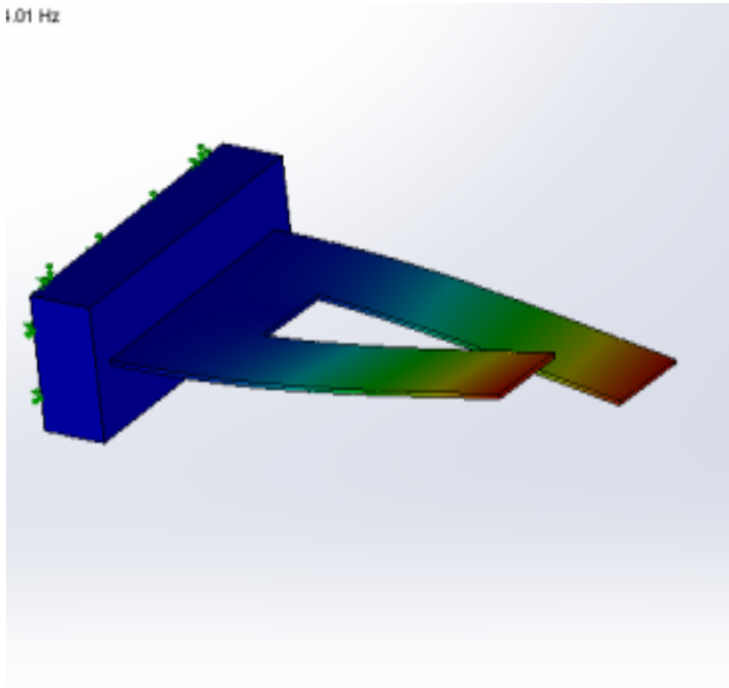
Here: $m_1=m_2=m=33M_c/140$ and $k_1=k_2=k=3EI/L^3$

The coupling spring: $k_c = \varepsilon k$

Equivalent two-DOF model



Mode 1: symmetric



Mode 2: Anti-symmetric

$$\lambda_{1,2}^{0} = \varepsilon + 1 \pm \sqrt{\varepsilon^2}$$

$$U_1^0 = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \qquad U_2^0 = \begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix}$$

Two-degree-of-freedom model with added mass

Mathematical model

The equation of motion with added mass (2Δ) positioned at the end of one of the cantilever becomes

$$m\begin{bmatrix} 1 & 0 \\ 0 & 1+2\Delta \end{bmatrix}\ddot{\mathbf{x}} + k\begin{bmatrix} 1+\varepsilon & -\varepsilon \\ -\varepsilon & 1+\varepsilon \end{bmatrix}\mathbf{x} = 0$$

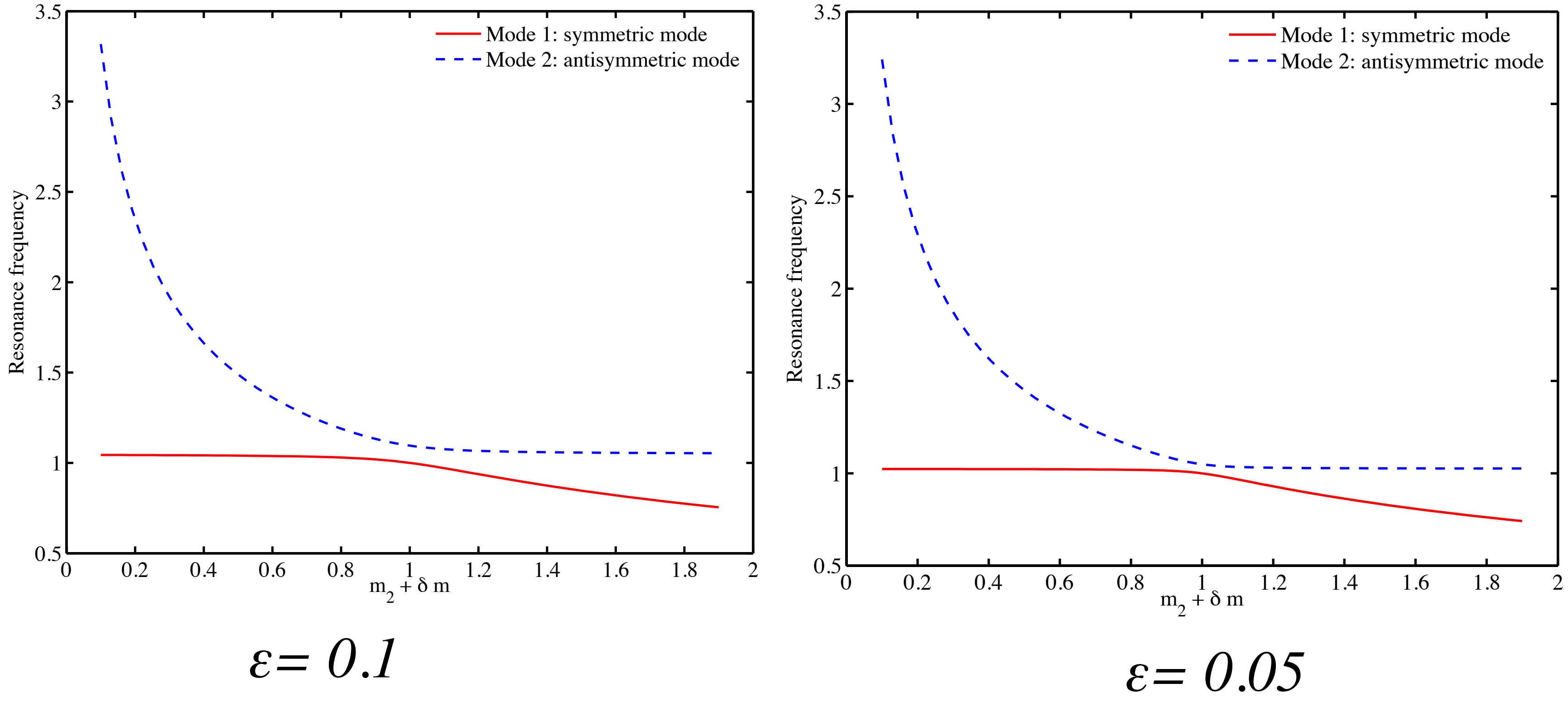
The resonance frequencies:

$$\boldsymbol{\lambda}_{1,2} = \frac{1+\Delta+\varepsilon+\in\Delta \pm \sqrt{(\Delta^2\varepsilon^2 + \Delta^2 + 2\varepsilon\Delta^2 + \varepsilon^2 + 2\varepsilon^2\Delta}}{1+2\Delta}$$

The eigenmodes $[1, U_{1,2}]$ where $U_{1,2} = \frac{\Delta+\varepsilon\Delta\pm\beta}{(1+2\Delta)\varepsilon}$

Where $\beta = \sqrt{(\Delta^2 \in \varepsilon^2 + \Delta^2 + 2\varepsilon\Delta^2 + \varepsilon^2 + 2\varepsilon^2\Delta}$

The mode veering phenomenon



Eigenvalues come close (closeness depends on the coupling strength) - but they do not cross – experimentally shown in:

du Bois, J. L., Adhikari, S. and Lieven, N. A. J., "Mode veering in stressed framed structures", Journal of Sound and Vibration, 322[4-5] (2009), pp. 1117-1124.

Extreme parametric sensitivity of the eigenmodes in the veering range

Leissa (1974), *Journal of Applied Mathematics and Physics (ZAMP)*: -

the (eigenfunctions) must undergo violent change – figuratively speaking, a dragonfly one instant, a butterfly the next, and something indescribable in between'.

Our aim is to exploit this parametric sensitivity and turn it in to a sensor device

Resonance shift and eigen-mode shift

Eigenmodes are known by many names: eigenvectors, mode shapes, eigenmodes, eigenfunctions or simply modes

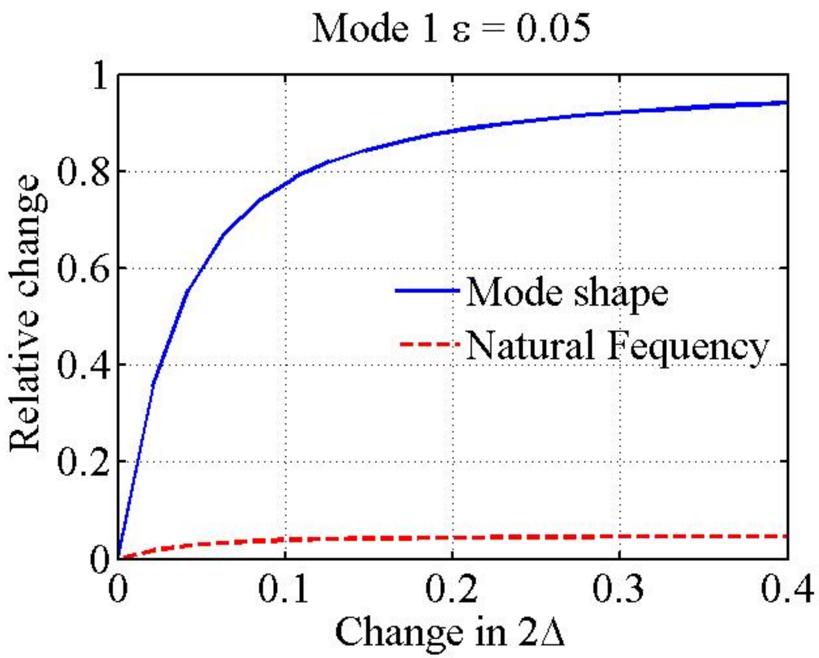
The classical resonance shift is:

$$\Delta\lambda = \frac{\lambda_i^0 - \lambda_i}{\lambda_i^0}$$

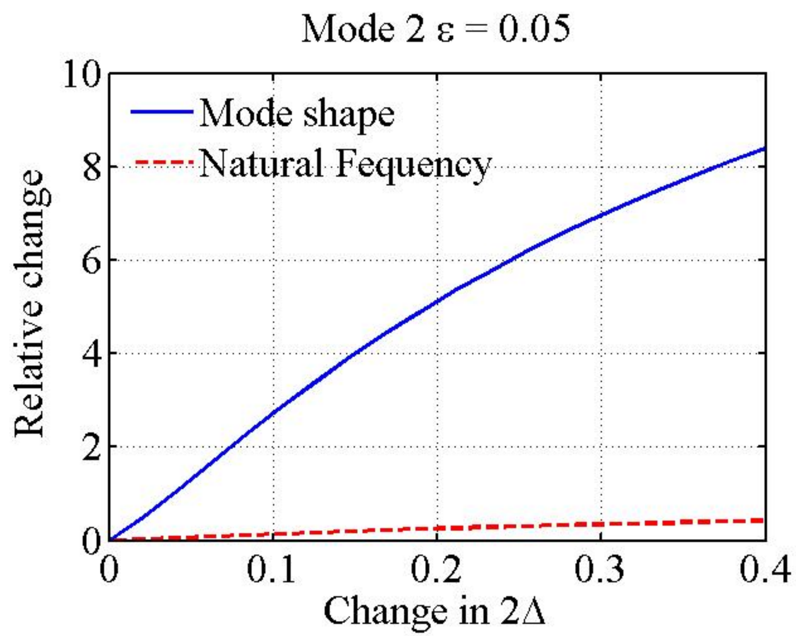
We introduce a new quantify - eigenmode-shift or simply the 'Mode-shift'

$$|\Delta u_i| = \frac{u_i^0 - u_i}{u_i^0}$$

Extreme parametric sensitivity of the eigenmodes in the veering range

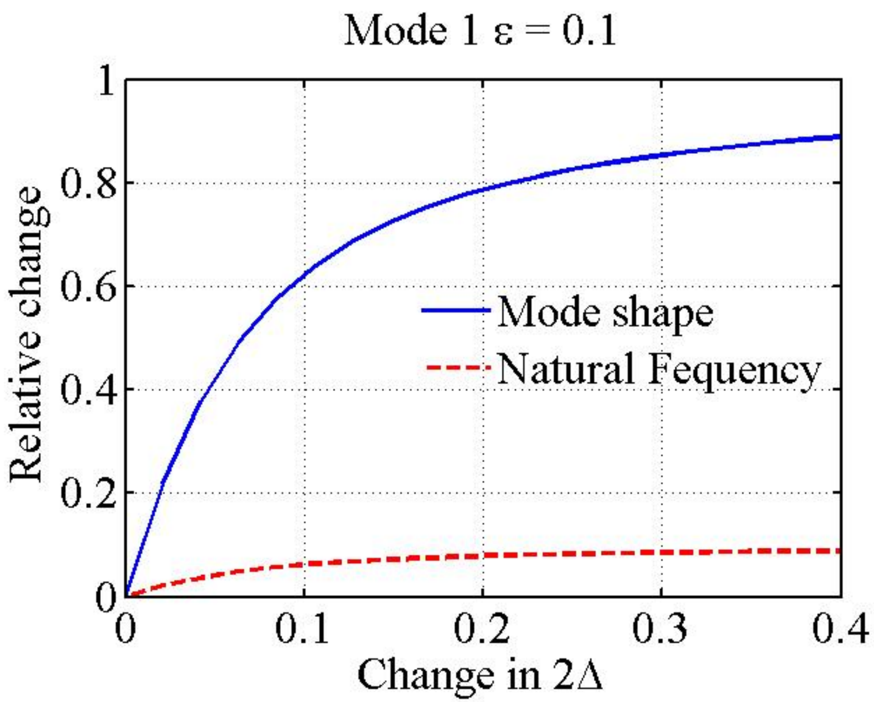


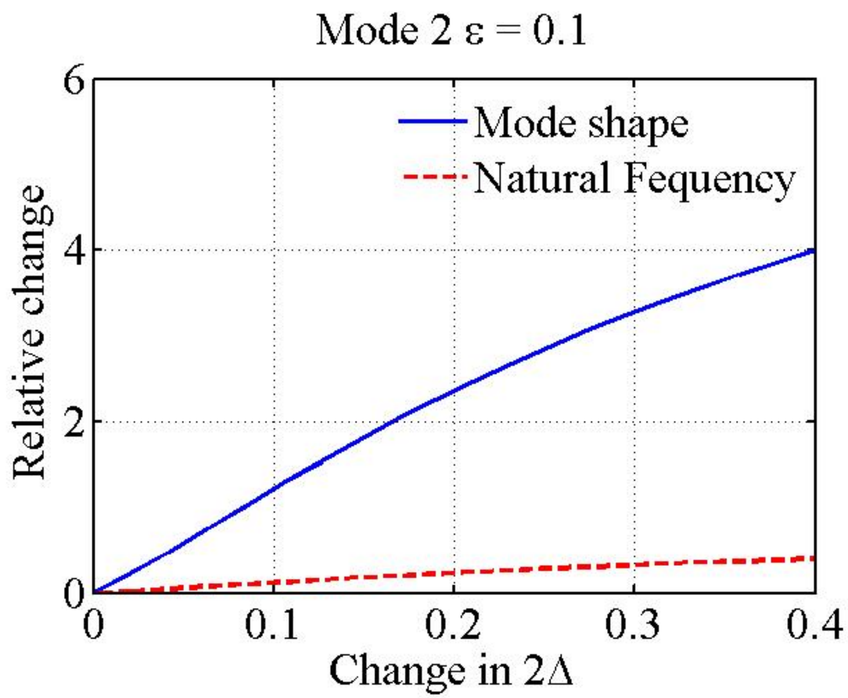
Highly sensitive



We need to use antisymmetric eigenmode

Sensitivity of the eigenmodes depends on coupling





Higher the coupling, lower the sensitivity

Mass detection from eigenmode-shift

Suppose δu is mode shift for the mass-loaded cantilever in the second mode of vibration (the anti-symmetric mode)

$$(U_2^0\text{-}U_{2\,=}) = 1\text{-}\,\frac{\Delta+\varepsilon\Delta+\beta}{(1+2\Delta)\varepsilon} = \delta\mathrm{u}$$

If we can measure the mode-shift δu, the mass can be identified from the above expressions as

$$\Delta = \frac{2\delta u\varepsilon+\varepsilon\delta u^2}{2(1+\delta u)(1-\delta u\varepsilon)}$$

Swansea University
Prifysgol Abertawe

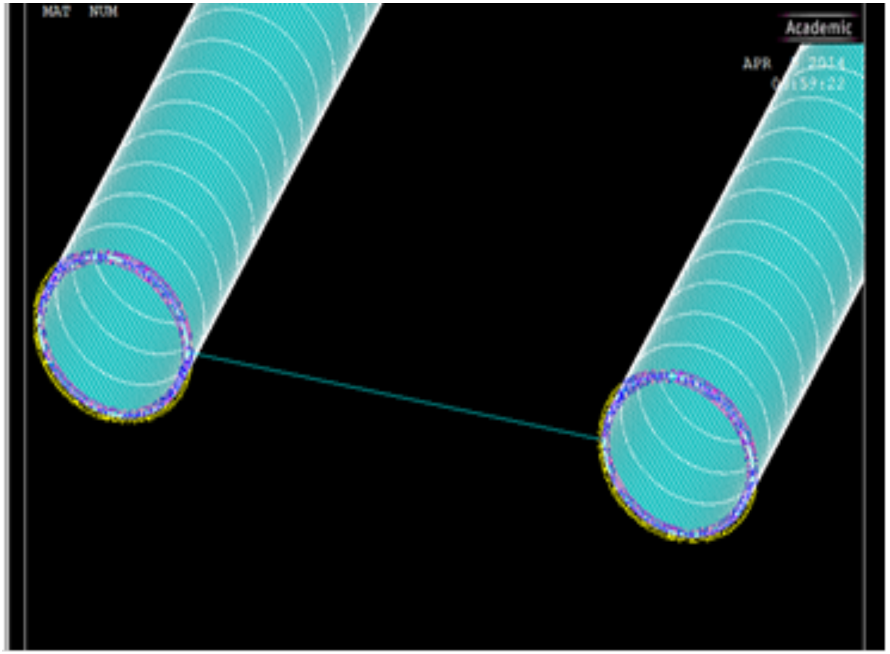
Finite Element Validation

Finite element model

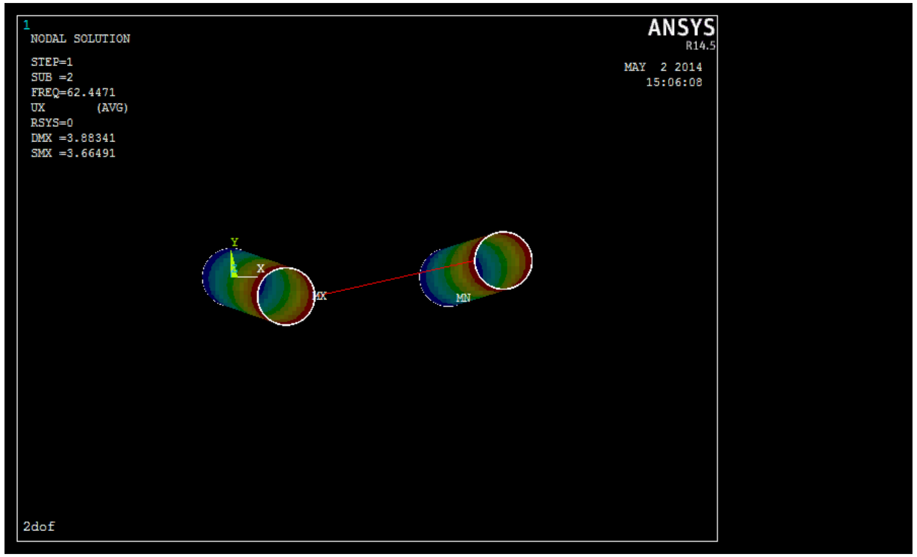
- Two hollow cylindrical cantilever beams

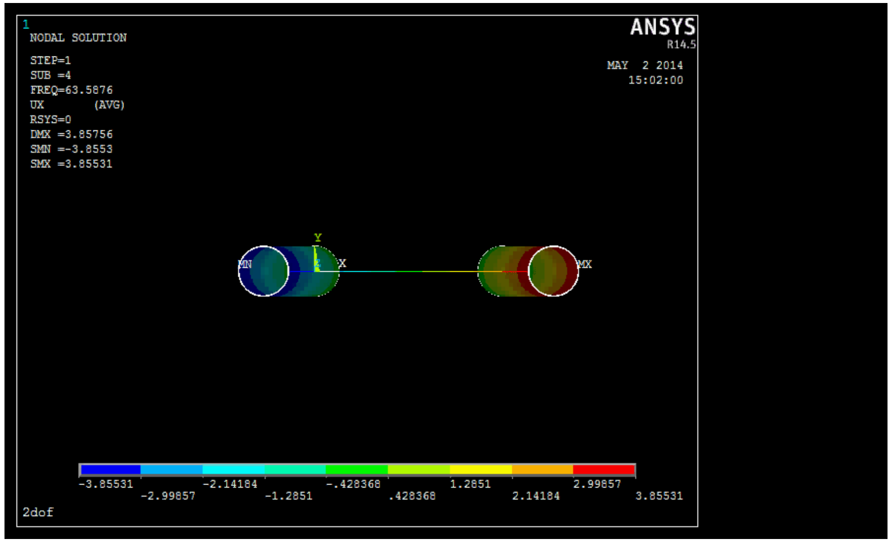
- The nodes at the end of cantilever are the eigenvectors

- The vector u_2 is obtained by taking an average of all the responses at the tip (homogenization)



Finite element modes

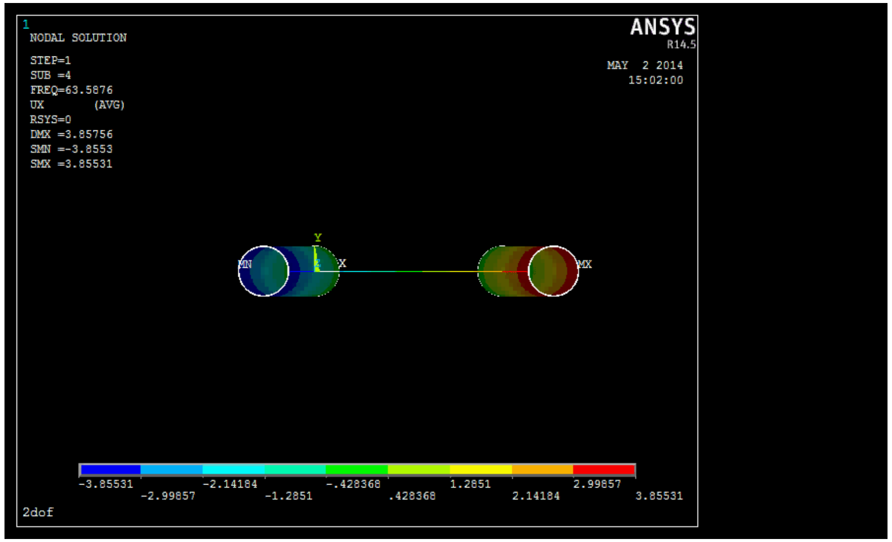




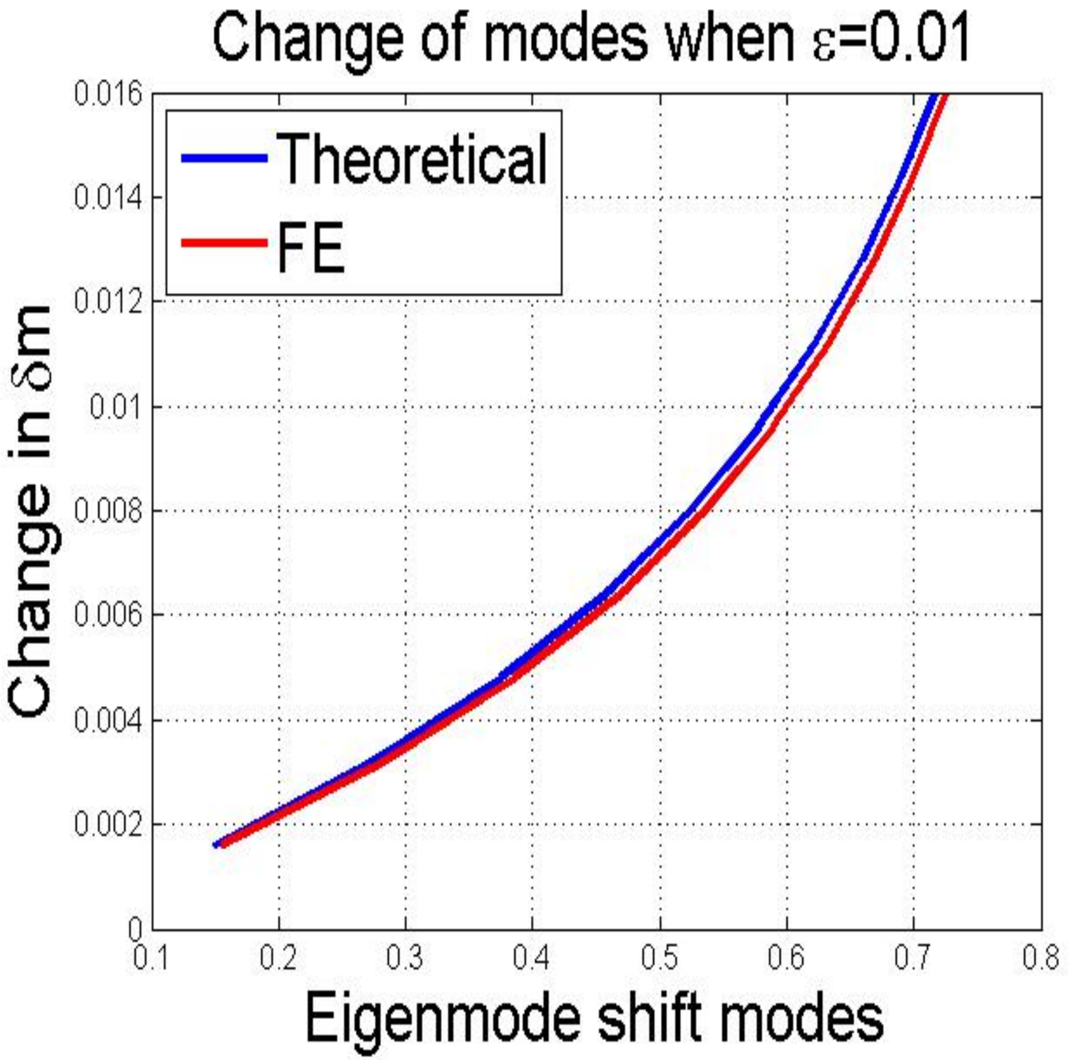
Finite element model

- The mode-shift δu is obtained by taking the difference in the second eigenmode between no mass ($\Delta = 0$) and mass loaded case ($\Delta \neq 0$)

- We use δu as 'input' and check if the original mass used for simulation is predicted by our sensing equation

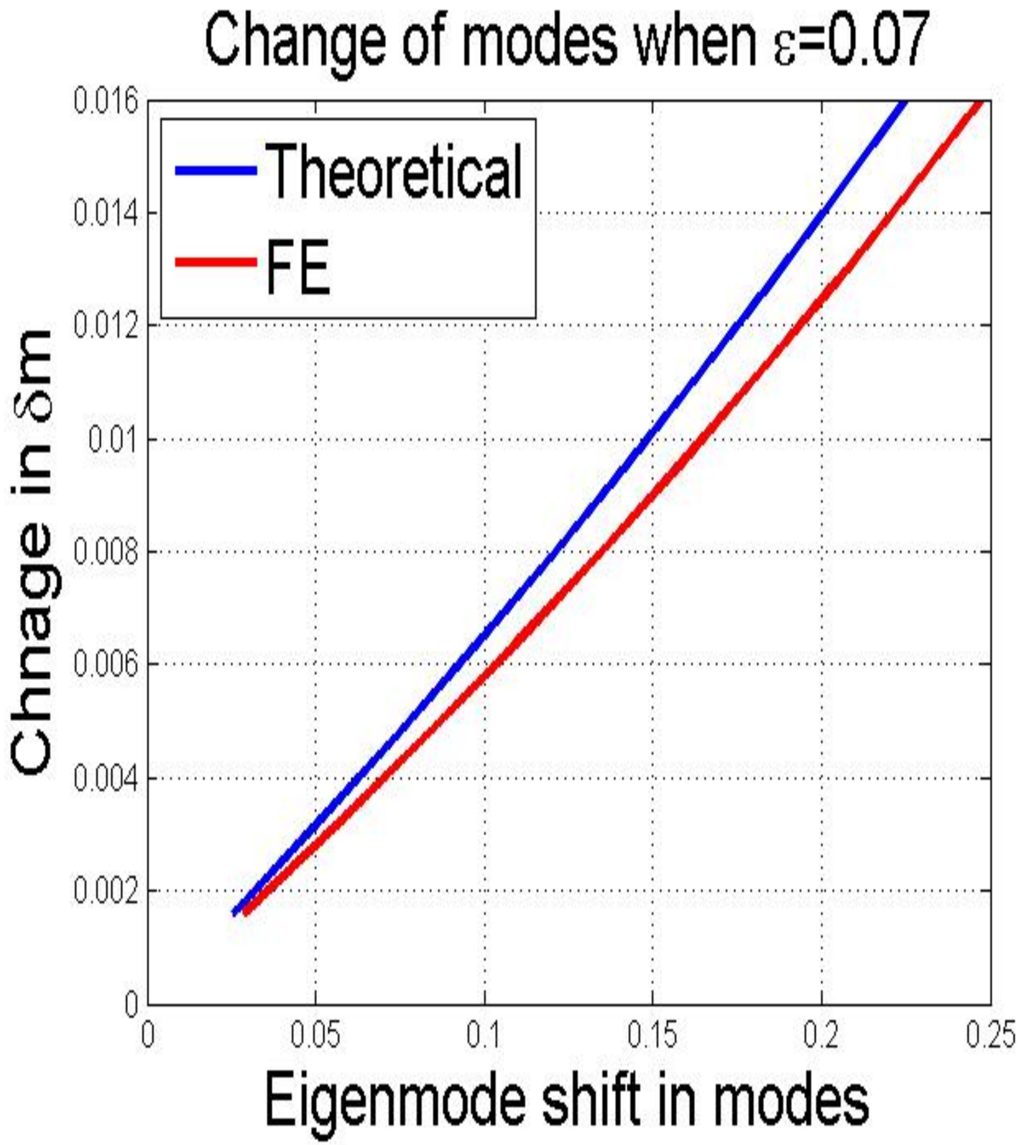


Validation results



Theoretical Δ	F.E Δ
0.0016	0.0017
0.0032	0.0033
0.0048	0.005
0.0064	0.0067
0.008	0.0083
0.0096	0.01
0.0112	0.0116
0.0128	0.0133
0.0144	0.015
0.016	0.0166

Validation results



Theoretical δΔ	FE Δ
0.0016	0.0012
0.0032	0.0025
0.0048	0.0038
0.0064	0.005
0.008	0.0063
0.0096	0.0075
0.0112	0.0087
0.0128	0.0099
0.0144	0.011
0.016	0.0122

Conclusions and future challanges

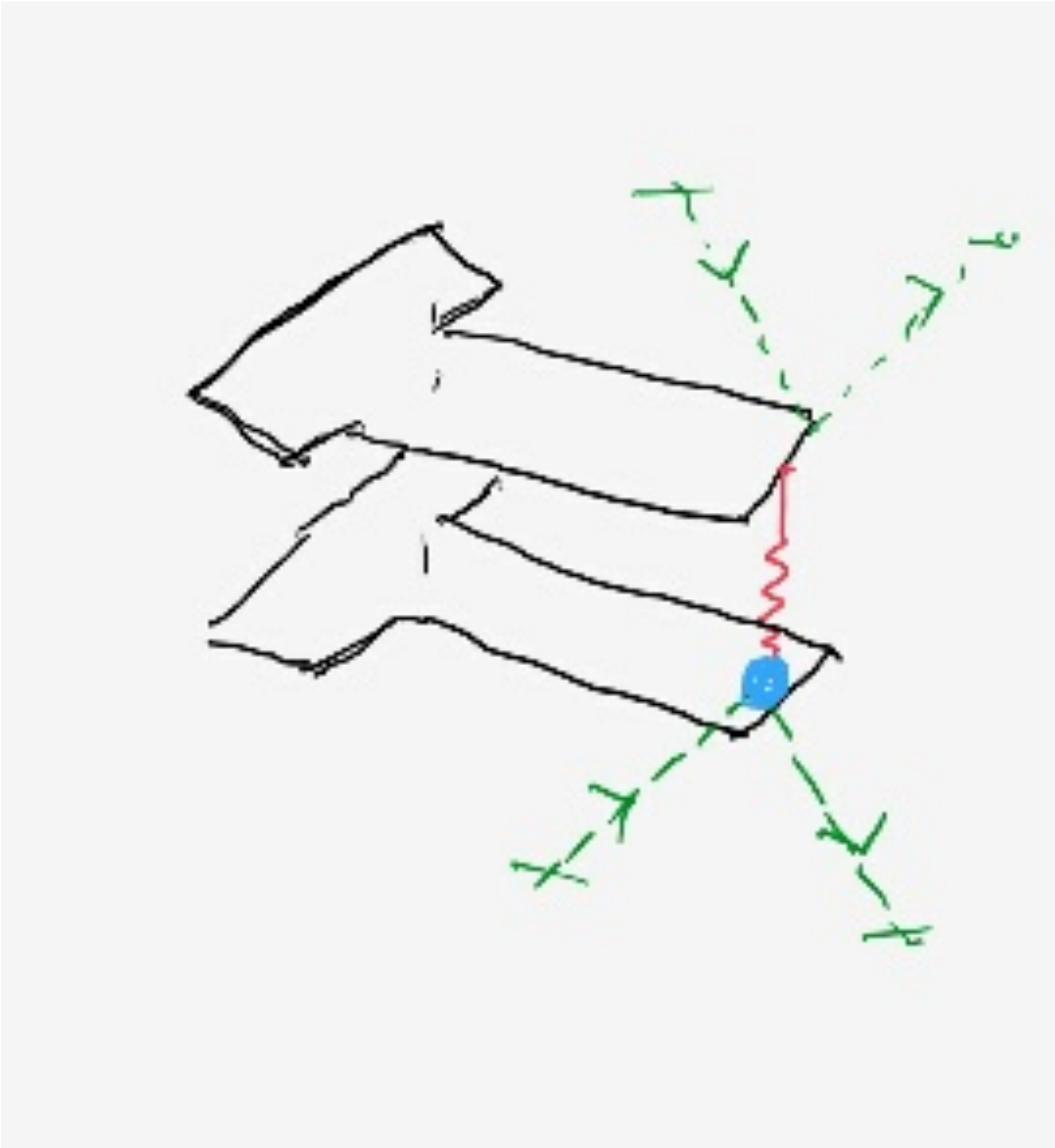
Conclusions

- A system using two cantilevers coupled by a ‘light’ spring is proposed
- Only one of the cantilever loaded with the mass to be identified
- Eigenmode-shift as opposed to conventional frequency-shift is suggested as a measured quantity for mass sensing
- 5-10 times increase in relative sensitivity can be achieved (depending on the coupling stiffness) – lower the coupling, the higher the sensitivity)
- The mode-shift δu is obtained by taking the difference in the second eigenmode between no mass ($\Delta = 0$) and the mass loaded case ($\Delta \neq 0$)
- We used δu as ‘input’ and validate if the original mass used for the simulation is predicted by our sensing equation using finite element simulation - good agreement was found

Future challenges

- Connect two identical cantilevers by a spring at the tip (we suggest 5-15% of the tip stiffness)
- The elastic coupling is the key! Without this, the technique does not work….
- Excite the system into the second mode (the anti-symmetric mode)
- Simultaneous readout of displacements of both the tips in the second mode
- Homogenization of the measured displacements at both the tips
- A point mass at the cantilever tip is assumed – the physical size of the attached body might introduce some errors
- Need to understand the role of Q-factor and noise

Future challenges



Acknowledgement

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