Random Eigenvalue Problems in Structural Dynamics

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Outline of the Presentation

- Random eigenvalue problem
- Perturbation Methods
- Asymptotic analysis of multidimensional integrals
- Moments and pdf of the eigenvalues
- Numerical Example & results
- Conclusions & open problems

Random Eigenvalue Problem

The random eigenvalue problem of undamped or proportionally damped linear systems:

$$\mathbf{K}(\mathbf{x})\boldsymbol{\phi}_j = \lambda_j \mathbf{M}(\mathbf{x})\boldsymbol{\phi}_j$$

λ_j eigenvalues; $\boldsymbol{\phi}_j$ eigenvectors; $\mathbf{M}(\mathbf{x}) \in \mathbb{R}^{N\times N}$ mass matrix and $\mathbf{K}(\mathbf{x}) \in \mathbb{R}^{N\times N}$ stiffness matrix. $\mathbf{x} \in \mathbb{R}^m$ is random parameter vector with pdf

$$p_{\mathbf{x}}(\mathbf{x}) = e^{-L(\mathbf{x})}$$

$-L(\mathbf{x})$ is the log-likelihood function.

The Aim

- To obtain the joint probability density function of the eigenvalues and the eigenvectors
- If the matrix $\mathbf{M}^{-1}\mathbf{K}$ is GUE (Gaussian unitary ensemble) or GOE (Gaussian orthogonal ensemble) an exact closed-form expression can be obtained for the joint pdf of the eigenvalues
- In general the system matrices for real structures are not GUE or GOE

Perturbation Method

Taylor series expansion of $\lambda_j(\mathbf{x})$ about $\mathbf{x} = \boldsymbol{\alpha}$

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha}) + \mathbf{d}_{\lambda_j}^T(\boldsymbol{\alpha})\,(\mathbf{x} - \boldsymbol{\alpha}) + \frac{1}{2}\,(\mathbf{x} - \boldsymbol{\alpha})^T\,\mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\,(\mathbf{x} - \boldsymbol{\alpha})$$

- In mean-centered approach $\boldsymbol{\alpha}$ is the mean of $\mathbf{x}$
- Alternatively, $\boldsymbol{\alpha}$ can be obtained such that the any moment of each eigenvalue is calculated most accurately

Multidimensional Integrals

We want to evaluate an m-dimensional integral over the unbounded domain $\mathbb{R}^m$:

$$\mathcal{J} = \int_{\mathbb{R}^m} e^{-f(\mathbf{x})} \, d\mathbf{x}$$

- Assume $f(\mathbf{x})$ is smooth and at least twice differentiable
- The maximum contribution to this integral comes from the neighborhood where $f(\mathbf{x})$ reaches its global minimum, say $\boldsymbol{\theta} \in \mathbb{R}^m$

Multidimensional Integrals

Therefore, at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 0, \forall k \quad \text{or} \quad \mathbf{d}_f(\boldsymbol{\theta}) = \mathbf{0}$$

Expand $f(\mathbf{x})$ in a Taylor series about $\boldsymbol{\theta}$:

$$\mathcal{J} = \int_{\mathbb{R}^m} e^{-\left\{f(\boldsymbol{\theta}) + \frac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^T \mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta}) + \varepsilon(\mathbf{x},\boldsymbol{\theta})\right\}} d\mathbf{x}$$

$$= e^{-f(\boldsymbol{\theta})} \int_{\mathbb{R}^m} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\theta})^T \mathbf{D}_f(\boldsymbol{\theta})(\mathbf{x}-\boldsymbol{\theta}) - \varepsilon(\mathbf{x},\boldsymbol{\theta})} d\mathbf{x}$$

Multidimensional Integrals

The error $\varepsilon(\mathbf{x}, \boldsymbol{\theta})$ depends on higher of derivatives of $f(\mathbf{x})$ at $\mathbf{x} = \boldsymbol{\theta}$. If they are small compared to $f(\boldsymbol{\theta})$ their contribution will negligible to the value of the integral. So we assume that $f(\boldsymbol{\theta})$ is large so that

$$\left| \frac{1}{f(\boldsymbol{\theta})} \mathcal{D}^{(j)}(f(\boldsymbol{\theta})) \right| \to 0 \quad \text{for} \quad j > 2$$

where $\mathcal{D}^{(j)}(f(\boldsymbol{\theta}))$ is jth order derivative of $f(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\theta}$. Under such assumptions $\varepsilon(\mathbf{x}, \boldsymbol{\theta}) \to 0$.

Multidimensional Integrals

- Use the coordinate transformation:

$$\boldsymbol{\xi} = (\mathbf{x} - \boldsymbol{\theta})\,\mathbf{D}_f^{-1/2}(\boldsymbol{\theta})$$

- The Jacobian: $\|\mathbf{J}\| = \|\mathbf{D}_f\,(\boldsymbol{\theta})\|^{-1/2}$
- The integral becomes:

$$\mathcal{J} \approx e^{-f(\boldsymbol{\theta})} \int_{\mathbb{R}^m} \|\mathbf{D}_f\,(\boldsymbol{\theta})\|^{-1/2}\, e^{-\frac{1}{2}\left(\boldsymbol{\xi}^T\boldsymbol{\xi}\right)}\, d\boldsymbol{\xi}$$

or

$$\boxed{\mathcal{J} \approx (2\pi)^{m/2} e^{-f(\boldsymbol{\theta})}\, \|\mathbf{D}_f\,(\boldsymbol{\theta})\|^{-1/2}}$$

Moments of Eigenvalues

An arbitrary rth order moment of the eigenvalues can be obtained from

$$\mu_j^{(r)} = \mathrm{E}\left[\lambda_j^r(\mathbf{x})\right] = \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x}) p_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

$$= \int_{\mathbb{R}^m} e^{-(L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x}))}\, d\mathbf{x}, \quad r = 1, 2, 3 \cdots$$

- Previous result can be used by choosing $f(\mathbf{x}) = L(\mathbf{x}) - r \ln \lambda_j(\mathbf{x})$

Moments of Eigenvalues

After some simplifications

$$\mu_j^{(r)} \approx (2\pi)^{m/2} \lambda_j^r(\boldsymbol{\theta}) e^{-L(\boldsymbol{\theta})}$$

$$\left\| \mathbf{D}_L(\boldsymbol{\theta}) + \frac{1}{r}\mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^T - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \right\|^{-1/2}$$

$$r = 1, 2, 3, \cdots$$

$\boldsymbol{\theta}$ is obtained from:

$$\mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) r = \lambda_j(\boldsymbol{\theta}) \mathbf{d}_L(\boldsymbol{\theta})$$

Moments of Eigenvalues

- Mean of the eigenvalues:

$$\widehat{\lambda}_j = \mu_j^{(1)} = \lambda_j(\boldsymbol{\theta}) e^{-L(\boldsymbol{\theta})} \left\| \mathbf{D}_L(\boldsymbol{\theta}) + \mathbf{d}_L(\boldsymbol{\theta}) \mathbf{d}_L(\boldsymbol{\theta})^T - \mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) / \lambda_j(\boldsymbol{\theta}) \right\|^{-1/2}$$

- Central moments of the eigenvalues:

$$\mathrm{E}\left[\left(\lambda_j - \widehat{\lambda}_j \right)^r \right] = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mu_j^{(k)} \widehat{\lambda}_j^{r-k}$$

Multivariate Gaussian Case

$$L(\mathbf{x}) = \frac{m}{2}\ln(2\pi) + \frac{1}{2}\ln\|\boldsymbol{\Sigma}\| + \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$$

so $\mathbf{d}_L(\mathbf{x}) = \boldsymbol{\Sigma}^{-1}\mathbf{x}$ and $\mathbf{D}_L(\mathbf{x}) = \boldsymbol{\Sigma}^{-1}$. Therefore:

$$\mu_j^{(r)} \approx \lambda_j^r(\boldsymbol{\theta}) e^{-\frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\mu})} \left\|\mathbf{I} + \frac{1}{r}\boldsymbol{\theta}\boldsymbol{\theta}^T\boldsymbol{\Sigma}^{-1} - \frac{r}{\lambda_j(\boldsymbol{\theta})}\boldsymbol{\Sigma}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta})\right\|^{-1/2}$$

where $\boldsymbol{\theta} = \frac{r}{\lambda_j(\boldsymbol{\theta})}\boldsymbol{\Sigma}\mathbf{d}_{\lambda_j}(\boldsymbol{\theta})$

Maximum Entropy pdf

Constraints for $u \in [0, \infty]$:

$$\int_0^\infty p_{\lambda_j}(u)du = 1$$

$$\int_0^\infty u^r p_{\lambda_j}(u)du = \mu_j^{(r)}, \quad r = 1, 2, 3, \cdots, n$$

Maximizing Shannon's measure of entropy $\mathcal{S} = -\int_0^\infty p_{\lambda_j}(u) \ln p_{\lambda_j}(u)du$, the pdf of λ_j is

$$p_{\lambda_j}(u) = e^{-\left\{\rho_0 + \sum_{i=1}^n \rho_i u^i\right\}} = e^{-\rho_0} e^{-\sum_{i=1}^n \rho_i u^i}, \quad u \geq 0$$

Maximum Entropy pdf

Taking first two moments, the resulting pdf is a truncated Gaussian density function

$$p_{\lambda_j}(u) = \frac{1}{\sqrt{2\pi}\sigma_j \, \Phi\left(\widehat{\lambda}_j/\sigma_j\right)} \exp\left\{ -\frac{\left(u-\widehat{\lambda}_j\right)^2}{2\sigma_j^2} \right\}$$

where $\sigma_j^2 = \mu_j^{(2)} - \widehat{\lambda}_j^2$

- Ensures that the probability of any eigenvalues becoming negative is zero

Central χ^2 Approximation

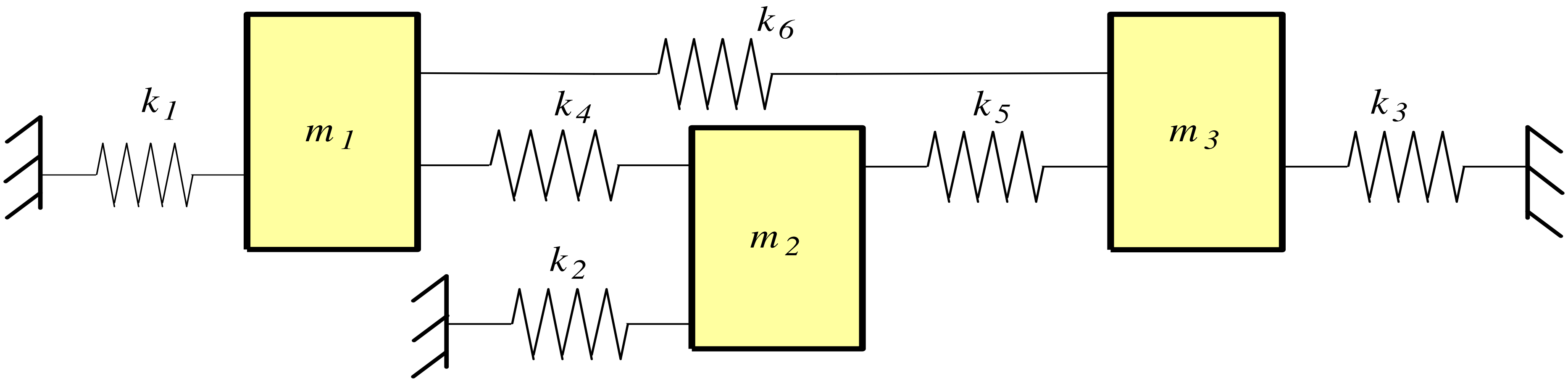
Pdf of jth eigenvalue

$$p_{\lambda_j}(u) \approx \frac{1}{\gamma_j} p_{\chi^2_{\nu_j}}\left(\frac{u-\eta_j}{\gamma_j}\right) = \frac{(u-\eta_j)^{\nu_j/2-1} e^{-(u-\eta_j)/2\gamma_j}}{(2\gamma_j)^{\nu_j/2}\Gamma(\nu_j/2)}$$

The constants η_j, γ_j, and ν_j are such that the first three moments of λ_j are the same.

Example System

Undamped three degree-of-freedom random system:



$$m_i = \overline{m}_i \left(1 + \epsilon_m x_i\right),\ i = 1, 2, 3$$

$$k_i = \overline{k}_i \left(1 + \epsilon_k x_{i+3}\right),\ i = 1, \cdots, 6$$

Vector of random variables: $\mathbf{x} = \{x_1, \cdots, x_9\}^T \in \mathbb{R}^9$

Example System

- $\mathbf{x}$ is standard Gaussian, $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$
- Strength parameters $\epsilon_m = \epsilon_k = 0.15$

Two parameter sets are considered:

- CASE 1: All eigenvalues are well separated
  For this case $\overline{m}_i = 1.0$ kg for $i = 1, 2, 3$; $\overline{k}_i = 1.0$ N/m for $i = 1, \cdots, 5$ and $k_6 = 3.0$ N/m
- CASE 2: Two eigenvalues are very close
  All parameter values are the same except $k_6 = 1.275$ N/m

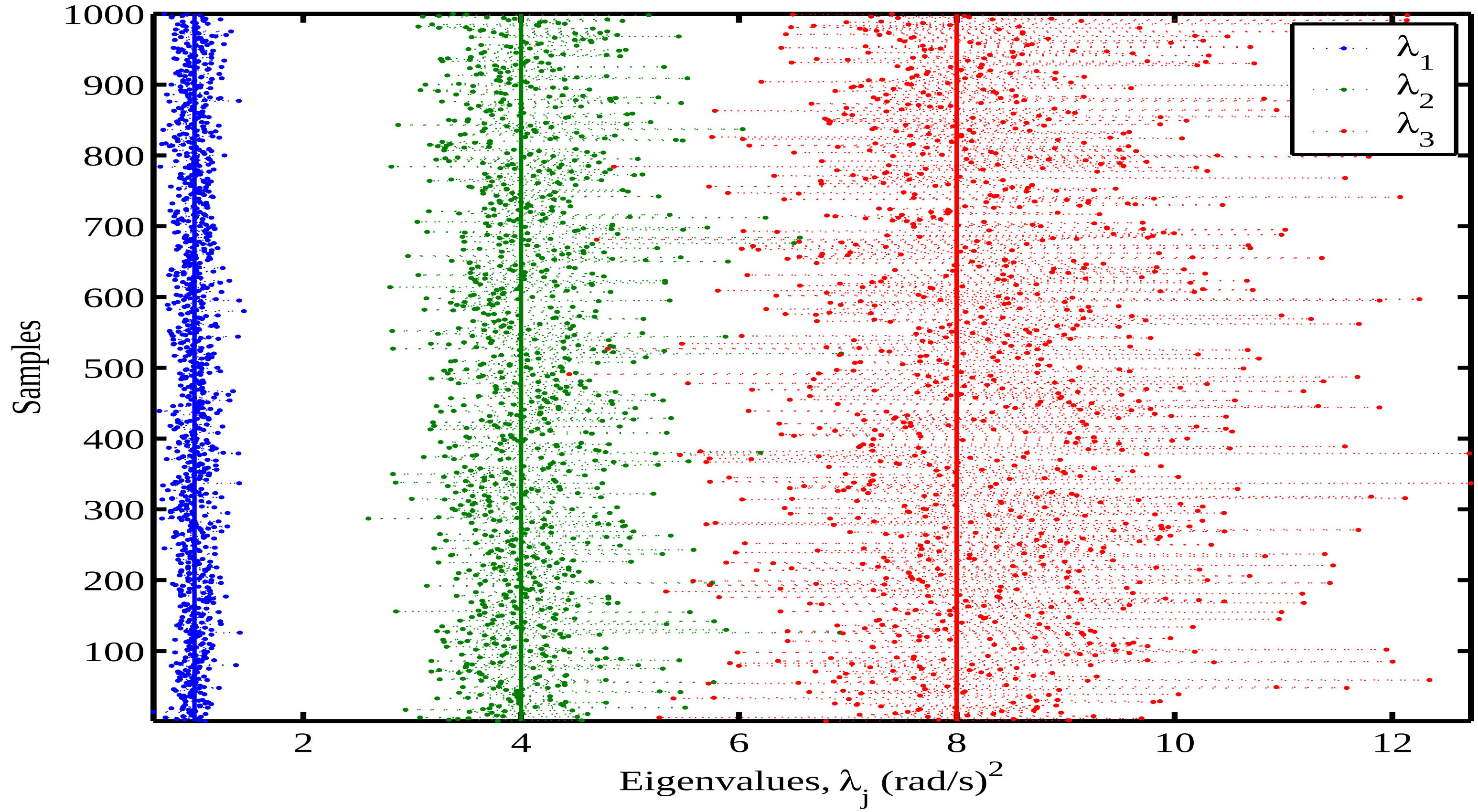
Example System

The percentage error:

$$\text{Error} = \frac{\mu^{(r)} - \{\mu^{(r)}\}_{\text{MCS}}}{\{\mu^{(r)}\}_{\text{MCS}}} \times 100$$

- 5000 samples are used in Monte Carlo simulation

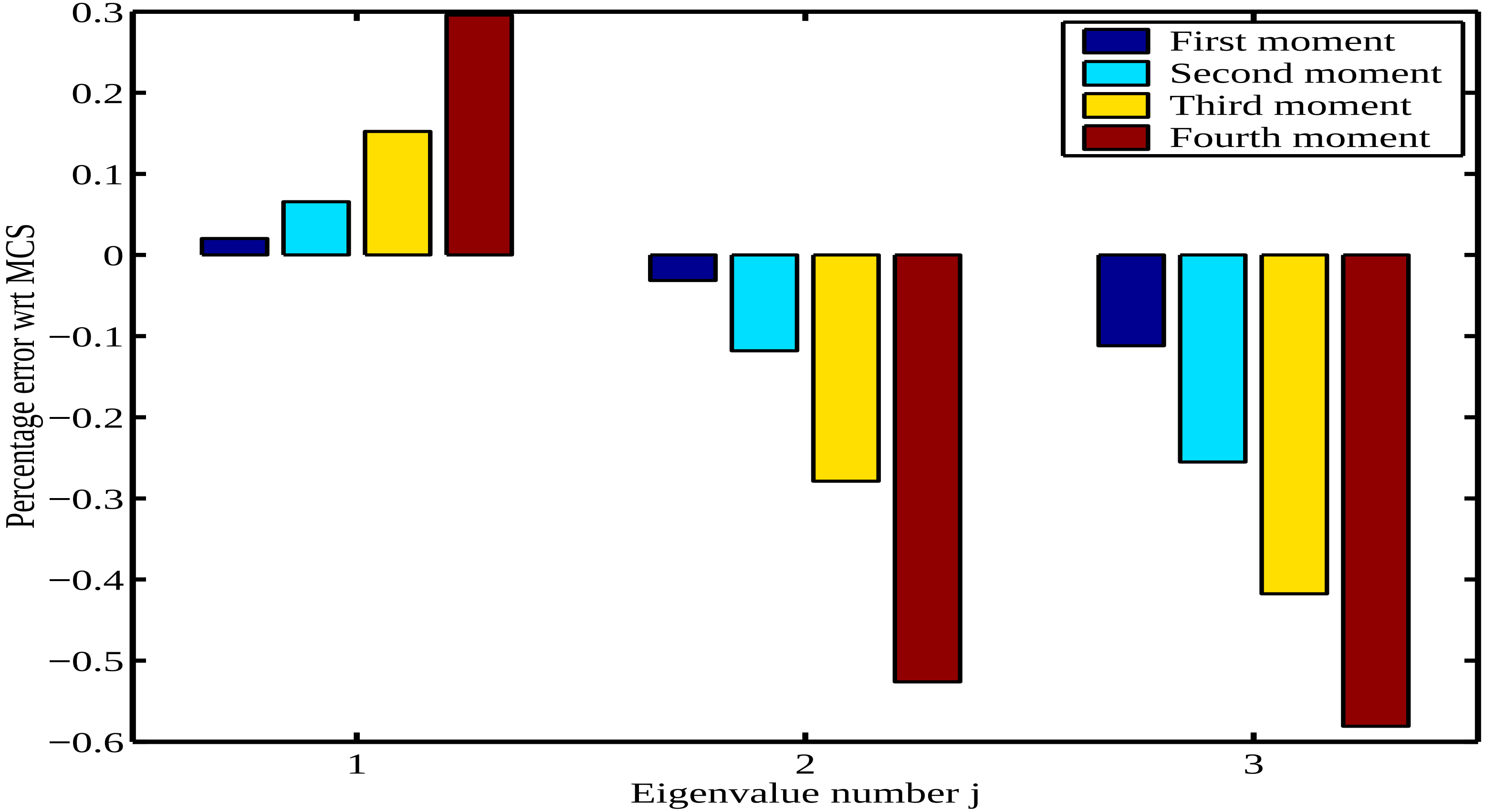
Case 1



Statistical scatter in the eigenvalues

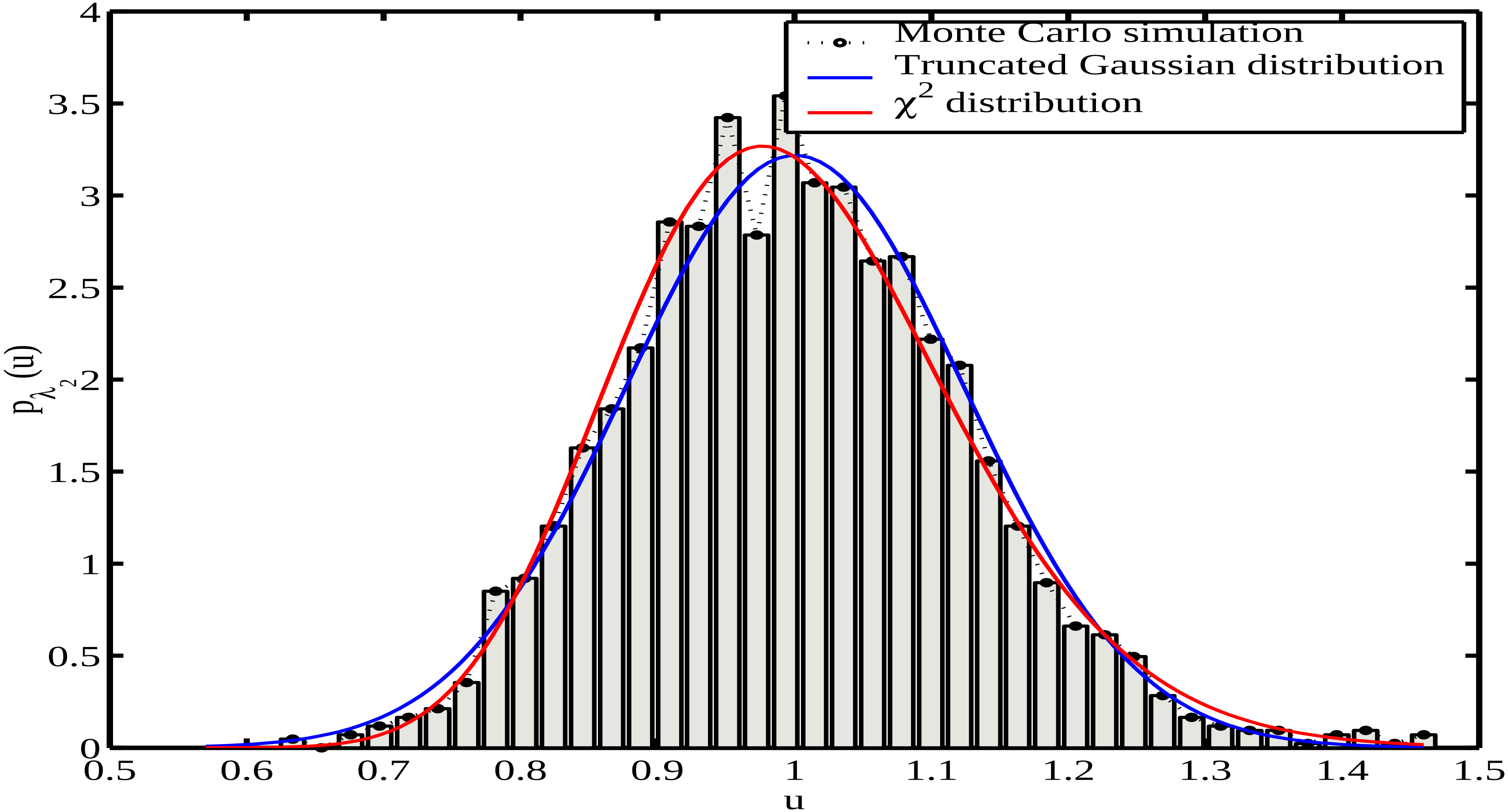
$\overline{\lambda}_1 = 1, \quad \overline{\lambda}_2 = 4,$ and $\overline{\lambda}_3 = 8$

Case 1



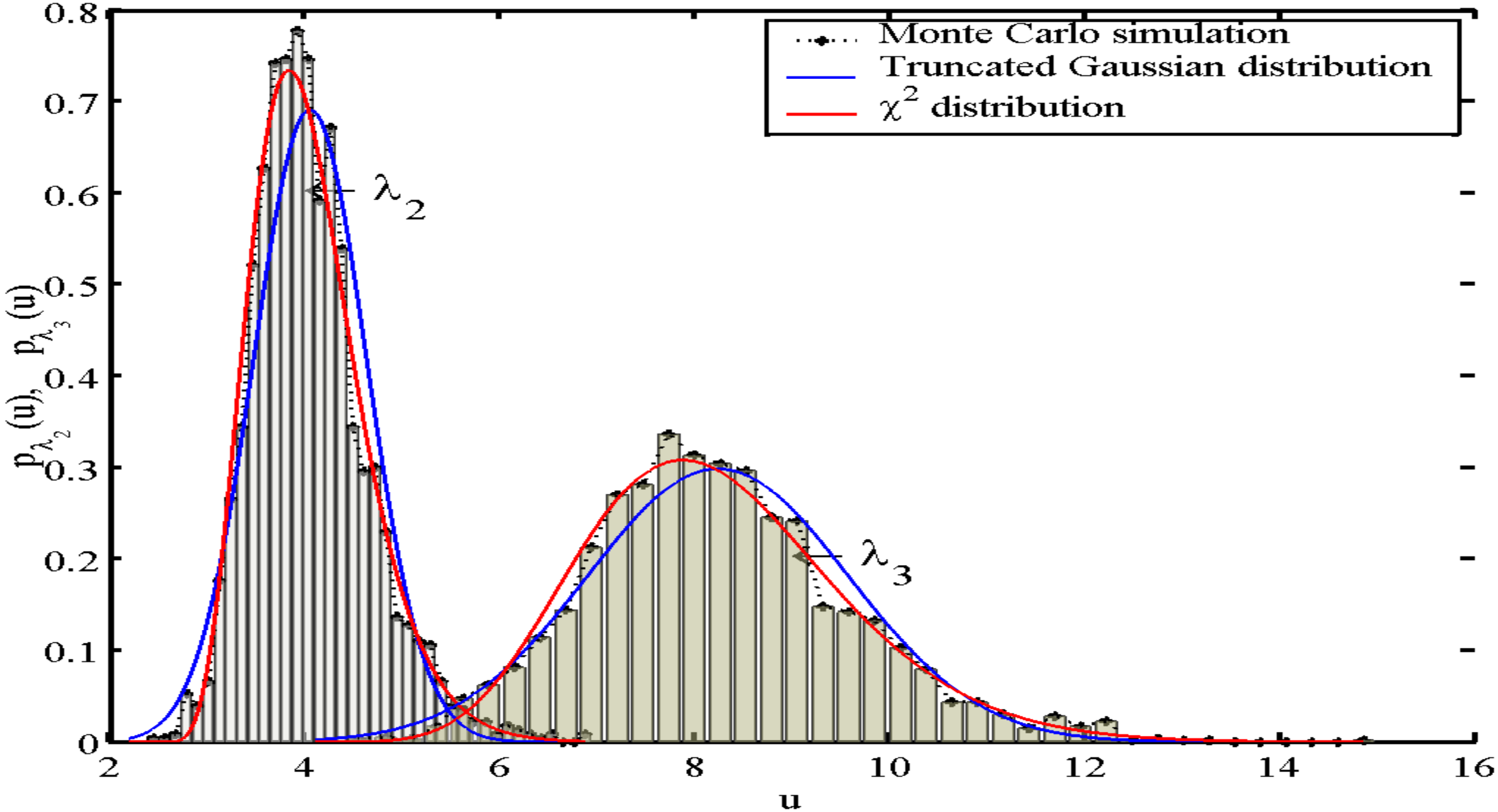
Percentage error for first four moments

Case 1



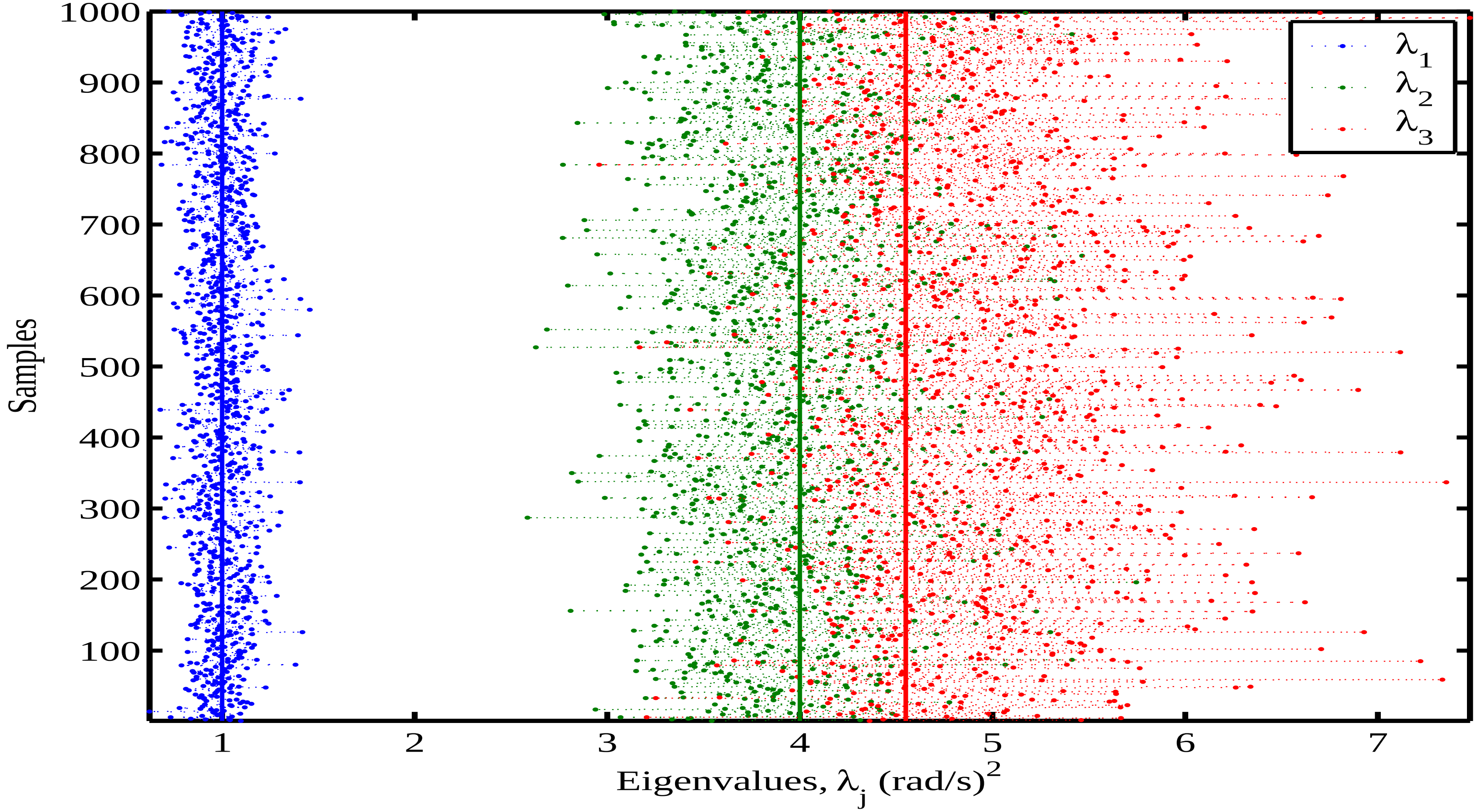
Pdf of the first eigenvalue

Case 1



Pdf of the second and third eigenvalues

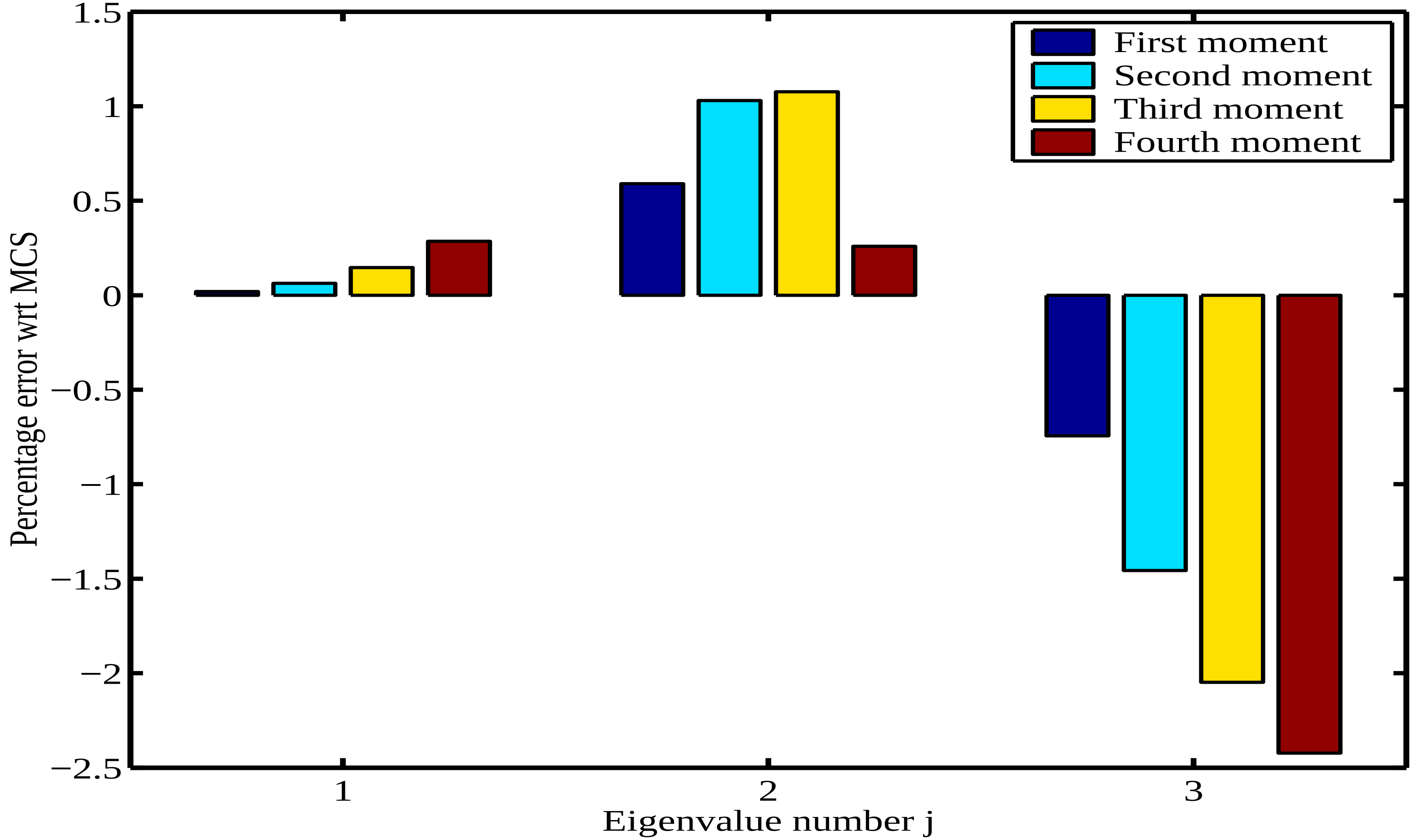
Case 2



Statistical scatter in the eigenvalues

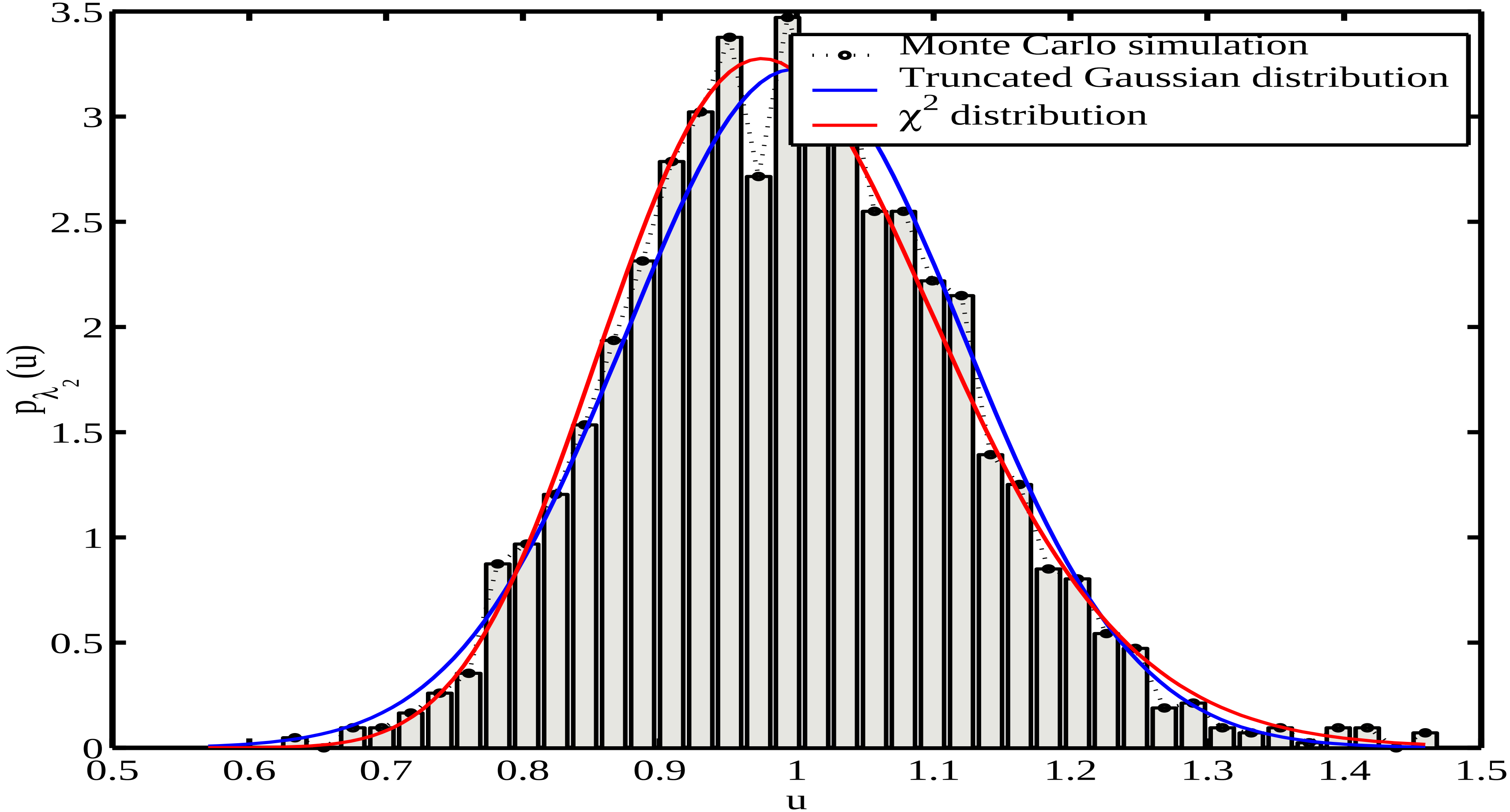
$\overline{\lambda}_1 = 1, \quad \overline{\lambda}_2 = 4,$ and $\overline{\lambda}_3 = 4.55$

Case 2



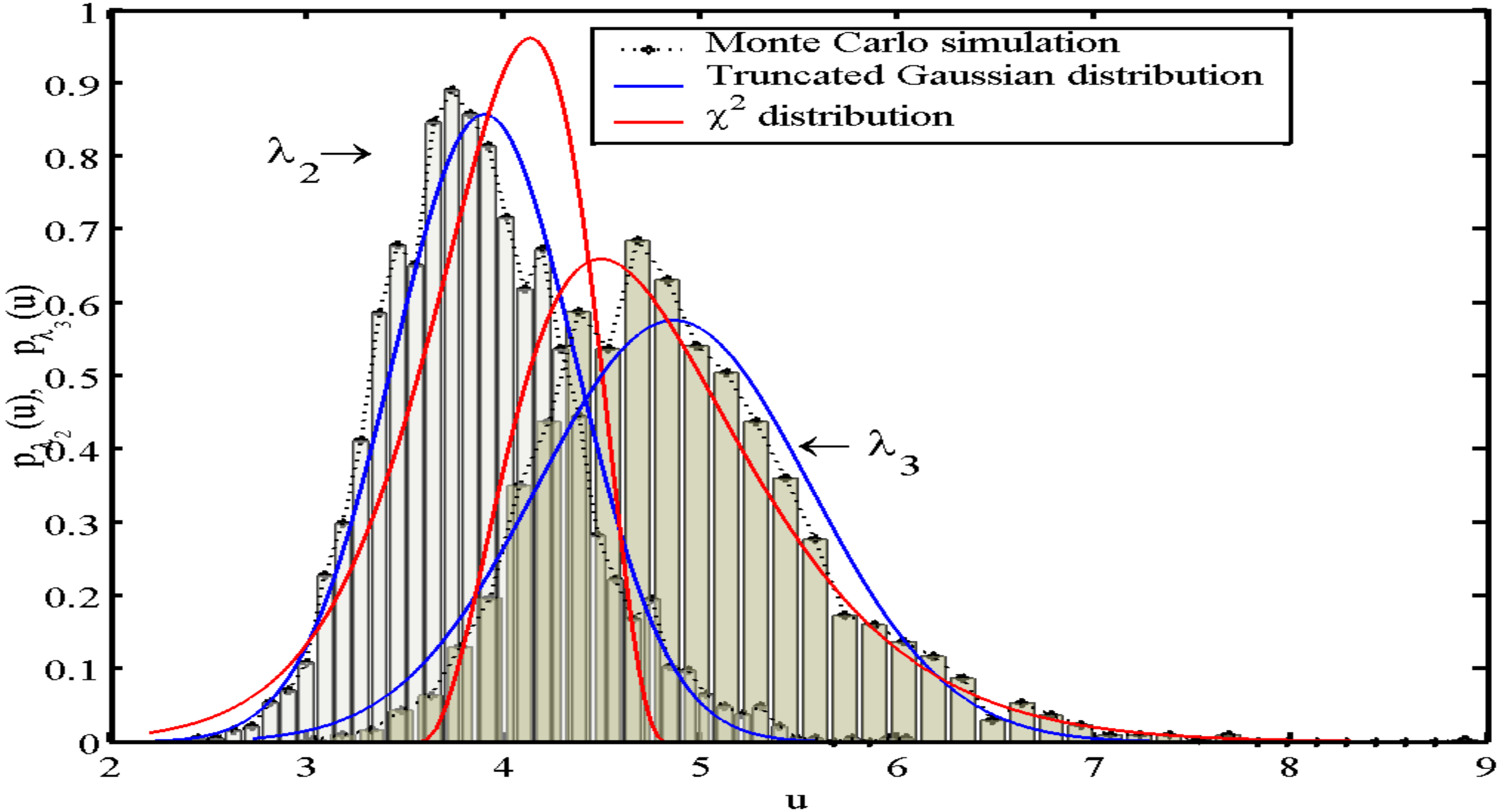
Percentage error for first four moments

Case 2



Pdf of the first eigenvalue

Case 2



Pdf of the second and third eigenvalues

Conclusions

- The statistics of the eigenvalues of linear stochastic dynamic systems has been considered
- A closed form expression is obtained for general order moments of the eigenvalues
- Pdf of the eigenvalues are obtained:
  - using maximum entropy method
  - in terms of central χ^2 density
- Proposed method works well when the eigenvalues are well separated

Open Problems

- Systems with closely-spaced/repeated eigenvalues
- Joint statistics (moments/pdf/cumulants) of the eigenvalues
- Joint statistics of the eigenvectors
- Joint statistics of the eigenvalues and eigenvectors
- Systems with non-proportional damping