Uncertainty Propagation Using Random Eigenfunction Expansion Method

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Outline of the talk

Stochastic Partial Differential Equations

We consider the stochastic elliptic partial differential equation (PDE)

$$\mathcal{L}_\theta\{a(\mathbf{r},\theta), u(\mathbf{r},\theta)\} = p(\mathbf{r}) \tag{1}$$

The stochastic operator $\mathcal{L}_\theta$ can be

- $\mathcal{L}_\theta \equiv \frac{\partial}{\partial x} AE(x,\theta)\frac{\partial}{\partial x}$ axial deformation of rods
- $\mathcal{L}_\theta \equiv \frac{\partial^2}{\partial x^2} EI(x,\theta)\frac{\partial^2}{\partial x^2}$ bending deformation of beams

Here $a : \mathbb{R}^d \times \Theta \to \mathbb{R}$ is a random field, which can be viewed as a set of random variables indexed by $\mathbf{r} \in \mathbb{R}^d$.

- We assume the random field $a(\mathbf{r},\theta)$ to be stationary and square integrable.
- Based on the physical problem, the random field $a(\mathbf{r},\theta)$ can be used to model different physical quantities (e.g., $AE(x,\theta)$, $EI(x,\theta)$).

Discretized stochastic PDE

- The random process $a(\mathbf{r}, \theta)$ can be expressed in a generalized fourier type of series known as the Karhunen-Loève expansion

$$a(\mathbf{r}, \theta) = a_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\nu_i} \xi_i(\theta) \varphi_i(\mathbf{r}) \tag{2}$$

- Here $a_0(\mathbf{r})$ is the mean function, $\xi_i(\theta)$ are uncorrelated standard Gaussian random variables, ν_i and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\mathcal{D}} C_a(\mathbf{r}_1, \mathbf{r}_2) \varphi_j(\mathbf{r}_1) \mathrm{d}\mathbf{r}_1 = \nu_j \varphi_j(\mathbf{r}_2), \quad \forall\ j = 1, 2, \cdots \tag{3}$$

Discrete equation for stochastic mechanics

- Truncating the KL expansion upto the M-th term and discretising the displacement field, the equation for static deformation can be expresses as

$$\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right] \mathbf{u}(\theta) = \mathbf{f} \tag{4}$$

- The aim is to efficiently solve for $\mathbf{u}(\theta)$.
- Note: Equation (4) can also be obtained with other considerations, such as random variable models.

Summary of solution techniques

- First- and second-order perturbation methods
- Neumann expansion method
- linear algebra based methods
- Spectral function approach
- General response surface based methods
  - Polynomial chaos (PC) expansion
  - High dimensional model representation (HDMR)
  - Gaussian process emulator (GPE)

Polynomial Chaos expansion

- Using the Polynomial Chaos expansion, the solution (a vector valued function) can be expressed as

$$\mathbf{u}(\theta) = \mathbf{u}_{i_0} h_0 + \sum_{i_1=1}^{\infty} \mathbf{u}_{i_1} h_1(\xi_{i_1}(\theta))$$
$$+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \mathbf{u}_{i_1,i_2} h_2(\xi_{i_1}(\theta), \xi_{i_2}(\theta))$$
$$+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \mathbf{u}_{i_1 i_2 i_3} h_3(\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta))$$
$$+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \sum_{i_4=1}^{i_3} \mathbf{u}_{i_1 i_2 i_3 i_4} \; h_4(\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta), \xi_{i_4}(\theta)) + \ldots,$$

Here $\mathbf{u}_{i_1,\ldots,i_p} \in \mathbb{R}^n$ are deterministic vectors to be determined.

Polynomial Chaos expansion

- After the finite truncation, concisely, the polynomial chaos expansion can be written as

$$\hat{\mathbf{u}}(\theta) = \sum_{k=1}^{P} H_k(\boldsymbol{\xi}(\theta))\mathbf{u}_k \tag{5}$$

  where $H_k(\boldsymbol{\xi}(\theta))$ are the polynomial chaoses.
- The value of the number of terms P depends on the number of basic random variables M and the order of the PC expansion r as

$$P = \sum_{j=0}^{r} \frac{(M+j-1)!}{j!(M-1)!} \tag{6}$$

Polynomial Chaos expansion

We need to solve a $nP \times nP$ linear equation to obtain all $\mathbf{u}_k \in \mathbb{R}^n$.

$$\begin{bmatrix} \mathbf{A}_{0,0} & \cdots & \mathbf{A}_{0,P-1} \\ \mathbf{A}_{1,0} & \cdots & \mathbf{A}_{1,P-1} \\ \vdots & \vdots & \vdots \\ \mathbf{A}_{P-1,0} & \cdots & \mathbf{A}_{P-1,P-1} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{P-1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{P-1} \end{Bmatrix} \tag{7}$$

P increases exponentially with M:

M	2	3	5	10	20	50	100
2nd order PC	5	9	20	65	230	1325	5150
3rd order PC	9	19	55	285	1770	23425	176850

Mathematical nature of the solution

- The elements of the solution vector are not simple polynomials, but ratio of polynomials in $\boldsymbol{\xi}(\theta)$.

Remark

If all $\mathbf{A}_i \in \mathbb{R}^{n\times n}$ *are matrices of rank n, then the elements of* $\mathbf{u}(\theta)$ *are the ratio of polynomials of the form*

$$\frac{p^{(n-1)}(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta))}{p^{(n)}(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta))} \tag{8}$$

where $p^{(n)}(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta))$ *is an n-th order complete multivariate polynomial of variables* $\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta)$.

Mathematical nature of the solution

Suppose we denote

$$\mathbf{A}(\theta) = \left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right] \in \mathbb{R}^{n\times n} \tag{9}$$

so that

$$\mathbf{u}(\theta) = \mathbf{A}^{-1}(\theta)\mathbf{f} \tag{10}$$

From the definition of the matrix inverse we have

$$\mathbf{A}^{-1} = \frac{\mathrm{Adj}(\mathbf{A})}{\det(\mathbf{A})} = \frac{\mathbf{C}_a^T}{\det(\mathbf{A})} \tag{11}$$

where $\mathbf{C}_a$ is the matrix of cofactors. The determinant of $\mathbf{A}$ contains a maximum of n number of products of A_{kj} and their linear combinations. Note from Eq. (9) that

$$A_{kj}(\theta) = A_{0_{kj}} + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_{i_{kj}} \tag{12}$$

Mathematical nature of the solution

- Since all the matrices are of full rank, the determinant contains a maximum of n number of products of linear combination of random variables in Eq. (12). On the other hand, each entries of the matrix of cofactors, contains a maximum of $(n-1)$ number of products of linear combination of random variables in Eq. (12). From Eqs. (10) and (11) it follows that

$$\mathbf{u}(\theta) = \frac{\mathbf{C}_a^T \mathbf{f}}{\det(\mathbf{A})} \tag{13}$$

  Therefore, the numerator of each element of the solution vector contains linear combinations of the elements of the cofactor matrix, which are complete polynomials of order $(n-1)$.
- *All the solution methods proposed for stochastic finite element analysis essentially aim to approximate the ratio of the polynomials given in Eq. (8).*

Possibilities of solution types

- We are looking to solve the following equation for $\mathbf{u}(\theta)$

$$\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right]\mathbf{u}(\theta) = \mathbf{f} \tag{14}$$

- Some possibilities include

$$\begin{aligned}\mathbf{u}(\theta) &= \sum_{k=1}^{P_1} H_k(\boldsymbol{\xi}(\theta))\mathbf{u}_k \\ \text{or} \quad &= \sum_{k=1}^{P_2} \Gamma_k(\boldsymbol{\xi}(\theta))\phi_k(\boldsymbol{\xi}(\theta)) \qquad (15) \\ \text{or} \quad &= \sum_{k=1}^{P_3} a_k \mathbf{U}_k(\boldsymbol{\xi}(\theta)) \quad \ldots \text{ etc.}\end{aligned}$$

Motivation behind the proposed approach

- In this paper we proposed the idea where both the coefficient vectors and the associated function are random. This implies that we are looking for a solution of the generic form

$$\mathbf{u}(\omega) = \sum_{k=1}^{P} \Gamma_k(\omega)\phi_k(\omega) \tag{16}$$

  for some scaler functions $\Gamma_k(\omega)$ and vectors $\phi_k(\omega)$.

- The random symmetric matrix appearing in the discretized stochastic finite element equation (14)

$$\mathbf{A}(\omega) = \mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\omega)\mathbf{A}_i \tag{17}$$

Random eigen function expansion

- The random eigenvalue problem associated with this matrix can be defined as

$$\mathbf{A}(\omega)\phi_k(\omega) = \lambda_k(\omega)\phi_k(\omega); \quad k = 1, 2, \ldots n \tag{18}$$

- We assume that the eigenvalues are distinct so that ϕ_k for $k = 1, 2, \ldots n$ forms a complete orthonormal basis. For notational convenience, define the matrix of eigenvalues and eigenvectors

$$\mathbf{\Lambda} = \text{diag}\,[\lambda_1, \lambda_2, \ldots, \lambda_n] \in \mathbb{R}^{n\times n} \quad \text{and} \quad \mathbf{\Phi} = [\phi_1, \phi_2, \ldots, \phi_n] \in \mathbb{R}^{n\times n} \tag{19}$$

Eigenvalues are ordered in the ascending order so that $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ according to a chosen norm.

Random eigen function expansion

- Since $\mathbf{\Phi}$ is an orthogonal matrix we have $\mathbf{\Phi}^{-1} = \mathbf{\Phi}^T$ so that the following identities involving the random matrices can be established

$$\mathbf{\Phi}(\omega)^T \mathbf{A}(\omega) \mathbf{\Phi}(\omega) = \mathbf{\Lambda}(\omega) \tag{20}$$

$$\mathbf{A}(\omega) = \mathbf{\Phi}(\omega)^{-T} \mathbf{\Lambda}(\omega) \mathbf{\Phi}(\omega)^{-1} \tag{21}$$

$$\text{and} \quad \mathbf{A}(\omega)^{-1} = \mathbf{\Phi}(\omega) \mathbf{\Lambda}(\omega)^{-1} \mathbf{\Phi}(\omega)^T \tag{22}$$

- The solution vector is given by

$$\mathbf{u}(\omega) = \left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\omega) \mathbf{A}_i \right]^{-1} \mathbf{f} \tag{23}$$

Random eigen function expansion

- Using the modal identities and considering that the matrix of eigenvalues is a diagonal matrix, we have

$$\mathbf{u}(\omega) = \mathbf{A}^{-1}(\omega)\mathbf{f} = \left[\mathbf{\Phi}(\omega)\mathbf{\Lambda}^{-1}(\omega)\mathbf{\Phi}(\omega)^T\right]\mathbf{f} = \sum_{k=1}^{n} \underbrace{\left[\frac{\phi_k^T(\omega)\mathbf{f}}{\lambda_k(\omega)}\right]}_{\Gamma_k(\omega)} \phi_k(\omega) \tag{24}$$

- This expression is 'exact' in the sense of probability 1. Comparing this with the sought form proposed in Eq. (16) we obtain that the random scalar functions are

$$\Gamma_k(\omega) = \frac{\phi_k^T(\omega)\mathbf{f}}{\lambda_k(\omega)}, \quad k = 1, 2, \cdots, n \tag{25}$$

and the vectors $\phi_k(\omega)$ are the random eigenvectors corresponding to the underlying random matrix $\mathbf{A}(\omega)$.

Some observations

- The eigenvalues increase in magnitude with increasing k and they appear in the denominator. Therefore the series can be truncated after a small number of terms, say $r < n$.
- Although the coefficient functions and the associated vectors are identified uniquely, the main difficulty in this approach is the solution of the random eigenvalue problem.
- In order to use the solution given here in a efficient manner, it is necessary to calculate the random eigensolutions.

The random eigenvalue problem

- The random eigenvalue problem is a research area in its own
- Several papers exist on this topic
- A number of techniques, such as, perturbation method, polynomial chaos, asymptotic method, reduced basis approach, random polynomials, hybrid perturbation-polynomial chaos method have been proposed
- We use the first-order methods for simplicity - but any existing techniques can be used

Random eigenvalues

- The eigenvalue problem corresponding to the deterministic stiffness matrix is given by

$$\mathbf{A}_0 \phi_{0_k} = \lambda_{0_k} \phi_{0_k}; \quad k = 1, 2, \ldots n \tag{26}$$

  Here λ_{0_k} and ϕ_{0_k} are the eigenvalues and eigenvectors corresponding the to deterministic system.

- The first order perturbation of the k-th eigenvalue is given by

$$\lambda_k = \lambda_{0_k} + \sum_{i=1}^{M} \xi_i \frac{\partial \lambda_k}{\partial \xi_i} \quad \text{where} \quad \frac{\partial \lambda_k}{\partial \xi_i} = \phi_{0_k}^T \frac{\partial \mathbf{A}}{\partial \xi_i} \phi_{0_k} \tag{27}$$

- For our **A** matrix, $\partial \mathbf{A} / \partial \xi_i = \mathbf{A}_i$

Random eigenvectors

- The expression for the first-order perturbation of the eigenvector can be given by

$$\phi_k = \phi_{0_k} + \sum_{i=1}^{M} \xi_i \frac{\partial \phi_k}{\partial \xi_i} \tag{28}$$

- The deterministic eigenvectors satisfy the following properties

$$\phi_{0_k}^T \phi_{0_k} = 1 \qquad \text{and} \qquad \phi_{0_k}^T \frac{\partial \phi_k}{\partial \xi_i} = 0 \tag{29}$$

- Different methods have been developed to calculate the derivatives of the eigenvectors. One of these methods expands the derivative of eigenvectors as a linear combination of deterministic eigenvectors

$$\frac{\partial \phi_k}{\partial \xi_i} = \sum_{m=1,\neq k}^{n} \alpha_{kim} \phi_{m0} \quad \text{where} \quad \alpha_{kim} = \frac{1}{\lambda_{0_k} - \lambda_{0_m}} \phi_{0_m}^T \frac{\partial \mathbf{A}}{\partial \xi_i} \phi_{0_k} \tag{30}$$

Random eigenvalues and eigenvectors

- Different expression need to be used when one more eigenvalues are repeated - we don't consider that case here.
- Summarising, random eigenvalues and eigenvectors can be explicitly expressed in closed-form as

$$\lambda_k(\omega) = \lambda_{0_k} + \sum_{i=1}^{M} \xi_i(\omega) \phi_{0_k}^T \mathbf{A}_i \phi_{0_k} \tag{31}$$

$$\phi_k(\omega) = \phi_{0_k} + \sum_{i=1}^{M} \xi_i(\omega) \sum_{m=1,\neq k}^{n} \left[\frac{\phi_{0_m}^T \mathbf{A}_i \phi_{0_k}}{\lambda_{0_k} - \lambda_{0_m}} \right] \phi_{0_m} \tag{32}$$

- Since these expressions are based on the first-order perturbation, they lead to errors in the eigenvalues and the eigenvectors. Consequently, when they are used, the proposed series expression becomes erroneous.
- We use thel Galerkin approach to address this error.

The Galerkin approach

- We consider the inner product norm $\langle \mathbf{u}(\omega), \mathbf{v}(\omega) \rangle = \int_\Omega P(\mathrm{d}\omega) \mathbf{u}^T(\omega)\mathbf{v}(\omega)$.
- Tthe approximate solution is expressed by the reduced series expansion

$$\hat{\mathbf{u}}(\omega) = \sum_{k=1}^{r} \left\{ c_k + \frac{\phi_k^T(\omega)\mathbf{f}}{\lambda_k(\omega)} \right\} \phi_k(\omega) \tag{33}$$

- The unknown constants c_k need to be obtained such that the error in series expansion in minimised according to our selected L_2 norm.
- The number of terms $r < n$ can be selected based on the spectral properties of the matrix $\mathbf{A}(\omega)$.

The Galerkin approach

- Substituting the approximate expression of in the governing equation, the error vector can be obtained as

$$\varepsilon(\omega) = \mathbf{A}(\omega) \sum_{k=1}^{r} \left\{ c_k + \frac{\phi_k^T(\omega)\mathbf{f}}{\lambda_k(\omega)} \right\} \phi_k(\omega) - \mathbf{f} \in \mathbb{R}^n \tag{34}$$

- We wish to obtain the coefficients c_k such that the error norm $\chi^2 = \langle \varepsilon(\omega), \varepsilon(\omega) \rangle$ is minimum. This can be achieved using the Galerkin approach so that the error is made orthogonal to the basis functions, that is, mathematically

$$\varepsilon(\omega) \perp \phi_j(\omega) \quad \text{or} \quad \left\langle \phi_j(\omega), \varepsilon(\omega) \right\rangle = 0 \quad \forall\, j = 1, 2, \ldots, r \tag{35}$$

- Imposing this condition and using the expression of $\varepsilon(\omega)$

$$\mathrm{E}\left[\phi_j^T(\omega)\mathbf{A}(\omega) \sum_{k=1}^{r} \left\{ c_k + \frac{\phi_k^T(\omega)\mathbf{f}}{\lambda_k(\omega)} \right\} \phi_k(\omega) - \phi_j^T(\omega)\mathbf{f} \right] = 0, \ \forall\, j \tag{36}$$

The Galerkin approach

- Interchanging the $\mathrm{E}[\bullet]$ and summation operations, and after some algebra the coefficients c_j can be obtained in closed-form as

$$c_k = \frac{1}{\mathrm{E}[\lambda_k(\omega)]} \left\{ \mathrm{E}\left[\phi_k^T(\omega)\mathbf{f}\right] - \sum_{m=1}^{r} \mathrm{E}\left[\frac{\left(\phi_k^T(\omega)\mathbf{A}(\omega)\phi_m(\omega)\right)\left(\phi_m^T(\omega)\mathbf{f}\right)}{\lambda_m(\omega)} \right] \right\}$$

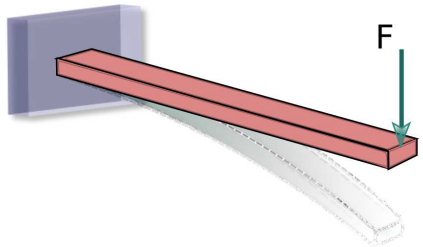
- Due to the nonlinearity, the expectation terms cannot be computed analytically even with first-order perturbation for the eigensolutions.
- A Monte Carlo simulation based approach is used to obtain these constants.
- Only a reduced number of constants $r < n$ need to be evaluated. Explicit closed-form expressions of eigenvalues and eigenvectors can be used to compute these constants.
- The sampling necessary to compute the constants can be coarser compared to sampling necessary for the series solution.

Summary of the proposed method

- The discretised equation: $\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\omega)\mathbf{A}_i\right]\mathbf{u}(\omega) = \mathbf{f}$
- The solution: $\mathbf{u}(\omega) = \sum_{k=1}^{r}\left\{c_k + \frac{\phi_k^T(\omega)\mathbf{f}}{\lambda_k(\omega)}\right\}\phi_k(\omega)$
- The random eigenvalues $\lambda_k(\omega) = \lambda_{0_k} + \sum_{i=1}^{M}\xi_i(\omega)\phi_{0_k}^T\mathbf{A}_i\phi_{0_k}$ and eigenvectors $\phi_k(\omega) = \phi_{0_k} + \sum_{i=1}^{M}\xi_i(\omega)\sum_{m=1,\neq k}^{n}\left[\frac{\phi_{0_m}^T\mathbf{A}_i\phi_{0_k}}{\lambda_{0_k}-\lambda_{0_m}}\right]\phi_{0_m}$
- The deterministic eigensolutions $\mathbf{A}_0\phi_{0_k} = \lambda_{0_k}\phi_{0_k}; \quad k = 1, 2, \ldots r.$ Choose r so that $\lambda_{0_1}/\lambda_{0_r}$ is sufficiently small.
- The constants for the Galerkin error minimisation: $c_k = \frac{1}{\mathrm{E}[\lambda_k(\omega)]}\left\{\mathrm{E}\left[\phi_k^T(\omega)\mathbf{f}\right] - \sum_{m=1}^{r}\mathrm{E}\left[\frac{\left(\phi_k^T(\omega)\mathbf{A}(\omega)\phi_m(\omega)\right)\left(\phi_m^T(\omega)\mathbf{f}\right)}{\lambda_m(\omega)}\right]\right\}$

The Euler-Bernoulli beam example

- An Euler-Bernoulli cantilever beam with stochastic bending modulus



- Length : 1.0 m, Cross-section : 30×5 mm^2, Young's Modulus: 69×10^9 Pa.
- We study the deflection of the beam under the action of a point load on the free end.

Problem details

- We assume that the bending modulus of the cantilever beam is a homogeneous stationary Gaussian random field of the form

$$EI(x,\theta) = EI_0(1 + a(x,\theta)) \tag{37}$$

where x is the coordinate along the length of the beam, EI_0 is the estimate of the mean bending modulus, $a(x,\theta)$ is a zero mean stationary random field.

- The autocorrelation function of this random field is assumed to be

$$C_a(x_1,x_2) = \sigma_a^2 e^{-(|x_1-x_2|)/\mu_a} \tag{38}$$

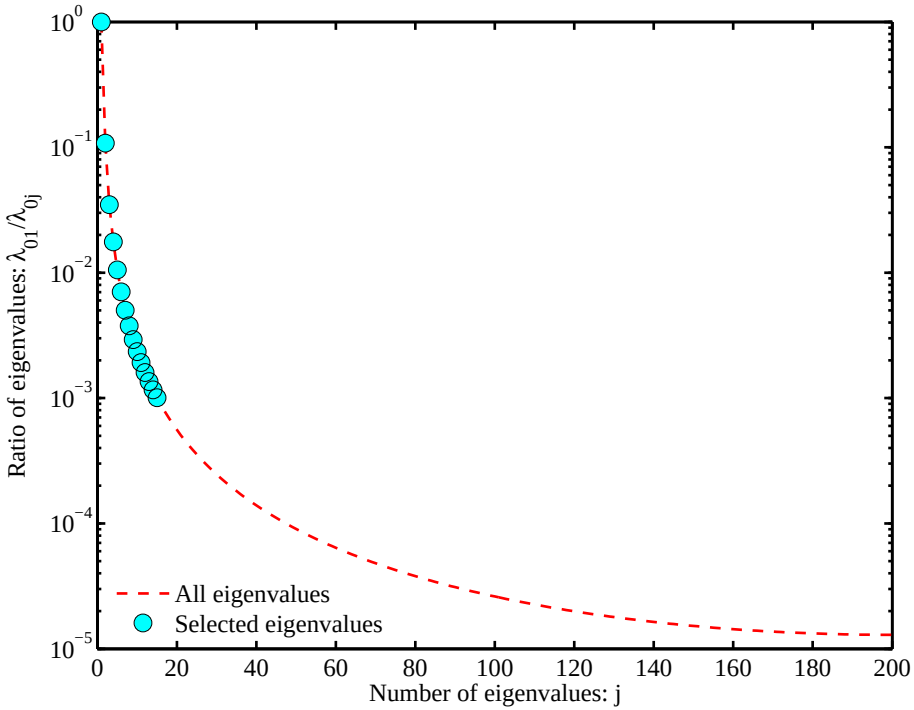
where μ_a is the correlation length and σ_a is the standard deviation.

- Two correlation lengths are considered in the numerical studies: $\mu_a = L/2$ and $\mu_a = L/5$.

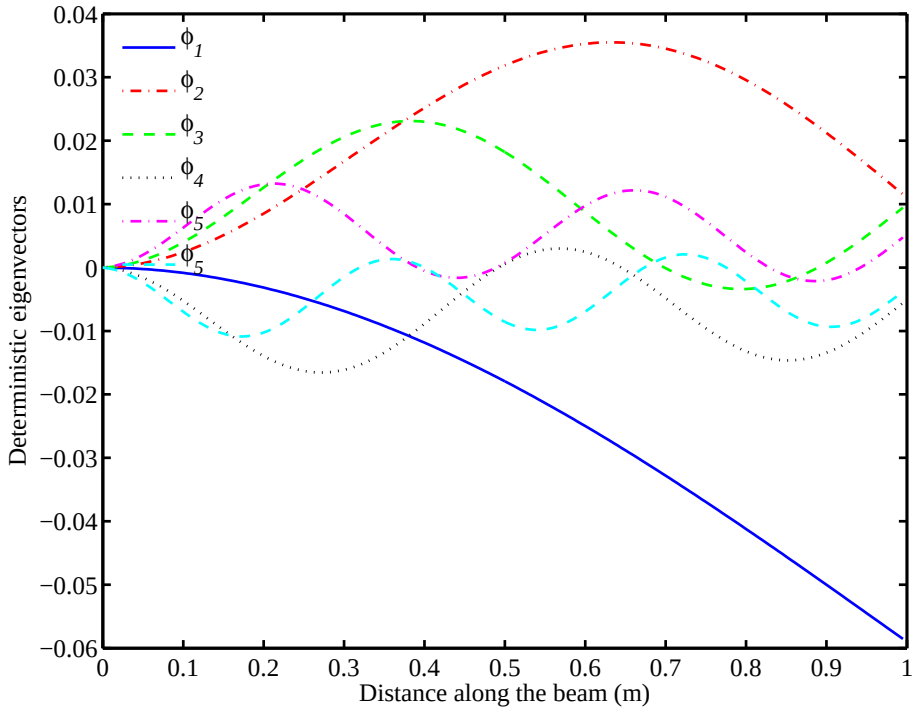
Problem details

- *Case 1:* The random field $a(x, \omega)$ is assumed to be a Gaussian random field with correlation length $\mu_a = L/2$. Four terms in the KL expansion are considered and the beam is divided into 200 elements. For this case we have $n = 400$ and $M = 4$. The results are compared with the different orders of polynomial chaos expansions and direct Monte Carlo simulation.
- *Case 2:* The random field $a(x, \omega)$ is assumed to be an Gaussian random field with correlation length $\mu_a = L/5$. The beam is divided into 200 elements and 14 random variables are considered in the discretisation of the random field. The value of M is selected such that $\nu_M/\nu_1 = 0.03$. For this case we have $n = 400$ and $M = 14$. The results are compared with direct Monte Carlo simulation.

Eigensolutions of the beam



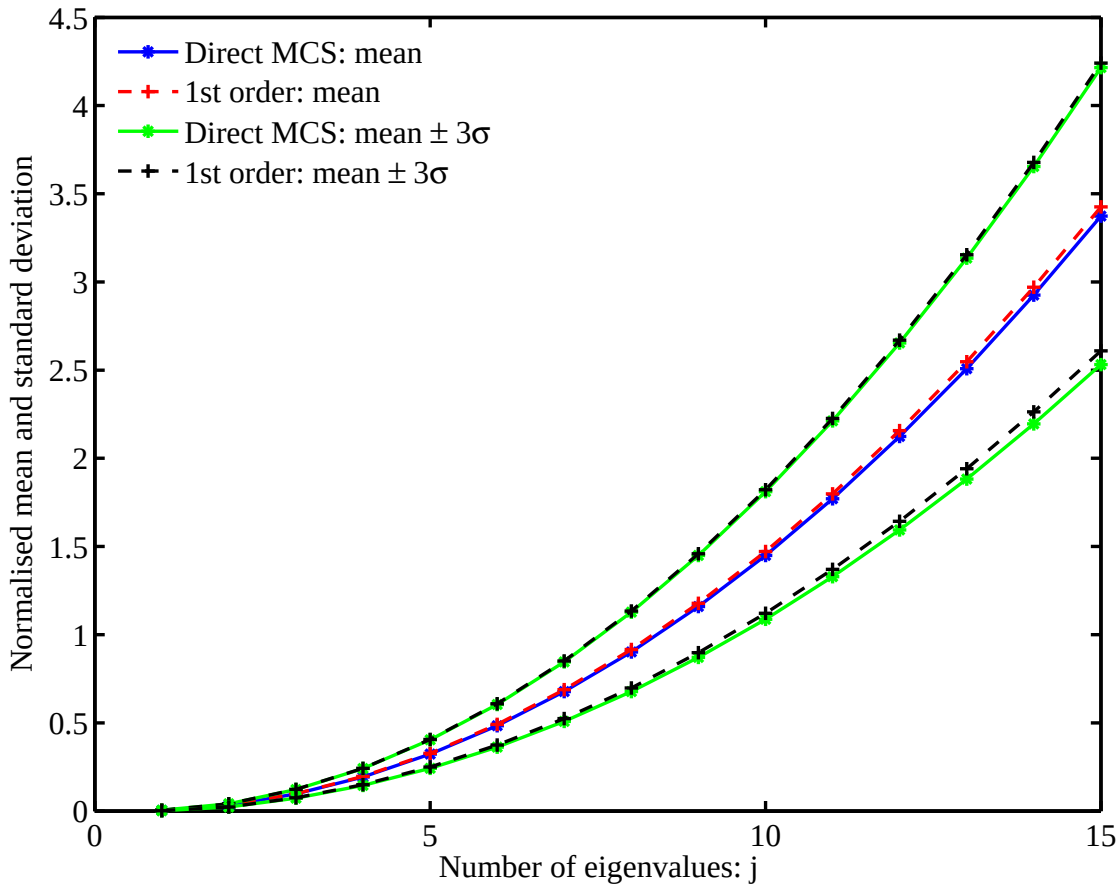
(a) Ratio of eigenvalues of the stiffness matrix $\mathbf{A}_0$.



(b) First six eigenvectors of $\mathbf{A}_0$ (transverse DOF).

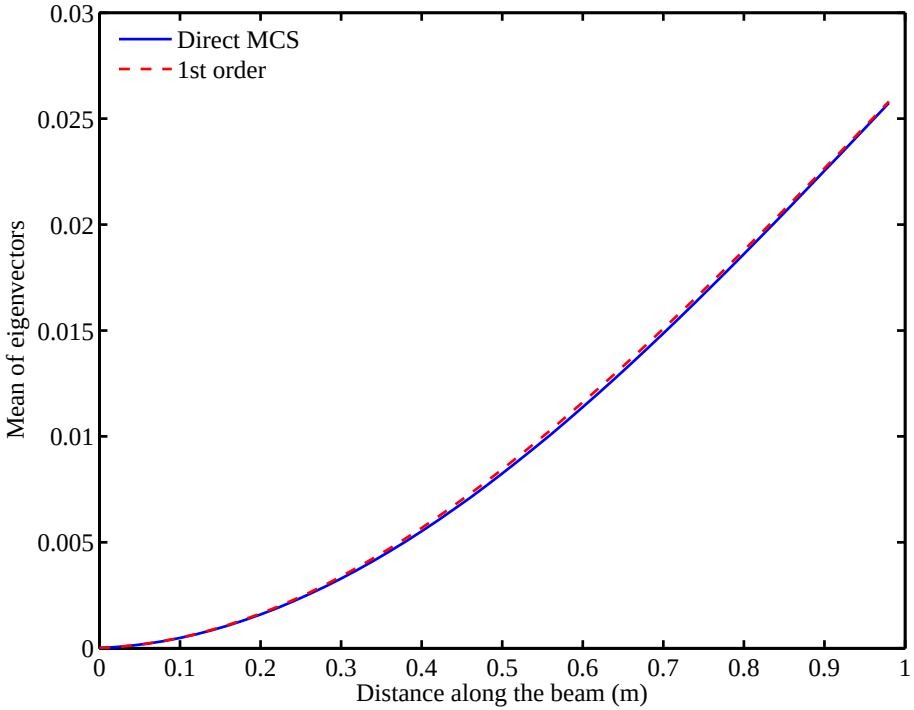
Figure : The eigenvalues and eigenvectors of the stiffness matrix $\mathbf{A}_0$. For $r = 15$, the ration eigenvalues $\lambda_{0_1}/\lambda_{0_r} < 10^{-3}$.

Moments of the eigenvalues

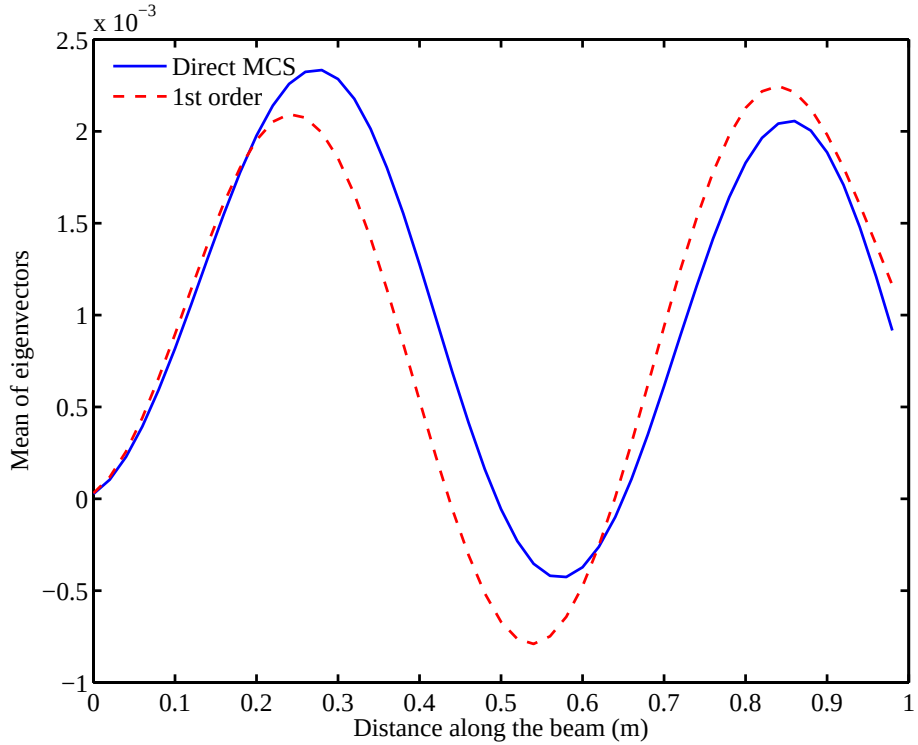


The mean and standard deviation bounds of the first 15 eigenvalues obtained from the first-order perturbation and direct Monte Carlo simulation for $\sigma_a = 0.2$.

Mean of the eigenvectors



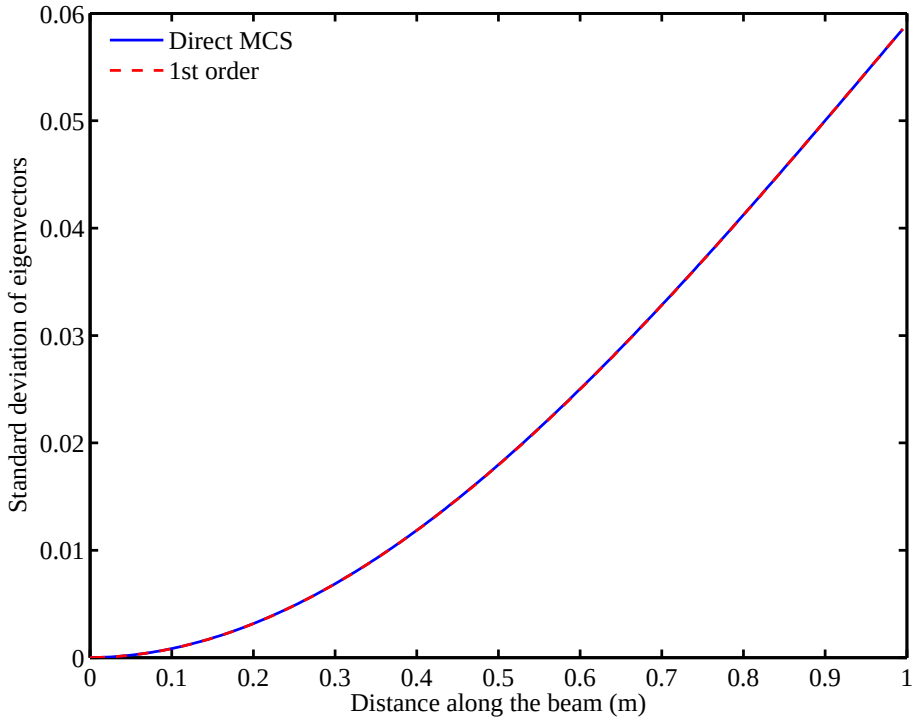
(a) Mean of the first eigenvector.



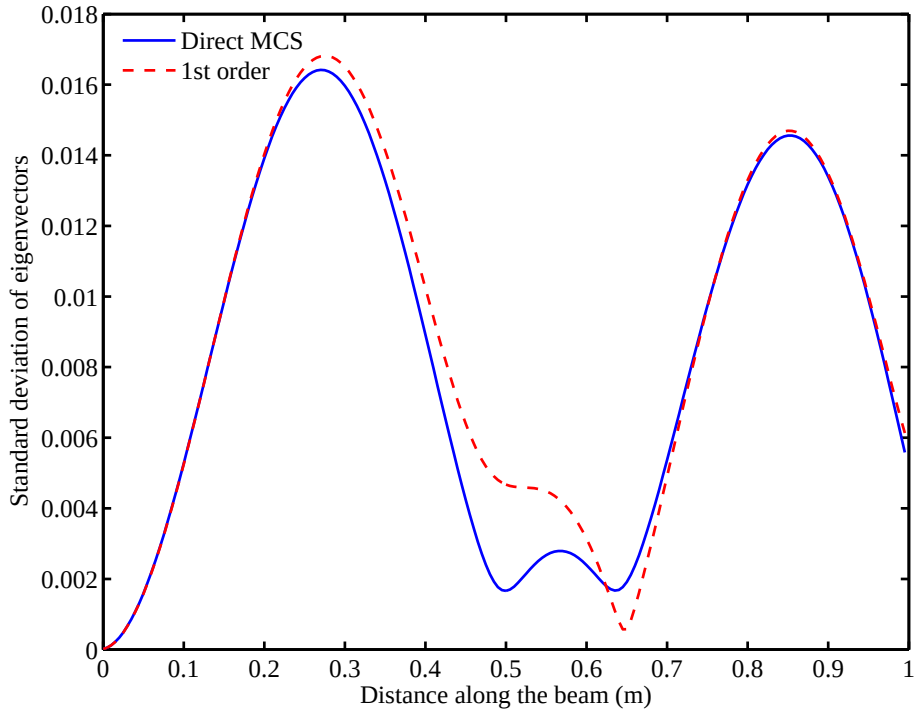
(b) Mean of the fourth eigenvector.

The mean of the eigenvectors (transverse DOF) obtained from the first-order perturbation and direct Monte Carlo simulation, $\sigma_a = 0.2$.

Standard deviation of the eigenvectors



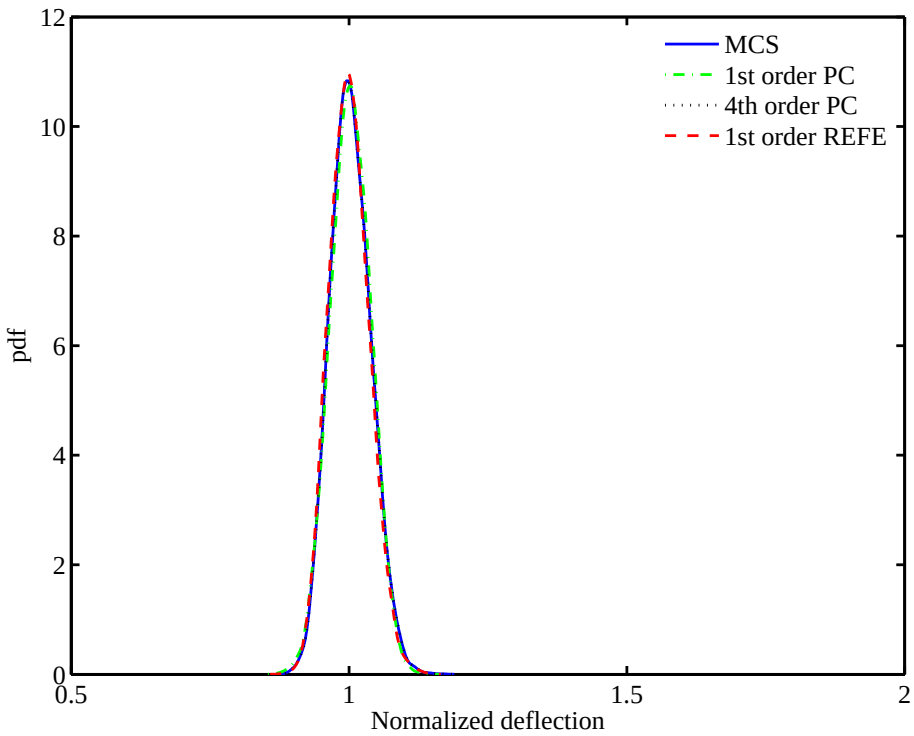
(c) Standard deviation of the first eigenvector.



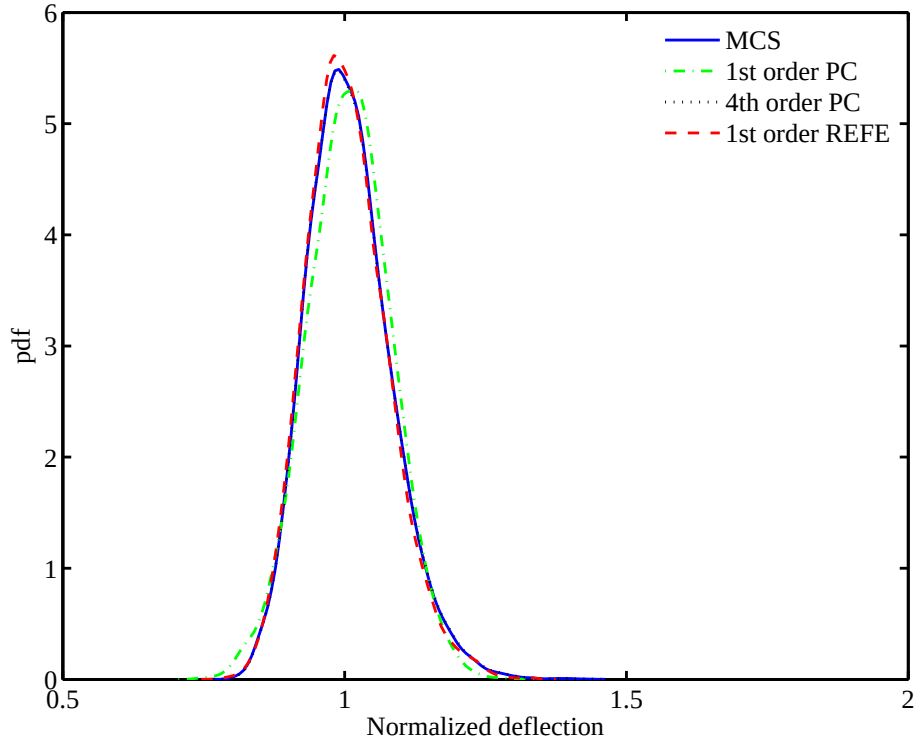
(d) Standard deviation of the fourth eigenvector.

The standard deviation of the eigenvectors (transverse DOF) obtained from the first-order perturbation and direct Monte Carlo simulation, $\sigma_a = 0.2$.

Probability density function of the response



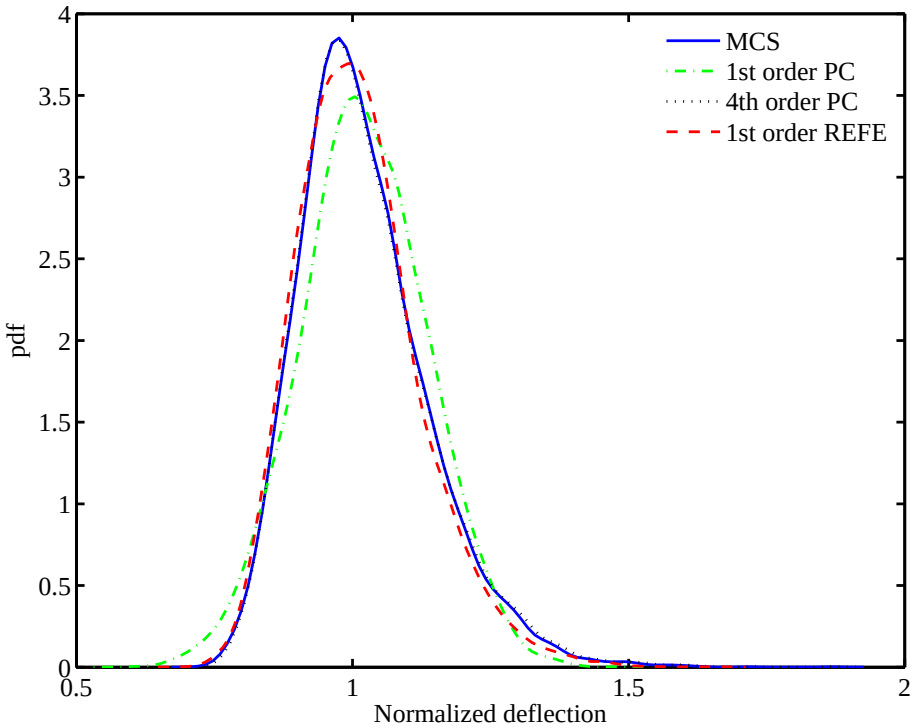
(e) Probability density function for $\sigma_a = 0.05$.



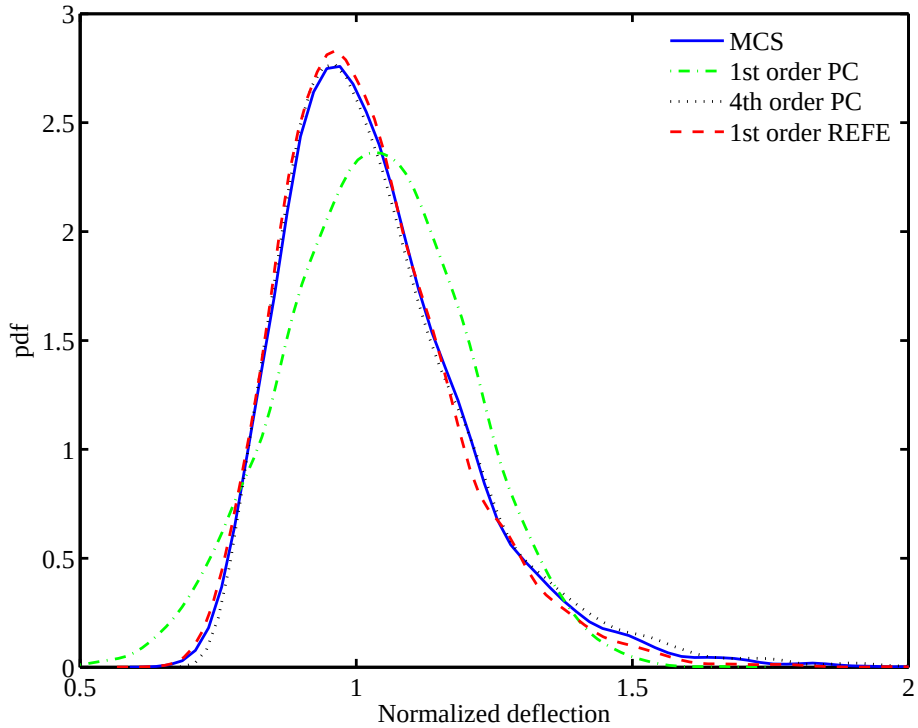
(f) Probability density function for $\sigma_a = 0.1$.

The probability density function of the normalized tip deflection of the cantilever beam under the action of a point load at the free end (10,000 sample MCS). First 15 modes are used in the calculation.

Probability density function of the response



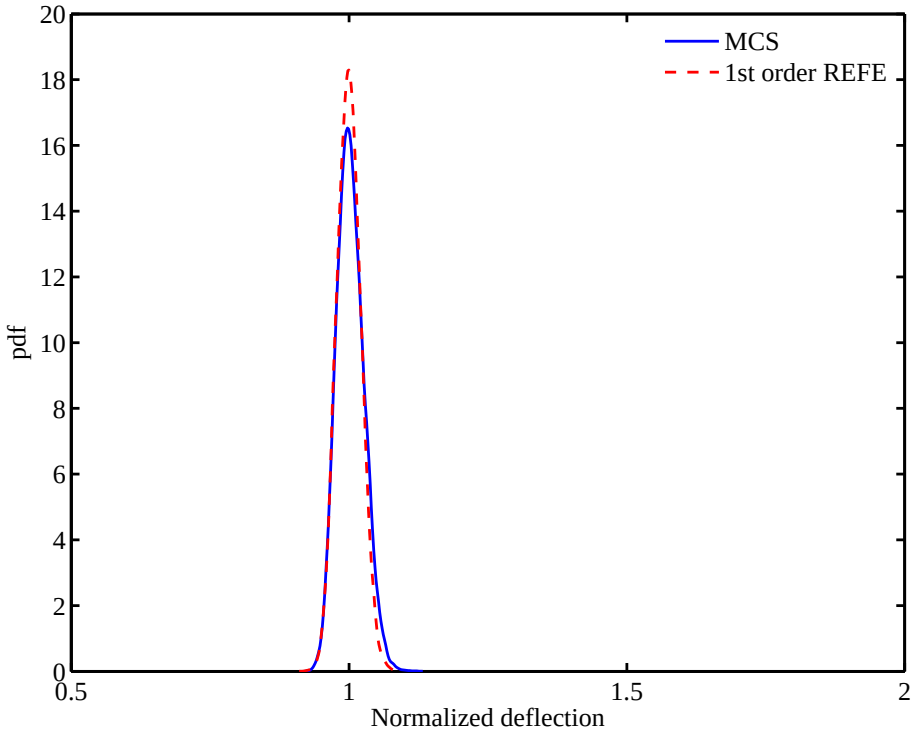
(g) Probability density function for $\sigma_a = 0.15$.



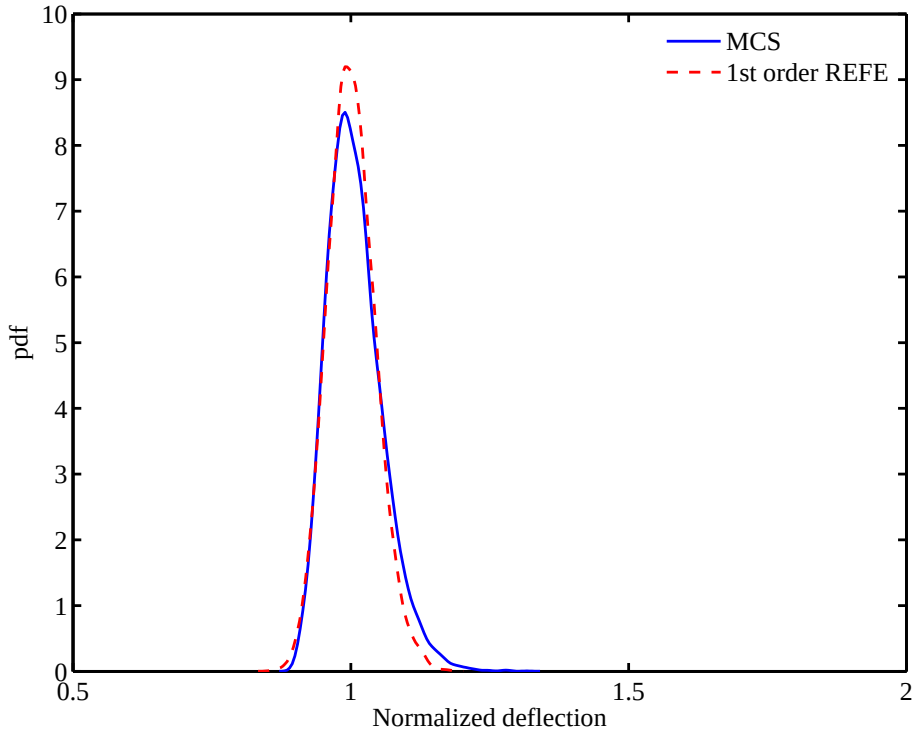
(h) Probability density function for $\sigma_a = 0.2$.

For $n = 400$ and $M = 4$, the fourth-order PC needs solution of a linear system of equation of size 28,000. Direct MCS: 141s; PC: 34s; REFE: 5s.

Probability density function of the response



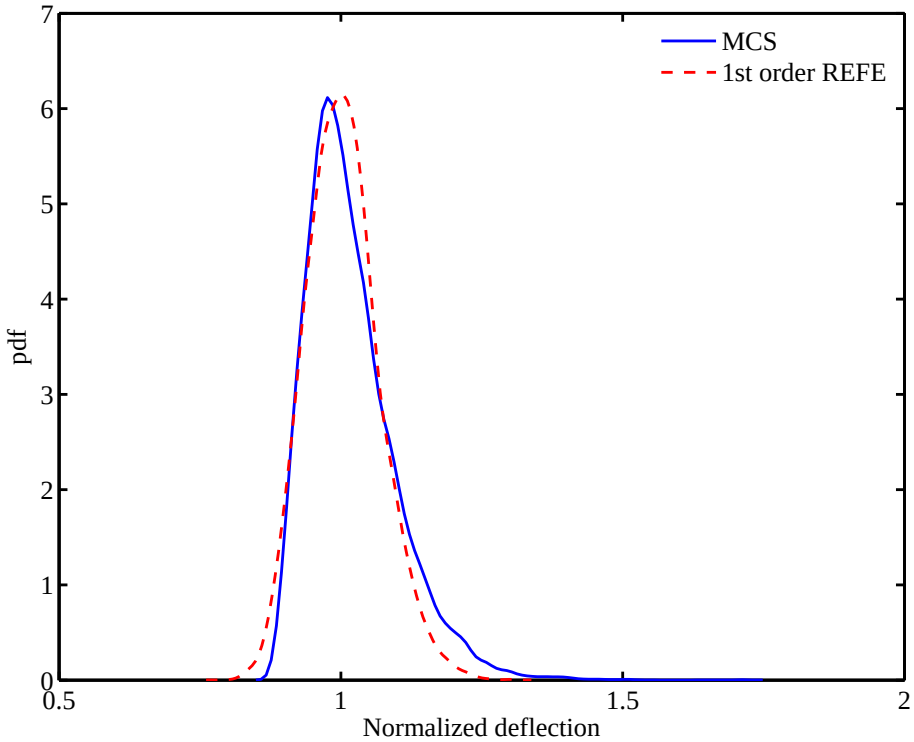
(i) Probability density function for $\sigma_a = 0.05$.



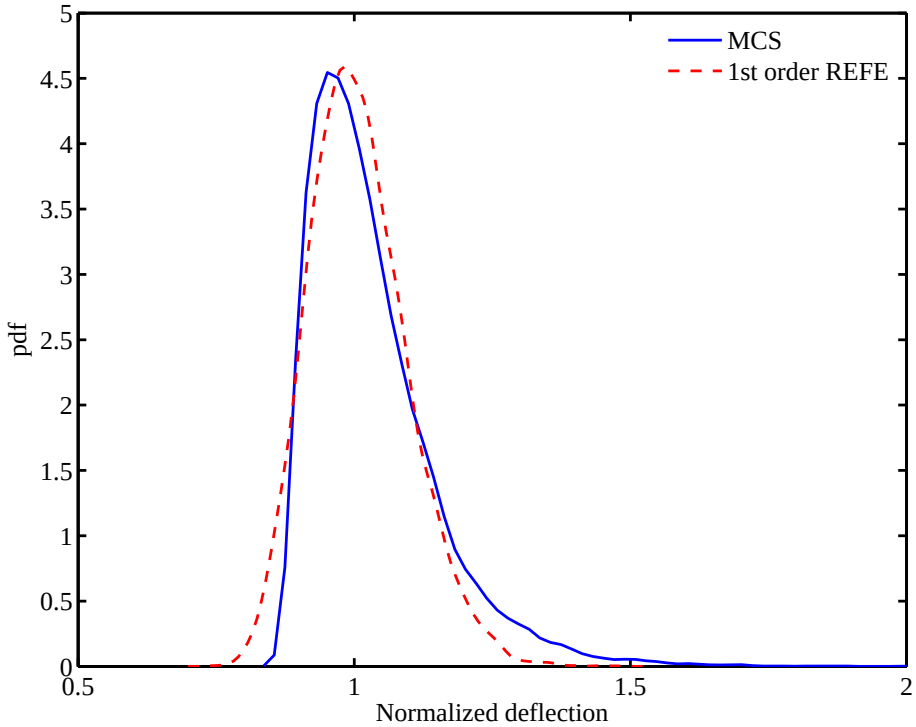
(j) Probability density function for $\sigma_a = 0.1$.

The probability density function of the normalized tip deflection of the cantilever beam under the action of a point load at the free end (10,000 sample MCS). First 15 modes are used in the calculation.

Probability density function of the response



(k) Probability density function for $\sigma_a = 0.15$.



(l) Probability density function for $\sigma_a = 0.2$.

For $n = 400$ and $M = 14$, the fourth-order PC would need the solution of a linear system of equation of size 1,224,000. Direct MCS: 2751s; REFE: 10s.

Conclusions

1. We consider discretized stochastic partial differential equations.
2. In the classical spectral stochastic finite element approach, the solution is projected into an infinite dimensional orthonormal basis functions and the associated constant vectors are obtained using the Galerkin type of error minimization approach
3. Here the solution is projected into a finite dimensional random vector basis and the associated coefficient functions are obtained in terms of the random eigenvalues and eigenvectors
4. A Galerkin error minimisation approach is proposed to obtain unknown constants which are used to minimise error arising due to approximate calculation of the eigensolutions.
5. Exact closed-form expression of these unknown constants are derived.

Conclusions

- The possibility of a reduced series expansion is discussed using the idea of conventional modal truncation.
- Encouraging agreements have been observed between the proposed reduced approach and direct Monte Carlo simulation
- Currently we are extending this idea to stochastic dynamic systems.

Discussions

- The proposed method takes advantage of the difference in the magnitude of the eigenvalues of the $\mathbf{A}_0$. This POD-like model reduction leads to a significantly smaller basis. This type of reduction is difficult to incorporate within the scope of PC as no information regarding the system matrices are used in constructing the orthogonal polynomial basis.