Distribution of Eigenvalues of Linear Stochastic Systems

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Outline of the talk

- Random eigenvalue problem
- Perturbation Methods
  - Mean-centered perturbation method
  - α-centered perturbation method
- Asymptotic analysis
- PDF of the eigenvalues
- Numerical Example
- Conclusions & Open Problems

Random eigenvalue problem

The random eigenvalue problem of undamped or proportionally damped linear systems:

$$\mathbf{K}(\mathbf{x})\phi_j = \lambda_j \mathbf{M}(\mathbf{x})\phi_j \qquad (1)$$

λ_j eigenvalues; ϕ_j eigenvectors; $\mathbf{M}(\mathbf{x}) \in \mathbb{R}^{N\times N}$ mass matrix and $\mathbf{K}(\mathbf{x}) \in \mathbb{R}^{N\times N}$ stiffness matrix. $\mathbf{x} \in \mathbb{R}^m$ is random parameter vector with pdf

$$p(\mathbf{x}) = (2\pi)^{-m/2} e^{-\mathbf{x}^T\mathbf{x}/2} \qquad (2)$$

The fundamental aim

- To obtain the joint probability density function of the eigenvalues and the eigenvectors.
- If the matrix $\mathbf{M}^{-1}\mathbf{K}$ is GUE (Gaussian unitary ensemble) or GOE (Gaussian orthogonal ensemble) an exact closed-form expression can be obtained for the joint pdf of the eigenvalues.
- In general the system matrices for real structures are not GUE or GOE.

Mean-centered perturbation

Assume that $\mathbf{M}(\mathbf{0}) = \mathbf{M}_0$ and $\mathbf{K}(\mathbf{0}) = \mathbf{K}_0$ are 'deterministic parts'. Deterministic eigenvalue problem: $\mathbf{K}_0\phi_{j0} = \lambda_{j_0}\mathbf{M}_0\phi_{j_0}$. The eigenvalues $\lambda_j(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}$ are non-linear functions of $\mathbf{x}$. Expanding $\lambda_j(\mathbf{x})$ by Taylor series about $\mathbf{x} = 0$:

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\mathbf{0}) + \mathbf{d}_{\lambda_j}^T(\mathbf{0})\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{D}_{\lambda_j}(\mathbf{0})\mathbf{x} \tag{3}$$

$\mathbf{d}_{\lambda_j}(\mathbf{0}) \in \mathbb{R}^m$: gradient vector, $\mathbf{D}_{\lambda_j}(\mathbf{0}) \in \mathbb{R}^{m\times m}$ the Hessian matrix of $\lambda_j(\mathbf{x})$ evaluated at $\mathbf{x} = \mathbf{0}$.

α-centered perturbation

We are looking for a point $\mathbf{x} = \boldsymbol{\alpha}$ in the $\mathbf{x}$-space such that the Taylor series expansion of $\lambda_j(\mathbf{x})$ about this point

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha}) + \mathbf{d}_{\lambda_j}^T(\boldsymbol{\alpha})(\mathbf{x} - \boldsymbol{\alpha}) + \frac{1}{2}(\mathbf{x} - \boldsymbol{\alpha})^T \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})(\mathbf{x} - \boldsymbol{\alpha}) \tag{4}$$

is optimal in some sense. The optimal point $\boldsymbol{\alpha}$ is selected such that the mean or the first moment of each eigenvalue is calculated most accurately.

α-centered perturbation

The mean of $\lambda_j(\mathbf{x})$ can be obtained as

$$\bar{\lambda}_j = \int_{\mathbb{R}^m} \lambda_j(\mathbf{x}) p(\mathbf{x}) \, d\mathbf{x} = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-h(\mathbf{x})} \, d\mathbf{x} \tag{5}$$

where $$h(\mathbf{x}) = \mathbf{x}^T \mathbf{x}/2 - \ln \lambda_j(\mathbf{x}) \tag{6}$$

Expand the function $h(\mathbf{x})$ in a Taylor series about a point where $h(\mathbf{x})$ attends its global minimum. By doing so the error in evaluating the integral (5) would be minimized.

α-centered perturbation

Therefore, the optimal point can be obtained as

$$\frac{\partial h(\mathbf{x})}{\partial x_k} = 0 \quad \text{or} \quad x_k = \frac{1}{\lambda_j(\mathbf{x})} \frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}, \quad \forall k \tag{7}$$

Combining for all k we have $\mathbf{d}_{\lambda_j}(\boldsymbol{\alpha}) = \lambda_j(\boldsymbol{\alpha})\boldsymbol{\alpha}$. Rearranging

$$\boldsymbol{\alpha} = \mathbf{d}_{\lambda_j}(\boldsymbol{\alpha})/\lambda_j(\boldsymbol{\alpha}) \tag{8}$$

This equation immediately gives a recipe for an iterative algorithm to obtain $\boldsymbol{\alpha}$.

α-centered perturbation

Substituting $\mathbf{d}_{\lambda_j}(\boldsymbol{\alpha})$ in Eq. (4)

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha})\left(1 - |\boldsymbol{\alpha}|^2\right) + \frac{1}{2}\boldsymbol{\alpha}^T \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\boldsymbol{\alpha} + \boldsymbol{\alpha}^T\left(\lambda_j(\boldsymbol{\alpha})\mathbf{I} - \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\right)\mathbf{x} + \frac{1}{2}\mathbf{x}^T \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\mathbf{x} \quad (9)$$

This, like the mean-centered approach, also results in a quadratic form in the random variable $\mathbf{x}$.

Eigenvalue statistics

Both approximations yield a quadratic form in Gaussian random variable of the form

$$\lambda_j(\mathbf{x}) \approx c_j + \mathbf{a}_j^T \mathbf{x} + \frac{1}{2}\mathbf{x}^T \mathbf{A}_j \mathbf{x} \qquad (10)$$

The moment generating function:

$$M_{\lambda_j}(s) = \mathrm{E}\left[e^{s\lambda_j(\mathbf{x})}\right] \approx \frac{e^{sc_j + \frac{s^2}{2}\mathbf{a}_j^T\left[\mathbf{I} - s\mathbf{A}_j\right]^{-1}\mathbf{a}_j}}{\sqrt{\left\|\mathbf{I} - s\mathbf{A}_j\right\|}} \qquad (11)$$

Eigenvalue statistics

Cumulants:

$$\kappa_r = \begin{cases} c_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{A}_j\right) & \text{if} \quad r = 1, \\ \frac{r!}{2}\mathbf{a}_j^T \mathbf{A}_j^{r-2}\mathbf{a}_j + \frac{(r-1)!}{2}\mathrm{Trace}\left(\mathbf{A}_j^r\right) & \text{if} \quad r \geq 2 \end{cases} \tag{12}$$

Thus

$$\bar{\lambda}_j = \kappa_1 = c_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{A}_j\right) \tag{13}$$

$$\mathrm{Var}\left[\lambda_j\right] = \kappa_2 = \mathbf{a}_j^T \mathbf{a}_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{A}_j^2\right) \tag{14}$$

Asymptotic analysis

We want to evaluate an integral of the following form:

$$\mathcal{J} = \int_{\mathbb{R}^m} f(\mathbf{x})p(\mathbf{x})\,d\mathbf{x} = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{\widetilde{h}(\mathbf{x})}\,d\mathbf{x} \tag{15}$$

where $\widetilde{h}(\mathbf{x}) = \ln f(\mathbf{x}) - \mathbf{x}^T\mathbf{x}/2$ (16)

Assume $f(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}$ is smooth and at least twice differentiable and $\widetilde{h}(\mathbf{x})$ reaches its global maximum at an unique point $\boldsymbol{\theta} \in \mathbb{R}^m$.

Asymptotic analysis

Therefore, at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial \widetilde{h}(\mathbf{x})}{\partial x_k} = 0 \text{ or } x_k = \frac{\partial}{\partial x_k} \ln f(\mathbf{x}), \forall k, \text{ or } \boldsymbol{\theta} = \frac{\partial}{\partial \mathbf{x}} \ln f(\boldsymbol{\theta}). \tag{17}$$

Further assume that $\widetilde{h}(\boldsymbol{\theta})$ is so large that

$$\left| \frac{1}{\widetilde{h}(\boldsymbol{\theta})} \mathcal{D}^j(\widetilde{h}(\boldsymbol{\theta})) \right| \to 0 \quad \text{for} \quad j > 2 \tag{18}$$

$\mathcal{D}^j(\widetilde{h}(\boldsymbol{\theta}))$: jth order derivative of $\widetilde{h}(\mathbf{x})$ at $\mathbf{x} = \boldsymbol{\theta}$.

Asymptotic analysis

Under previous assumptions, using second-order Taylor series of $\widetilde{h}(\mathbf{x})$ the integral (12) can be evaluated asymptotically as

$$\mathcal{J} \approx \frac{e^{\widetilde{h}(\boldsymbol{\theta})}}{\sqrt{\|\widetilde{\mathbf{H}}(\boldsymbol{\theta})\|}} = f(\boldsymbol{\theta}) e^{-\left(\boldsymbol{\theta}^T \boldsymbol{\theta}/2\right)} \|\widetilde{\mathbf{H}}(\boldsymbol{\theta})\|^{-1/2} \quad (19)$$

$\widetilde{\mathbf{H}}(\boldsymbol{\theta})$ is the Hessian matrix of $\widetilde{h}(\mathbf{x})$ at $\mathbf{x} = \boldsymbol{\theta}$.

Asymptotic analysis

An arbitrary rth order moment of the eigenvalues

$$\mu'_r = \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x}) p(\mathbf{x})\, d\mathbf{x}, \quad r = 1, 2, 3 \cdots \tag{20}$$

Comparing this with Eq. (12) it is clear that

$$f(\mathbf{x}) = \lambda_j^r(\mathbf{x}) \quad \text{and} \quad \widetilde{h}(\mathbf{x}) = r \ln \lambda_j(\mathbf{x}) - \mathbf{x}^T\mathbf{x}/2 \tag{21}$$

The optimal point $\boldsymbol{\theta}$ can be obtained from (14) as

$$\boldsymbol{\theta} = r\, \mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) / \lambda_j(\boldsymbol{\theta}) \tag{22}$$

Asymptotic analysis

The rth moment:

$$\mu'_r = \lambda_j^r(\boldsymbol{\theta}) e^{-\frac{|\boldsymbol{\theta}|^2}{2}} \left\| \mathbf{I} + \frac{1}{r}\boldsymbol{\theta}\boldsymbol{\theta}^T - \frac{r}{\lambda_j(\boldsymbol{\theta})} \mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \right\|^{-1/2} \tag{23}$$

The mean of the eigenvalues (substitute $r = 1$):

$$\bar{\lambda}_j = \lambda_j(\boldsymbol{\theta}) e^{-\frac{|\boldsymbol{\theta}|^2}{2}} \left\| \mathbf{I} + \boldsymbol{\theta}\boldsymbol{\theta}^T - \mathbf{D}_{\lambda_j}(\boldsymbol{\theta})/\lambda_j(\boldsymbol{\theta}) \right\|^{-1/2} \tag{24}$$

Central moments:

$\mathrm{E}\left[(\lambda_j - \bar{\lambda}_j)^r\right] = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mu'_k \bar{\lambda}_j^{r-k}.$

Pdf of the eigenvalues

Theorem 1 $\lambda_j(\mathbf{x})$ *is distributed as a non-central χ^2 random variable with noncentrality parameter δ^2 and degrees-of-freedom m' if and only if (a) $\mathbf{A}_j^2 = \mathbf{A}_j$, (b) $\mathrm{Trace}\left(\mathbf{A}_j\right) = m'$ and (c) $\mathbf{a}_j = \mathbf{A}_j \mathbf{a}_j$, $\delta^2 = c_j = \mathbf{a}_j^T \mathbf{a}_j / 4$.*

This implies that the the Hessian matrix $\mathbf{A}_j$ should be an idempotent matrix. In general this requirement is not expected to be satisfied for eigenvalues of real structural systems.

Central χ^2 approximation *(Pearson's)*

Pdf of the jth eigenvalue

$$p_{\lambda_j}(u) \approx \frac{1}{\widetilde{\gamma}} p_{\chi^2_\nu}\left(\frac{u-\widetilde{\eta}}{\widetilde{\gamma}}\right) = \frac{(u-\widetilde{\eta})^{\nu/2-1} e^{-(u-\widetilde{\eta})/2\widetilde{\gamma}}}{(2\widetilde{\gamma})^{\nu/2}\Gamma(\nu/2)} \tag{25}$$

where

$$\widetilde{\eta} = \frac{-2{\kappa_2}^2 + \kappa_1\kappa_3}{\kappa_3}, \ \widetilde{\gamma} = \frac{\kappa_3}{4\kappa_2}, \text{ and } \nu = 8\frac{{\kappa_2}^3}{{\kappa_3}^2} \tag{26}$$

Non-central χ^2 approximation

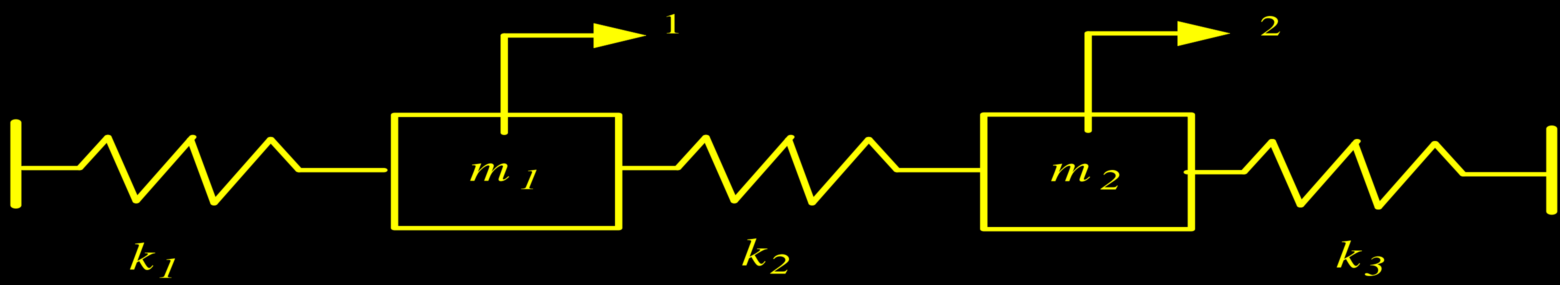
Pdf of the jth eigenvalue

$$p_{\lambda_j}(u) \approx \frac{1}{\gamma_j} p_{Q_j}\left(\frac{u-\eta_j}{\gamma_j}\right) \qquad (27)$$

where $p_{Q_j}(u) \;=\; \frac{e^{-(\delta_j+u/2)}u^{m/2-1}}{2^{m/2}} \sum_{r=0}^{\infty} \frac{(\delta u)^r}{r!\,2^r\Gamma(m/2+r)}$,
$\eta_j = c_j - \frac{1}{2}a_j^T \mathbf{A}_j^{-1} a_j$, $\gamma_j = \frac{\mathrm{Trace}\left(\mathbf{A}_j\right)}{2m}$, $\delta_j^2 = \boldsymbol{\rho}_j^T \boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_j = \mathbf{A}_j^{-1} a_j$.

Numerical example

Undamped two degree-of-system system: $m_1 = 1$ Kg, $m_2 = 1.5$ Kg, $\bar{k}_1 = 1000$ N/m, $\bar{k}_2 = 1100$ N/m and $k_3 = 100$ N/m.



Only the stiffness parameters k_1 and k_2 are uncertain: $k_i = \bar{k}_i(1 + \epsilon_i x_i)$, $i = 1, 2$. $\mathbf{x} = \{x_1, x_2\}^T \in \mathbb{R}^2$ and the 'strength parameters' $\epsilon_1 = \epsilon_2 = 0.25$.

Numerical example

Following six methods are compared

1. *Mean-centered first-order perturbation*
2. *Mean-centered second-order perturbation*
3. *α-centered first-order perturbation*
4. *α-centered second-order perturbation*
5. *Asymptotic method*
6. *Monte Carlo Simulation (10K samples)* - can be considered as benchmark.

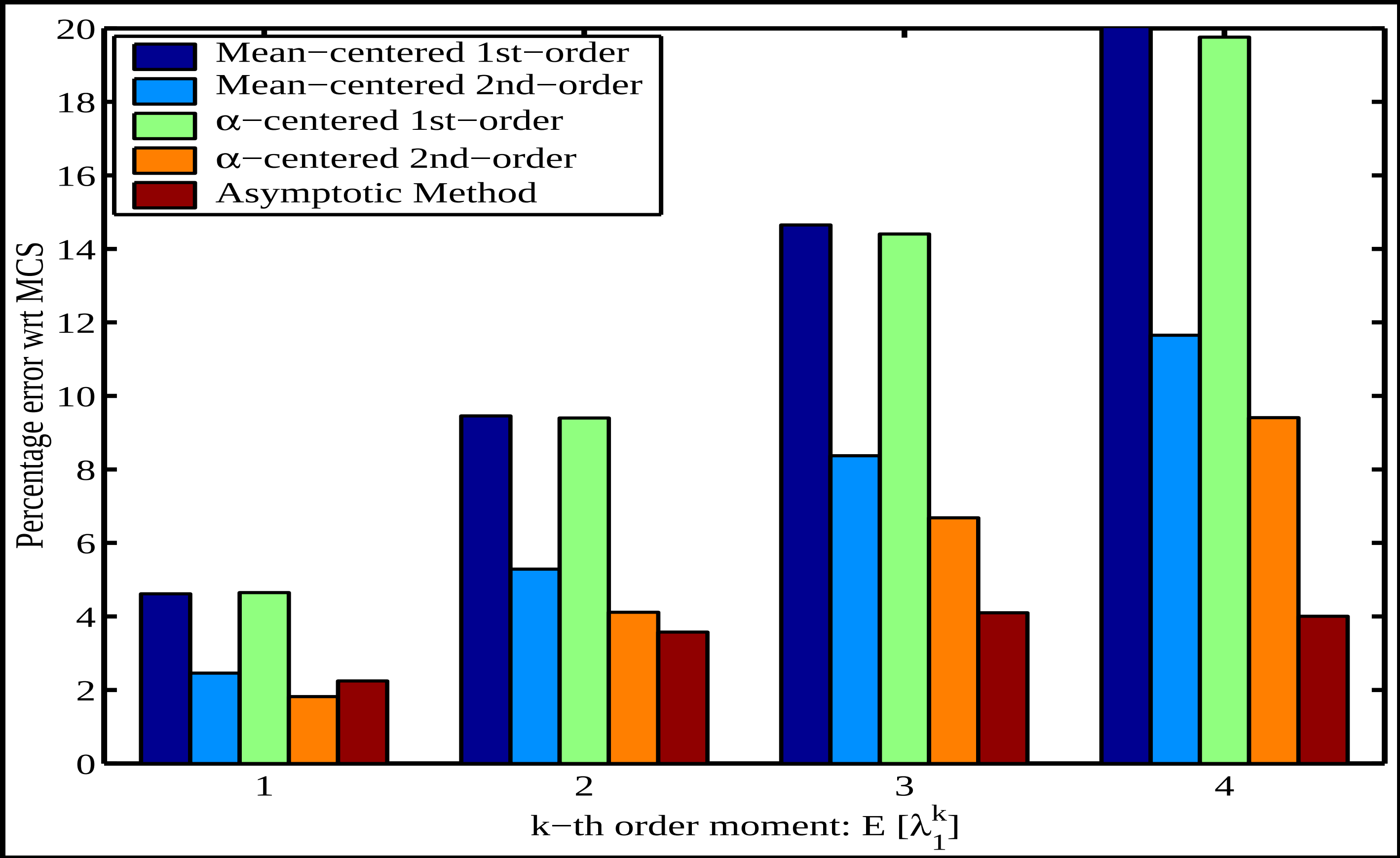
Numerical example

The percentage error:

$$\text{Error}_{i\text{th method}} = \frac{\{\mu'_k\}_{i\text{th method}} - \{\mu'_k\}_{\text{MCS}}}{\{\mu'_k\}_{\text{MCS}}} \times 100$$

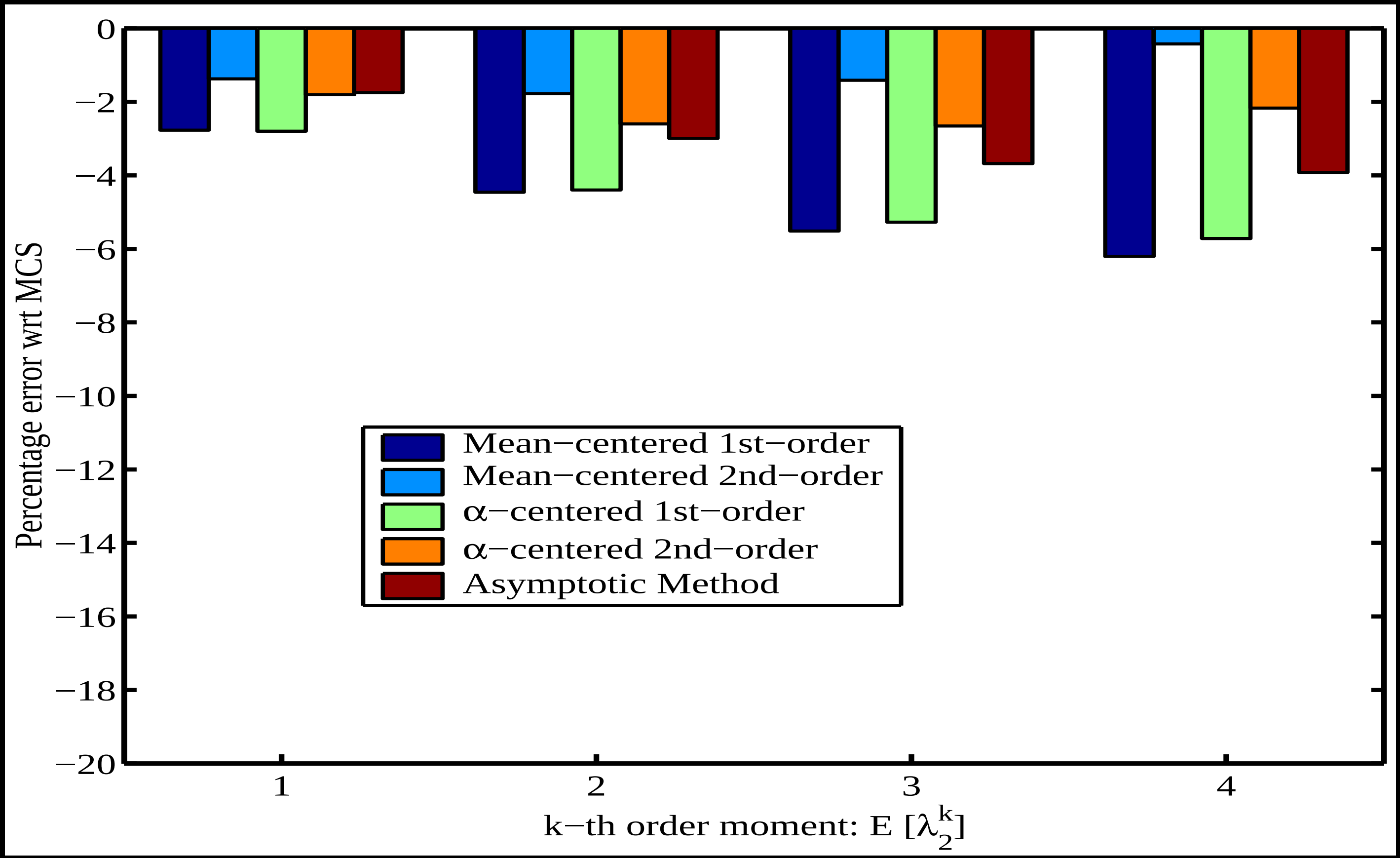
$i = 1, \cdots 5.$

Numerical example



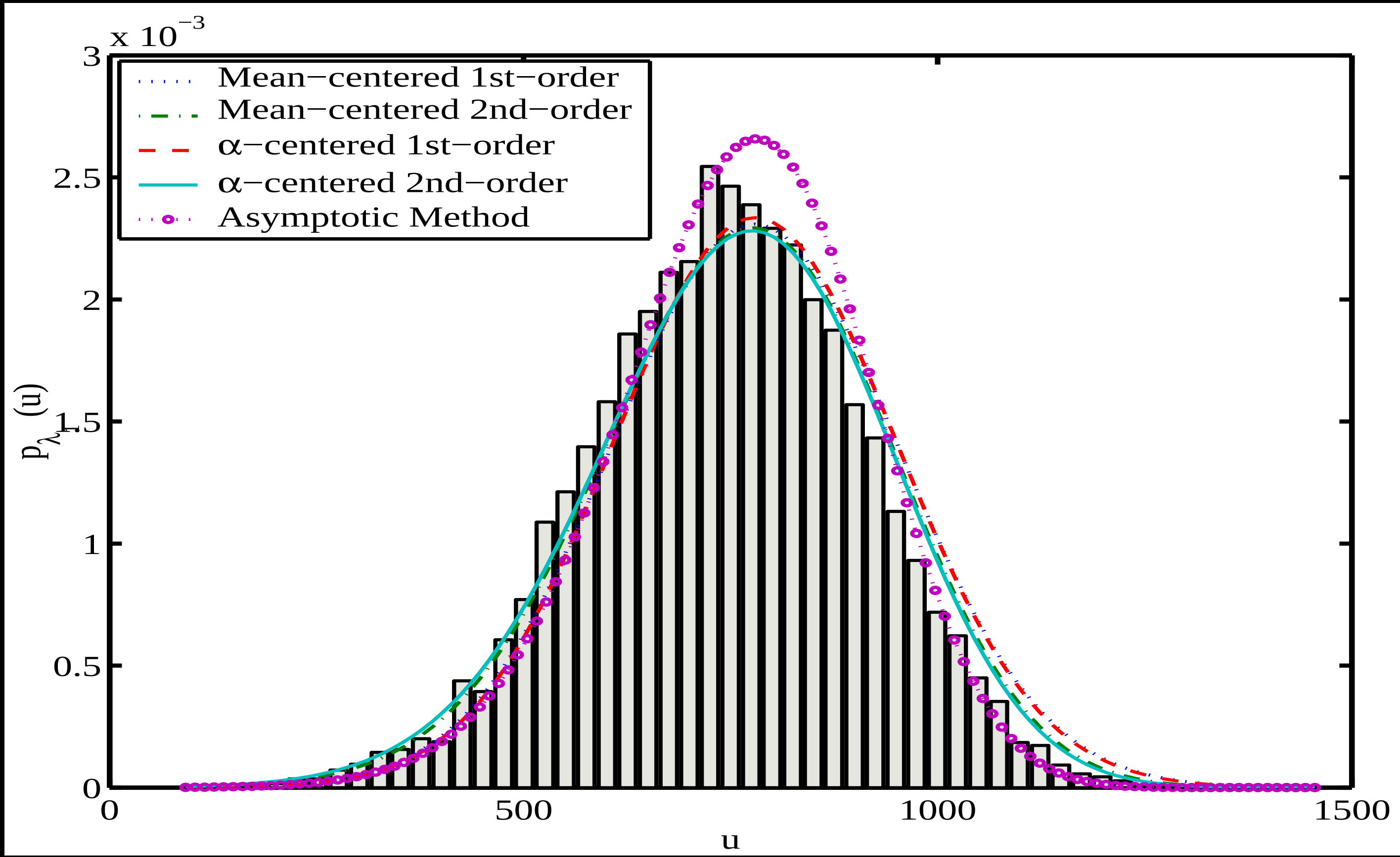
Percentage error for the first four raw moments of the first eigenvalue

Numerical example



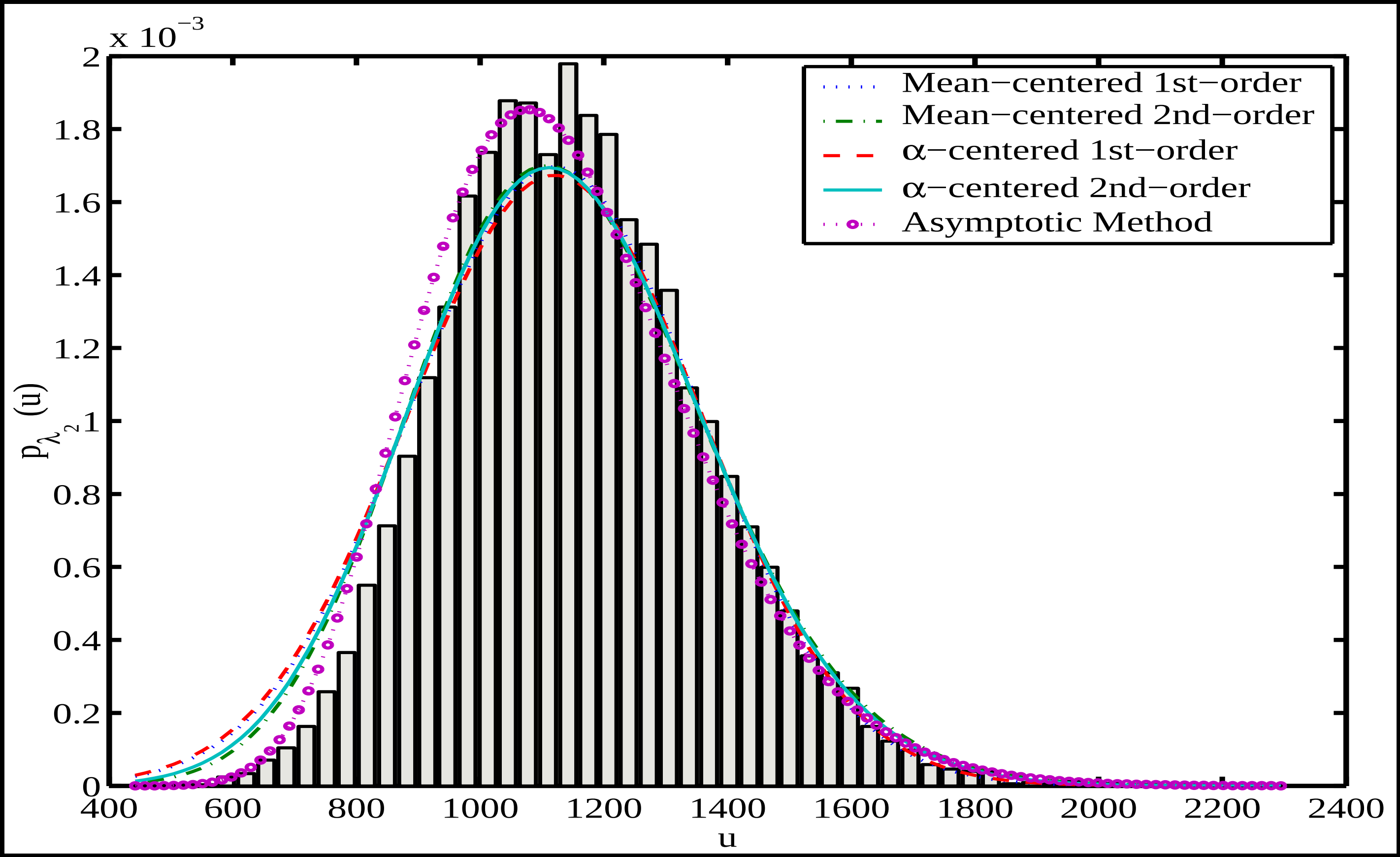
Percentage error for the first four raw moments of the second eigenvalue

Numerical example



Probability density function of the first eigenvalue

Numerical example



Probability density function of the second eigenvalue

Conclusions

- Two methods, namely (a) optimal point expansion method, and (b) asymptotic moment method, are proposed.
- The optimal point is obtained so that the mean of the eigenvalues are estimated most accurately.
- The asymptotic method assumes that the eigenvalues are large compared to their 3rd order or higher derivatives.
- Pdf of the eigenvalues are obtained in terms of central and non-central χ^2 densities.

Open problems

- Joint statistics (moments/pdf/cumulants) of the eigenvalues with non-Gaussian system parameters.
- Statistics of the difference and ratio of the eigenvalues.
- Statistics of a single eigenvector (for GUE/GOE and general matrices).
- Joint statistics of the eigenvectors.
- Joint statistics of the eigenvalues and eigenvectors.