# Transient Dynamics of Structures With Uncertain Parameters

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## Outline of the talk

1. **Introduction**
   - SPDE for dynamical systems
2. **Discretization Techniques**
3. **Spectral decomposition in vector space**
   - Projection in a finite dimensional vector-space
4. **Numerical illustration**
   - The Euler-Bernoulli beam
   - Computational Cost
5. **Conclusion & Future Work**

## Stochastic PDEs for structural dynamics

We consider the stochastic partial differential equation (PDE) pertinent to the structural dynamics problem as

$$\rho(\mathbf{r},\theta)\frac{\partial^2 U(\mathbf{r},t,\theta)}{\partial t^2} + \mathcal{L}_\alpha \frac{\partial U(\mathbf{r},t,\theta)}{\partial t} + \mathcal{L}_\beta U(\mathbf{r},t,\theta) = p(\mathbf{r},t) \qquad (1)$$

The stochastic operator $\mathcal{L}_\beta$ can be

- $\mathcal{L}_\beta \equiv \frac{\partial}{\partial x} AE(x,\theta)\frac{\partial}{\partial x}$ axial deformation of rods
- $\mathcal{L}_\beta \equiv \frac{\partial^2}{\partial x^2} EI(x,\theta)\frac{\partial^2}{\partial x^2}$ bending deformation of beams

$\mathcal{L}_\alpha$ denotes the stochastic damping, which is mostly proportional in nature.

Here $\alpha, \beta : \mathbb{R}^d \times \Theta \to \mathbb{R}$ are stationary square integrable random fields, which can be viewed as a set of random variables indexed by $\mathbf{r} \in \mathbb{R}^d$.

Based on the physical problem the random field $a(\mathbf{r},\theta)$ can be used to model different physical quantities (e.g., $AE(x,\theta)$, $EI(x,\theta)$).

## Discretized Stochastic PDE

- A random process $a(\mathbf{r},\theta)$ can be expressed in a generalised Fourier type of series known as the Karhunen-Loève expansion

$$a(\mathbf{r},\theta) = a_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\nu_i}\xi_i(\theta)\varphi_i(\mathbf{r}) \tag{2}$$

  Here $a_0(\mathbf{r})$ is the mean function, $\xi_i(\theta)$ are uncorrelated standard Gaussian random variables, $\nu_i$ and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\mathcal{D}} C_a(\mathbf{r}_1,\mathbf{r}_2)\varphi_j(\mathbf{r}_1)\mathrm{d}\mathbf{r}_1 = \nu_j\varphi_j(\mathbf{r}_2), \quad \forall\, j = 1, 2, \cdots \tag{3}$$

- For non-Gaussian random fields (e.g. uniform, lognormal), Eq. 2 can represented with a PC type expansion and different sets of orthogonal polynomials from the Weiner-Askey scheme can be utilized to represent the trial basis.

## Discrete equation for stochastic mechanics

- The stochastic PDE along with the boundary conditions results in:

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}(\theta,t)+\mathbf{C}(\theta)\dot{\mathbf{u}}(\theta,t)+\mathbf{K}(\theta)\mathbf{u}(\theta,t)=\mathbf{f}(t) \tag{4}$$

- $\mathbf{M}(\theta)=\mathbf{M}_0+\sum_{i=1}^{p}\mu_i(\theta_i)\mathbf{M}_i\in\mathbb{R}^{n\times n}$ is the random mass matrix, $\mathbf{K}(\theta)=\mathbf{K}_0+\sum_{i=1}^{p}\nu_i(\theta_i)\mathbf{K}_i\in\mathbb{R}^{n\times n}$ is the random stiffness matrix, $\mathbf{C}(\theta)\in\mathbb{R}^{n\times n}$ as the random damping matrix and $\mathbf{f}(t)$ is the forcing vector
- The mass and stiffness matrices have been expressed in terms of their deterministic components ($\mathbf{M}_0$ and $\mathbf{K}_0$) and the corresponding random contributions ($\mathbf{M}_i$ and $\mathbf{K}_i$) obtained from discretising the stochastic field with a finite number of random variables ($\mu_i(\theta_i)$ and $\nu_i(\theta_i)$) and their corresponding spatial basis functions.
- Proportional damping model is considered for which $\mathbf{C}(\theta)=\zeta_1\mathbf{M}(\theta)+\zeta_2\mathbf{K}(\theta)$, where $\zeta_1$ and $\zeta_2$ are scalars.

## Time domain representation

If the time steps are fixed to $\Delta t$, then the equation of motion can be written as

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{C}(\theta)\dot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{K}(\theta)\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t}. \tag{5}$$

Following the Newmark method based on constant average acceleration scheme, the above equations can be represented as

$$[a_0\mathbf{M}(\theta) + a_1\mathbf{C}(\theta) + \mathbf{K}(\theta)]\,\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta) \tag{6}$$

$$\text{and,}\quad \mathbf{p}^{eqv}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t} + f\left(\mathbf{u}_t(\theta), \dot{\mathbf{u}}_t(\theta), \ddot{\mathbf{u}}_t(\theta), \mathbf{M}(\theta), \mathbf{C}(\theta)\right) \tag{7}$$

where $\mathbf{p}^{eqv}_{t+\Delta t}(\theta)$ is the equivalent force at time $t + \Delta t$ which consists of contributions of the system response at the previous time step.

## Newmark's method

The expressions for the velocities $\dot{\mathbf{u}}_{t+\Delta t}(\theta)$ and accelerations $\ddot{\mathbf{u}}_{t+\Delta t}(\theta)$ at each time step is a linear combination of the values of the system response at previous time steps (Newmark method) as

$$\ddot{\mathbf{u}}_{t+\Delta t}(\theta) = a_0 \left[\mathbf{u}_{t+\Delta t}(\theta) - \mathbf{u}_t(\theta)\right] - a_2 \dot{\mathbf{u}}_t(\theta) - a_3 \ddot{\mathbf{u}}_t(\theta) \tag{8}$$

$$\text{and,} \quad \dot{\mathbf{u}}_{t+\Delta t}(\theta) = \dot{\mathbf{u}}_t(\theta) + a_6 \ddot{\mathbf{u}}_t(\theta) + a_7 \ddot{\mathbf{u}}_{t+\Delta t}(\theta) \tag{9}$$

where the integration constants $a_i, \; i = 1, 2, \ldots, 7$ are independent of system properties and depends only on the chosen time step and some constants:

$$a_0 = \frac{1}{\alpha \Delta t^2}; \qquad a_1 = \frac{\delta}{\alpha \Delta t}; \qquad a_2 = \frac{1}{\alpha \Delta t}; \qquad a_3 = \frac{1}{2\alpha} - 1; \tag{10}$$

$$a_4 = \frac{\delta}{\alpha} - 1; \qquad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right); \qquad a_6 = \Delta t(1 - \delta); \qquad a_7 = \delta \Delta t \tag{11}$$

## Newmark's method

Following this development, the linear structural system in (6) can be expressed as

$$\underbrace{\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right]}_{\mathbf{A}(\theta)} \mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta). \tag{12}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. For the case of proportional damping, the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written similar to the case of frequency domain as

$$\mathbf{A}_0 = [a_0 + a_1\zeta_1]\,\mathbf{M}_0 + [a_1\zeta_2 + 1]\,\mathbf{K}_0 \tag{13}$$

$$\text{and,} \quad \mathbf{A}_i = [a_0 + a_1\zeta_1]\,\mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p_1 \tag{14}$$

$$= [a_1\zeta_2 + 1]\,\mathbf{K}_i \quad \text{for} \quad i = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\,.$$

## General mathematical representation

In general the main equation which need to be solved can be expressed as

$$\left(\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta_i)\mathbf{A}_i\right)\mathbf{u}(\theta) = \mathbf{f}(\theta) \tag{15}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. These can be real or complex matrices.

## Polynomial Chaos expansion

- After the finite truncation, concisely, the polynomial chaos expansion can be written as

$$\hat{\mathbf{u}}(\theta) = \sum_{k=1}^{P} H_k(\xi(\theta))\mathbf{u}_k \tag{16}$$

where $H_k(\xi(\theta))$ are the polynomial chaoses and $\mathbf{u}_k \in \mathbb{R}^n$ are deterministic vectors to be determined.

- The value of the number of terms $P$ depends on the number of basic random variables $M$ and the order of the PC expansion $r$ as

$$P = \sum_{j=0}^{r} \frac{(M+j-1)!}{j!(M-1)!} \tag{17}$$

## Polynomial Chaos expansion

We need to solve a $nP \times nP$ linear equation to obtain all $\mathbf{u}_k$ for every frequency point:

$$\begin{bmatrix} \mathbf{A}_{0,0} & \cdots & \mathbf{A}_{0,P-1} \\ \mathbf{A}_{1,0} & \cdots & \mathbf{A}_{1,P-1} \\ \vdots & \vdots & \vdots \\ \mathbf{A}_{P-1,0} & \cdots & \mathbf{A}_{P-1,P-1} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{P-1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{P-1} \end{Bmatrix} \tag{18}$$

P increases exponentially with $M$:

| $M$ | 2 | 3 | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|---|---|
| 2nd order PC | 5 | 9 | 20 | 65 | 230 | 1325 | 5150 |
| 3rd order PC | 9 | 19 | 55 | 285 | 1770 | 23425 | 176850 |

## Polynomial Chaos expansion: Some Observations

- The basis is a function of the pdf of the random variables only. For example, Hermite polynomials for Gaussian pdf, Legender's polynomials for uniform pdf.
- The physics of the underlying problem (static, dynamic, heat conduction, transients....) cannot be incorporated in the basis.
- For an $n$-dimensional output vector, the number of terms in the projection can be more than $n$ (depends on the number of random variables). This implies that many of the vectors $\mathbf{u}_k$ are linearly dependent.
- The physical interpretation of the coefficient vectors $\mathbf{u}_k$ is not immediately obvious.
- The functional form of the response is a pure polynomial in random variables.

## Projection in a finite dimensional vector-space

Suppose the solution of Eq. (15) is given by

$$\hat{\mathbf{u}}_{t+\Delta t}(\theta) = \left[ \mathbf{A}_0 + \sum_{i=1}^{M} \Gamma_i(\boldsymbol{\xi}(\theta))\mathbf{A}_i \right]^{-1} \mathbf{f}^{eqv}_{t+\Delta t}(\theta) \tag{19}$$

Orthogonal decompostion of the deterministic system yields

$$\boldsymbol{\lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{R}^{n \times n}; \boldsymbol{\Phi} = [\phi_1, \phi_2, \ldots, \phi_n] \in \mathbb{R}^{n \times n} \tag{20}$$

where the eigenpairs are ordered in the ascending order:
$\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n}$. We introduce the transformations
$\widetilde{\mathbf{A}}_i = \boldsymbol{\Phi}^T \mathbf{A}_i \boldsymbol{\Phi} \in \mathbb{R}^{n \times n}; i = 0, 1, 2, \ldots, M$.
The orthonormality of $\boldsymbol{\Phi}$ one has

$$\hat{\mathbf{u}}_{t+\Delta t}(\theta) = \left[ \boldsymbol{\Phi}^{-T} \boldsymbol{\Lambda}_0 \boldsymbol{\Phi}^{-1} + \sum_{i=1}^{M} \Gamma_i(\boldsymbol{\xi}(\theta)) \boldsymbol{\Phi}^{-T} \widetilde{\mathbf{A}}_i \boldsymbol{\Phi}^{-1} \right]^{-1} \mathbf{f}^{eqv}_{t+\Delta t}(\theta) \tag{21}$$

$$\text{where} \qquad \boldsymbol{\xi}(\theta) = \left[\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta)\right]^T.$$

## Projection in a finite dimensional vector-space

Now we separate the diagonal and off-diagonal terms of the $\widetilde{\mathbf{A}}_i$ matrices as

$$\widetilde{\mathbf{A}}_i = \mathbf{\Lambda}_i + \mathbf{\Delta}_i, \quad i = 1, 2, \ldots, M \tag{22}$$

Here the diagonal matrix

$$\mathbf{\Lambda}_i = \mathrm{diag}\left[\widetilde{\mathbf{A}}\right] = \mathrm{diag}\left[\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}\right] \in \mathbb{R}^{n\times n} \tag{23}$$

and $\mathbf{\Delta}_i = \widetilde{\mathbf{A}}_i - \mathbf{\Lambda}_i$ is an off-diagonal only matrix. We can write :

$$\mathbf{\Psi}\left(\xi(\theta)\right) = \left[\underbrace{\mathbf{\Lambda}_0 + \sum_{i=1}^{M} \Gamma_i(\xi(\theta))\mathbf{\Lambda}_i}_{\mathbf{\Lambda}(\Gamma_i(\xi(\theta)))} + \underbrace{\sum_{i=1}^{M} \Gamma_i(\xi(\theta))\mathbf{\Delta}_i}_{\mathbf{\Delta}(\Gamma_i(\xi(\theta)))}\right]^{-1} . \tag{24}$$

## Projection in a finite dimensional vector-space

The diagonal matrix $\mathbf{\Lambda}(\xi(\theta))$ is treated as the preconditioner in the stochastic Krylov space, such that the solution can be projected onto a very few basis functions.
Hence the left preconditioned stochastic Krylov space becomes

$$\begin{aligned}\mathcal{K}_m(\mathbf{\Lambda}^{-1}\mathbf{\Psi}, \mathbf{\Lambda}^{-1}\mathbf{f}^{eqv}_{t+\Delta t}) = \mathrm{span}\{&\mathbf{\Phi}^T\mathbf{\Lambda}^{-1}\mathbf{\Phi}\mathbf{f}^{eqv}_{t+\Delta t}, \mathbf{\Phi}^T(\mathbf{\Lambda}^{-1}\mathbf{\Delta})\mathbf{\Lambda}^{-1}\mathbf{\Phi}\mathbf{f}^{eqv}_{t+\Delta t},\\ &\mathbf{\Phi}^T(\mathbf{\Lambda}^{-1}\mathbf{\Delta})^2\mathbf{\Lambda}^{-1}\mathbf{\Phi}\mathbf{f}^{eqv}_{t+\Delta t}, \ldots, \mathbf{\Phi}^T(\mathbf{\Lambda}^{-1}\mathbf{\Delta})^{m-1}\mathbf{\Lambda}^{-1}\mathbf{\Phi}\mathbf{f}^{eqv}_{t+\Delta t}\}\end{aligned} \tag{25}$$

The equivalent infinite Neumann matrix series representation of the above equation is

$$\mathbf{\Psi}(\xi(\theta)) = \sum_{s=0}^{\infty}(-1)^s\left[\mathbf{\Lambda}^{-1}(\xi(\theta))\,\mathbf{\Delta}(\xi(\theta))\right]^s\mathbf{\Lambda}^{-1}(\xi(\theta)) \tag{26}$$

## Projection in a finite dimensional vector-space

Taking an arbitrary $r$-th element of $\mathbf{u}(t, \theta)$, Eqn. (21) can be rearranged to have

$$u^r_{t+\Delta t}(\theta) = \sum_{k=1}^{n} \Phi_{rk} \left( \sum_{j=1}^{n} \Psi_{kj} \left(\xi(\theta)\right) \left( \phi_j^T \mathbf{f}^{eqv}_{t+\Delta t} \right) \right) \tag{27}$$

Defining

$$\mathcal{L}_k \left(t, \xi(\theta)\right) = \sum_{j=1}^{n} \Psi_{kj} \left(\xi(\theta)\right) \left( \phi_j^T \mathbf{f}^{eqv}_{t+\Delta t} \right) \tag{28}$$

and collecting all the elements in Eqn. (27) for $r = 1, 2, \ldots, n$ one has

$$\mathbf{u}_{t+\Delta t}(\theta) = \sum_{k=1}^{n} \mathcal{L}_k \left(t, \xi(\theta)\right) \phi_k \tag{29}$$

## Projection in a finite dimensional vector-space

A few observations :

- The matrix power series is different from the classical Neumann series in that the elements of the former are not simple polynomials in $\xi_i(\theta)$ but are in terms of the ratio of polynomials.
- The convergence of the series depends on the spectral radius of

$$\mathbf{R}\left(\boldsymbol{\xi}(\theta)\right) = \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right) \tag{30}$$

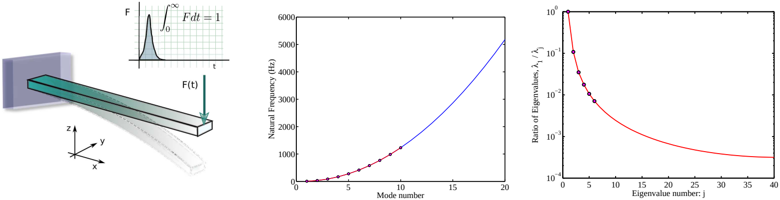
- A generic term of the matrix **R** is

$$R_{rs} = \frac{\Delta_{rs}}{\Lambda_{rr}} = \frac{\sum_{i=1}^{M}\Gamma_i(\xi)\Delta_{i_{rs}}}{\Lambda_{0_r} + \sum_{i=1}^{M}\Gamma_i(\xi)\Lambda_{i_r}} = \frac{\sum_{i=1}^{M}\Gamma_i(\xi)\widetilde{A}_{i_{rs}}}{\Lambda_{0_r} + \sum_{i=1}^{M}\Gamma_i(\xi)\widetilde{A}_{i_{rr}}}; r \neq s \tag{31}$$

which shows that the spectral radius of **R** is controlled by the diagonal dominance of the $\widetilde{\mathbf{A}}_i$ matrices.

# The Euler-Bernoulli beam example

- An Euler-Bernoulli cantilever beam with stochastic bending modulus for a specified value of the correlation length and for different degrees of variability of the random field.




(a) Euler-Bernoulli beam (b) Natural frequency distribution. (c) Eigenvalue ratio of KL decomposition

- Length : 1.0 $m$, Cross-section : $39 \times 5.93\ mm^2$, Young's Modulus: $2 \times 10^{11}\ Pa$.
- Load: Unit impulse at $t = 0$ on the free end of the beam.

## Problem details

- The bending modulus of the cantilever beam is taken to be a homogeneous stationary Gaussian random field of the form

$$EI(x,\theta) = EI_0(1 + a(x,\theta)) \tag{32}$$

where $x$ is the coordinate along the length of the beam, $EI_0$ is the estimate of the mean bending modulus, $a(x,\theta)$ is a zero mean stationary random field.

- The covariance kernel associated with this random field is

$$C_a(x_1, x_2) = \sigma_a^2 e^{-(|x_1 - x_2|)/\mu_a} \tag{33}$$

where $\mu_a$ is the correlation length and $\sigma_a$ is the standard deviation.

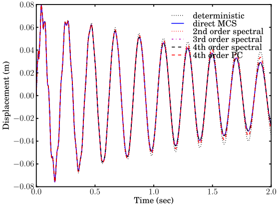
- A correlation length of $\mu_a = L/5$ is considered in the present numerical study.

## Problem details

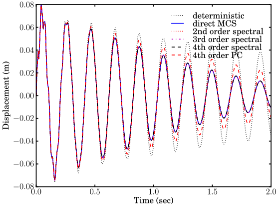
The random field is assumed to be Gaussian. The results are compared with the polynomial chaos expansion.

- The number of degrees of freedom of the system is n = 200.
- The K.L. expansion is truncated at a finite number of terms such that 90% variability is retained.
- direct MCS have been performed with 10,000 random samples and for three different values of standard deviation of the random field, $\sigma_a = 0.05, 0.1, 0.2$.
- Constant modal damping is taken with 1% damping factor for all modes.
- Time domain response of the free end of the beam is sought under the action of a unit impulse at $t = 0$
- Upto $4^{th}$ order spectral functions have been considered in the present problem. Comparison have been made with $4^{th}$ order Polynomial chaos results.

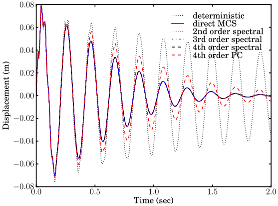
# Mean deflection of the beam




(d) Mean, $\sigma_a = 0.05$.



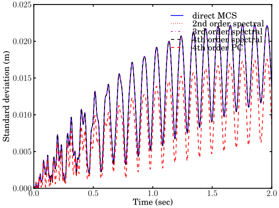

(e) Mean, $\sigma_a = 0.1$.



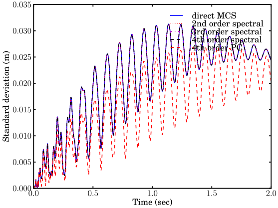

(f) Mean, $\sigma_a = 0.2$.

- Time domain response of the deflection of the tip of the cantilever for three values of standard deviation $\sigma_a$ of the underlying random field.
- Spectral functions approach approximates the solution accurately.
- For long time-integration, the discrepancy of the $4^{th}$ order PC results increases.

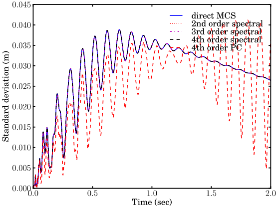
# Standard deviation of the beam response




(g) Standard deviation of deflection, $\sigma_a = 0.05$.

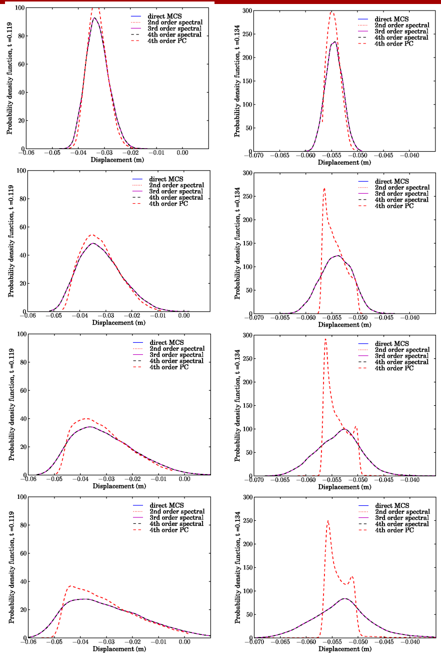



(h) Standard deviation of deflection, $\sigma_a = 0.1$.




(i) Standard deviation of deflection, $\sigma_a = 0.2$.

- The standard deviation of the tip deflection of the beam.
- Since the standard deviation comprises of higher order products of the Hermite polynomials associated with the PC expansion, the higher order moments are less accurately replicated and tend to deviate more significantly.



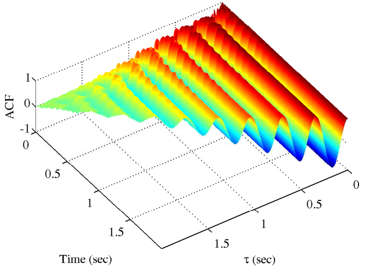

Comparison of PDF of deflection of the beam at t = 0.119s and t = 0.134s for $\sigma_a = \{0.05, 0.10, 0.15, 0.20\}$.

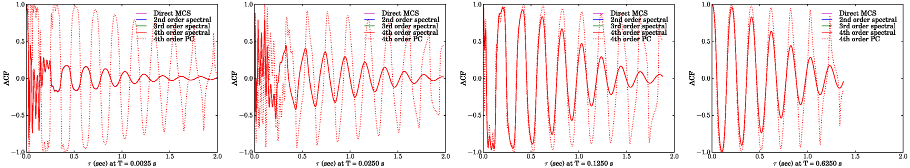
- good agreement of the density functions for different orders of spectral functions.
- the $4^{th}$ order PC fails to produce the appropriate distribution function.
- an increase in order of the PC method to improve the higher order moments of the response is expensive, as the dimension of the resulting linear algebraic system increases with it exponentially.

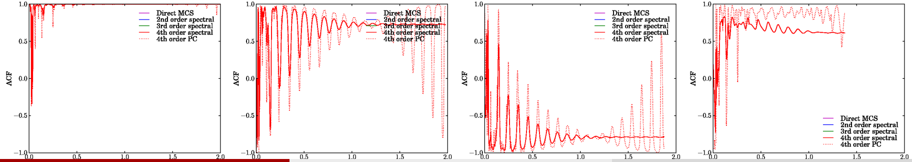
## Autocorrelation surface of the response

$$ACF(t,\tau) = \frac{E\left[(u_t - \mu_t)(u_{t+\tau} - \mu_{t+\tau})\right]}{\sigma_t \; \sigma_{t+\tau}}$$

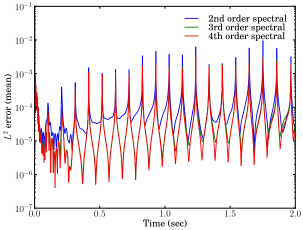
for $T, \tau \in [0, 2.0]$
and input parametric uncertainty of $\sigma_a = 0.20$
Below:
$ACF$ at specific values of $T$ (top) and $\tau$ (bottom).

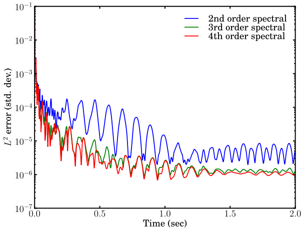


# Error Analysis



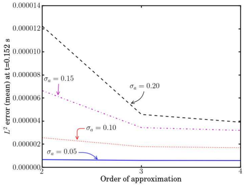



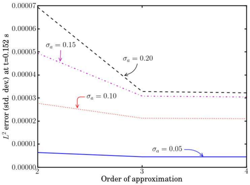

Convergence of the error with spectral function order.

A $L^2$ relative error norm is defined as

$$\epsilon_{\Sigma_j}^{(m)}(t) = \frac{\left\| \Sigma_{j\,SF}^{(m)}(t) - \Sigma_{j\,MCS}(t) \right\|_{L^2(\mathcal{D})}}{\left\| \Sigma_{j\,MCS}(t) \right\|_{L^2(\mathcal{D})}}; \quad j = 1, 2$$

for different spectral function order ($m$) and $j = 1 \Rightarrow$ mean, $j = 2 \Rightarrow$: std dev.




Higher order functions provide a better aprroximation of the solution at an enhanced cost.

## Computational cost

The calculation times are shown for a single time step (done with an $200 \times 200$ system and 4 Gaussian random variables).

| Calculation | Avg Time(s) | Min Time(s) | Max Time(s) |
|---|---|---|---|
| Direct MCS | 13.589 | 13.506 | 13.798 |
| 2nd order spectral | 1.375 | 1.345 | 1.396 |
| 3rd order spectral | 1.445 | 1.414 | 1.465 |
| 4th order spectral | 1.500 | 1.481 | 1.523 |
| 4th order PC | 5.117 | 4.975 | 5.327 |

- All calculations were performed using a single processor core while the optimized *ATLAS, LAPACK* and *BLAS* libraries were used on 8 processor cores for the last case.
- The $4^{th}$ order spectral function is 9 times more efficient than direct MCS and 3.5 times more efficient than $4^{th}$ order PC.
- Computational time increases with spectral function order.

# Conclusion

- The true nature of the solution is a ratio of two polynomials of random variables where the denominator has higher degree than the numerator. The proposed spectral basis functions have this correct mathematical form.
- The proposed method utilizes the eigen-spectrum of the deterministic coefficient matrices $\mathbf{A}_0$ to achieve a POD-like model reduction which helps to work with a significantly smaller subspace dimension.
- The polynomial basis used in the PC method has no adaptive characteristics and remains the same for all time steps, however, in reality, the non-linearity in the stochastic domain is compounded with each incremental time step.
- The spectral functions used in the present approach changes with each time step which allows a better estimation of the response variables.

## Future Work

The future work that can be pursued along this direction may include :

- Trying to reducing the computational burden of integration over the probability space using the efficient variance reduction and/or sampling techniques.
- A-priori error analysis and a rigorous study of the convergence behavior can give important intuitive guidance in moving towards a choice of a more efficient set of basis functions suitable for this class of stochastic problems.
- Extension of the presented idea to the class of non-linear (geometric) dynamics problems.
- Extension of the present approach to study the behavior of time dependent diffusion problems using the dynamic loading.