Random Eigenvalue Problem for Linear Dynamic Systems

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Outline of the talk

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- Mean-centered perturbation method
- α-centered perturbation method

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Random eigenvalue problem

The random eigenvalue problem of undamped or proportionally damped linear systems:

$$\mathbf{K}(\mathbf{x})\boldsymbol{\phi}_j = \lambda_j \mathbf{M}(\mathbf{x})\boldsymbol{\phi}_j. \tag{1}$$

λ_j eigenvalues; $\boldsymbol{\phi}_j$ eigenvectors; $\mathbf{M}(\mathbf{x}) \in \mathbb{R}^{N\times N}$ mass matrix and $\mathbf{K}(\mathbf{x}) \in \mathbb{R}^{N\times N}$ stiffness matrix.
$\mathbf{x} \in \mathbb{R}^m$ is random parameter vector with pdf

$$p(\mathbf{x}) = (2\pi)^{-m/2} e^{-\mathbf{x}^T\mathbf{x}/2}. \tag{2}$$

The Fundamental aim

To obtain the joint probability density function of the eigenvalues and the eigenvectors.

If the matrix $\mathbf{M}^{-1}\mathbf{K}$ is GUE (Gaussian unitary ensemble) or GOE (Gaussian orthogonal ensemble) an exact closed-form expression can be obtained for the joint pdf of the eigenvalues.

In general the system matrices for real structures are not GUE or GOE

Mean-centered perturbation method

Assume that $\mathbf{M}(\mathbf{0}) = \mathbf{M}_0$ and $\mathbf{K}(\mathbf{0}) = \mathbf{K}_0$ are 'deterministic parts' (in general different from the mean matrices). The deterministic eigenvalue problem $\mathbf{K}_0\boldsymbol{\phi}_{j0} = \lambda_{j0}\mathbf{M}_0\boldsymbol{\phi}_{j0}$.
The eigenvalues $\lambda_j(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}$ are non-linear functions of $\mathbf{x}$. Here $\lambda_j(\mathbf{x})$ is replaced by its Taylor series about the point $\mathbf{x} = 0$

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\mathbf{0}) + \mathbf{d}_{\lambda_j}^T(\mathbf{0})\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{D}_{\lambda_j}(\mathbf{0})\mathbf{x}. \tag{3}$$

$\mathbf{d}_{\lambda_j}(\mathbf{0}) \in \mathbb{R}^m$ and $\mathbf{D}_{\lambda_j}(\mathbf{0}) \in \mathbb{R}^{m\times m}$ are respectively the gradient vector and the Hessian matrix of $\lambda_j(\mathbf{x})$ evaluated at $\mathbf{x} = \mathbf{0}$.

α-centered perturbation method

We are looking for a point $\mathbf{x} = \boldsymbol{\alpha}$ in the $\mathbf{x}$-space such that the Taylor series expansion of $\lambda_j(\mathbf{x})$ about this point

$$\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha}) + \mathbf{d}^T_{\lambda_j}(\boldsymbol{\alpha})\,(\mathbf{x}-\boldsymbol{\alpha}) + \frac{1}{2}\,(\mathbf{x}-\boldsymbol{\alpha})^T\,\mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\,(\mathbf{x}-\boldsymbol{\alpha}) \tag{4}$$

is optimal in some sense. The optimal point α is selected such that the mean or the first moment of each eigenvalue is calculated most accurately.

α-centered perturbation method

The mean of $\lambda_j(\mathbf{x})$ can be obtained as

$$\bar{\lambda}_j = \int_{\mathbb{R}^m} \lambda_j(\mathbf{x}) p(\mathbf{x})\, d\mathbf{x} = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-h(\mathbf{x})}\, d\mathbf{x} \tag{5}$$

$$\text{where} \quad h(\mathbf{x}) = \mathbf{x}^T\mathbf{x}/2 - \ln \lambda_j(\mathbf{x}). \tag{6}$$

Expand the function $h(\mathbf{x})$ in a Taylor series about a point where $h(\mathbf{x})$ attends its global minimum. By doing so the error in evaluating the integral (5) would be minimized. Therefore, the optimal point can be obtained as

$$\frac{\partial h(\mathbf{x})}{\partial x_k} = 0 \quad \text{or} \quad x_k = \frac{1}{\lambda_j(\mathbf{x})} \frac{\partial \lambda_j(\mathbf{x})}{\partial x_k}, \quad \forall k. \tag{7}$$

α-centered perturbation method

Combining for all k we have $\mathbf{d}_{\lambda_j}(\boldsymbol{\alpha}) = \lambda_j(\boldsymbol{\alpha})\boldsymbol{\alpha}$. Rearranging

$$\boldsymbol{\alpha} = \mathbf{d}_{\lambda_j}(\boldsymbol{\alpha})/\lambda_j(\boldsymbol{\alpha}). \tag{8}$$

This equation immediately gives a recipe for an iterative algorithm to obtain $\boldsymbol{\alpha}$. Substituting $\mathbf{d}_{\lambda_j}(\boldsymbol{\alpha})$ in Eq. (4)

$$\begin{aligned}\lambda_j(\mathbf{x}) \approx \lambda_j(\boldsymbol{\alpha})\left(1 - |\boldsymbol{\alpha}|^2\right) + \frac{1}{2}\boldsymbol{\alpha}^T \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\boldsymbol{\alpha} \\ + \boldsymbol{\alpha}^T\left(\lambda_j(\boldsymbol{\alpha})\mathbf{I} - \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\right)\mathbf{x} + \frac{1}{2}\mathbf{x}^T \mathbf{D}_{\lambda_j}(\boldsymbol{\alpha})\mathbf{x}.\end{aligned} \tag{9}$$

Eigenvalue statistics using theory of quadratic forms

Both approximations yield a quadratic form in Gaussian random variable $\lambda_j(\mathbf{x}) \approx c_j + \mathbf{a}_j^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{A}_j\mathbf{x}$.
The moment generating function:

$$M_{\lambda_j}(s) = \mathrm{E}\left[e^{s\lambda_j(\mathbf{x})}\right] \approx \frac{e^{sc_j+\frac{s^2}{2}\mathbf{a}_j^T\left[\mathbf{I}-s\mathbf{A}_j\right]^{-1}\mathbf{a}_j}}{\sqrt{\|\mathbf{I}-s\mathbf{A}_j\|}} \tag{10}$$

Cumulants:

$$\kappa_r = \begin{cases} c_j + \frac{1}{2}\mathrm{Trace}\left(\mathbf{A}_j\right) & \text{if} \quad r = 1, \\ \frac{r!}{2}\mathbf{a}_j^T\mathbf{A}_j^{r-2}\mathbf{a}_j + \frac{(r-1)!}{2}\mathrm{Trace}\left(\mathbf{A}_j^r\right) & \text{if} \quad r \geq 2. \end{cases} \tag{11}$$

Asymptotic analysis

We want to evaluate an integral of the following form:

$$\mathcal{J} = \int_{\mathbb{R}^m} f(\mathbf{x}) p(\mathbf{x})\, d\mathbf{x} = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{\widetilde{h}(\mathbf{x})}\, d\mathbf{x} \tag{12}$$

$$\text{where} \quad \widetilde{h}(\mathbf{x}) = \ln f(\mathbf{x}) - \mathbf{x}^T\mathbf{x}/2. \tag{13}$$

Assume $f(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}$ is smooth and at least twice differentiable and $\widetilde{h}(\mathbf{x})$ reaches its global maximum at an unique point $\boldsymbol{\theta} \in \mathbb{R}^m$. Therefore, at $\mathbf{x} = \boldsymbol{\theta}$

$$\frac{\partial \widetilde{h}(\mathbf{x})}{\partial x_k} = 0 \text{ or } x_k = \frac{\partial}{\partial x_k} \ln f(\mathbf{x}), \forall k, \text{ or } \boldsymbol{\theta} = \frac{\partial}{\partial \mathbf{x}} \ln f(\boldsymbol{\theta}). \tag{14}$$

Asymptotic analysis

Further assume that $\widetilde{h}(\boldsymbol{\theta})$ is so large that

$$\left| \frac{1}{\widetilde{h}(\boldsymbol{\theta})} \mathcal{D}^j(\widetilde{h}(\boldsymbol{\theta})) \right| \to 0 \quad \text{for} \quad j > 2 \tag{15}$$

where $\mathcal{D}^j(\widetilde{h}(\boldsymbol{\theta}))$ is jth order derivative of $\widetilde{h}(\mathbf{x})$ evaluated at $\mathbf{x} = \boldsymbol{\theta}$. Under such assumptions, using second-order Taylor series of $\widetilde{h}(\mathbf{x})$ the integral (12) can be evaluated as

$$\mathcal{J} \approx \frac{e^{\widetilde{h}(\boldsymbol{\theta})}}{\sqrt{\|\widetilde{\mathbf{H}}(\boldsymbol{\theta})\|}} = f(\boldsymbol{\theta}) e^{-\left(\boldsymbol{\theta}^T \boldsymbol{\theta}/2\right)} \|\widetilde{\mathbf{H}}(\boldsymbol{\theta})\|^{-1/2}. \tag{16}$$

Asymptotic analysis

An arbitrary rth order moment of the eigenvalues

$$\mu_r' = \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x}) p(\mathbf{x})\, d\mathbf{x}, \quad r = 1, 2, 3 \cdots \tag{17}$$

Comparing this with Eq. (12) it is clear that

$$f(\mathbf{x}) = \lambda_j^r(\mathbf{x}) \quad \text{and} \quad \widetilde{h}(\mathbf{x}) = r \ln \lambda_j(\mathbf{x}) - \mathbf{x}^T\mathbf{x}/2. \tag{18}$$

The optimal point $\boldsymbol{\theta}$ can be obtained from (14) as

$$\boldsymbol{\theta} = r\, \mathbf{d}_{\lambda_j}(\boldsymbol{\theta})/\lambda_j(\boldsymbol{\theta}). \tag{19}$$

Asymptotic analysis

Using the asymptotic approximation, the rth moment:

$$\mu'_r = \lambda_j^r(\boldsymbol{\theta}) e^{-\frac{|\boldsymbol{\theta}|^2}{2}} \left\| \mathbf{I} + \frac{1}{r}\boldsymbol{\theta}\boldsymbol{\theta}^T - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \right\|^{-1/2}. \qquad (20)$$

The mean of the eigenvalues (by substituting $r = 1$):

$$\bar{\lambda}_j = \lambda_j(\boldsymbol{\theta}) e^{-\frac{|\boldsymbol{\theta}|^2}{2}} \left\| \mathbf{I} + \boldsymbol{\theta}\boldsymbol{\theta}^T - \mathbf{D}_{\lambda_j}(\boldsymbol{\theta})/\lambda_j(\boldsymbol{\theta}) \right\|^{-1/2}. \qquad (21)$$

Central moments: $\mathrm{E}\left[(\lambda_j - \bar{\lambda}_j)^r\right] = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mu'_k \bar{\lambda}_j^{r-k}$.

Pdf of the eigenvalues

Theorem 1 *$\lambda_j(\mathbf{x})$ is distributed as a non-central χ^2 random variable with noncentrality parameter δ^2 and degrees-of-freedom m' if and only if (a) $\mathbf{A}_j^2 = \mathbf{A}_j$, (b) $\mathrm{Trace}\,(\mathbf{A}_j) = m'$ and (c) $\mathbf{a}_j = \mathbf{A}_j\mathbf{a}_j$, $\delta^2 = c_j = \mathbf{a}_j^T\mathbf{a}_j/4$.*

This implies that the the Hessian matrix $\mathbf{A}_j$ should be an idempotent matrix. In general this requirement is not expected to be satisfied for eigenvalues of real structural systems.

Pearson's approximation (central χ^2)

Pdf of the jth eigenvalue

$$p_{\lambda_j}(u) \approx \frac{1}{\widetilde{\gamma}} p_{\chi^2_\nu}\left(\frac{u-\widetilde{\eta}}{\widetilde{\gamma}}\right) = \frac{(u-\widetilde{\eta})^{\nu/2-1} e^{-(u-\widetilde{\eta})/2\widetilde{\gamma}}}{(2\widetilde{\gamma})^{\nu/2}\Gamma(\nu/2)}. \qquad (22)$$

where

$$\widetilde{\eta} = \frac{-2{\kappa_2}^2 + \kappa_1\kappa_3}{\kappa_3},\ \widetilde{\gamma} = \frac{\kappa_3}{4\kappa_2},\ \text{and } \nu = 8\frac{{\kappa_2}^3}{{\kappa_3}^2}. \qquad (23)$$

Non-central χ^2 approximation

Pdf of the jth eigenvalue

$$p_{\lambda_j}(u) \approx \frac{1}{\gamma_j} p_{Q_j}\left(\frac{u-\eta_j}{\gamma_j}\right) \tag{24}$$

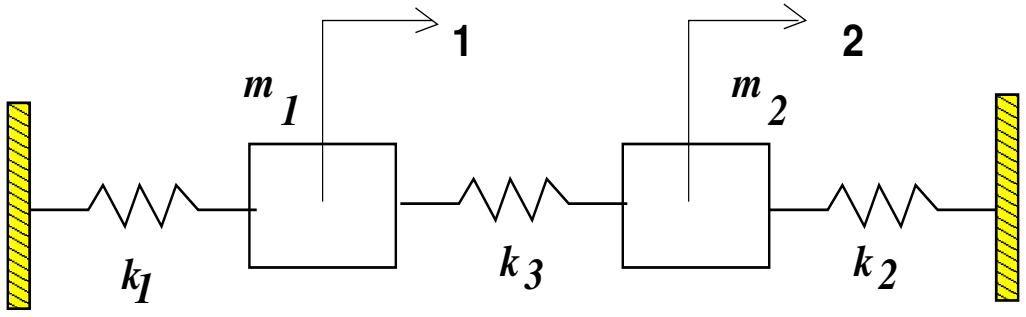
where

$$p_{Q_j}(u) = \frac{e^{-(\delta_j+u/2)} u^{m/2-1}}{2^{m/2}} \sum_{r=0}^{\infty} \frac{(\delta u)^r}{r!\, 2^r \Gamma(m/2+r)}. \tag{25}$$

where $\eta_j = c_j - \frac{1}{2} a_j^T \mathbf{A}_j^{-1} a_j$, $\gamma_j = \frac{\mathrm{Trace}(\mathbf{A}_j)}{2m}$, $\delta_j^2 = \boldsymbol{\rho}_j^T \boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_j = \mathbf{A}_j^{-1} a_j$.

Numerical Example

Undamped two degree-of-system system: $m_1 = 1$ Kg, $m_2 = 1.5$ Kg, $\bar{k}_1 = 1000$ N/m, $\bar{k}_2 = 1100$ N/m and $k_3 = 100$ N/m.



Only the stiffness parameters k_1 and k_2 are uncertain: $k_i = \bar{k}_i(1 + \epsilon_i x_i)$, $i = 1, 2$. $\mathbf{x} = \{x_1, x_2\}^T \in \mathbb{R}^2$ and the 'strength parameters' $\epsilon_1 = \epsilon_2 = 0.25$.

Numerical Example

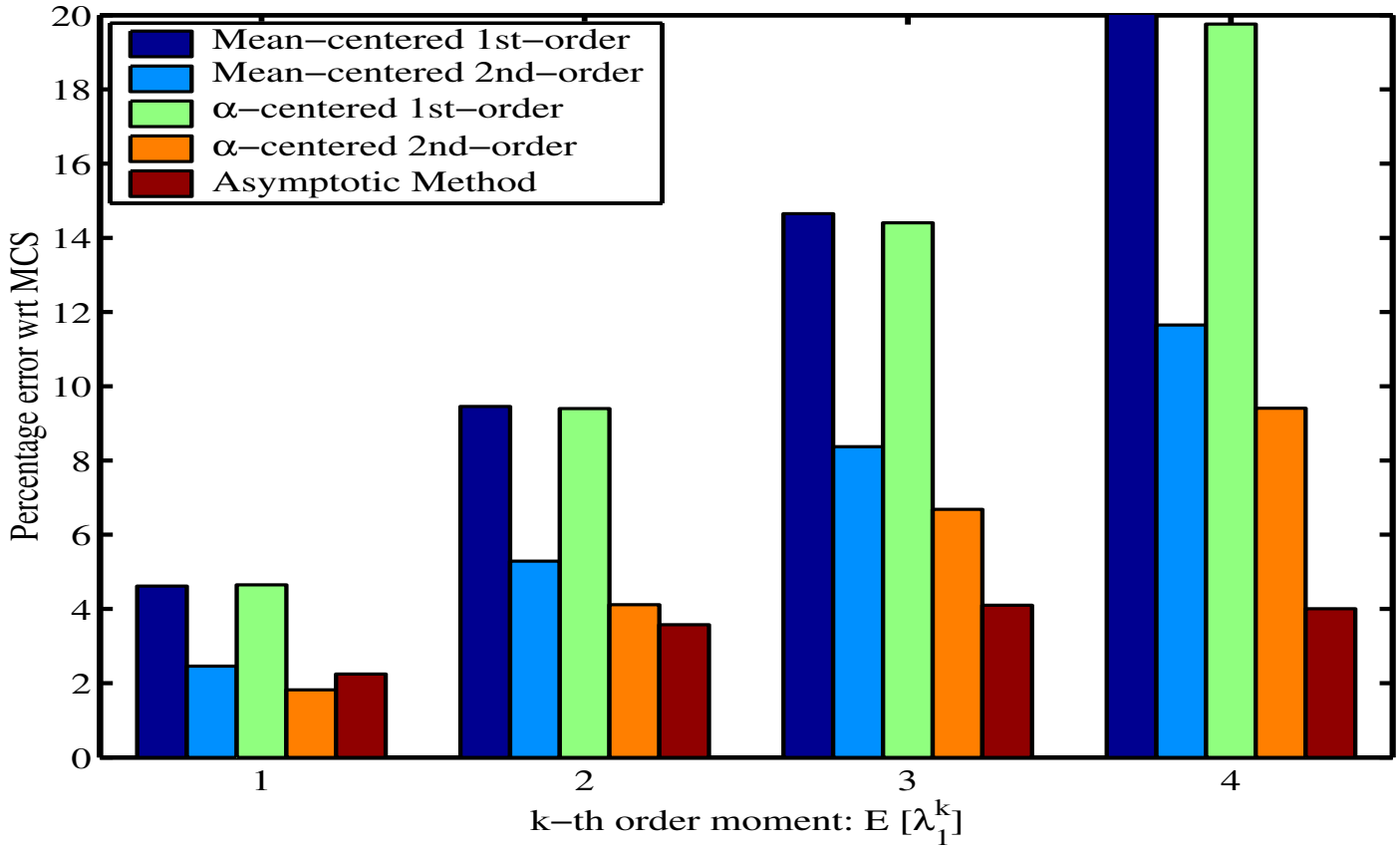
Following six methods are compared

1. *Mean-centered first-order perturbation*
2. *Mean-centered second-order perturbation*
3. α*-centered first-order perturbation*
4. α*-centered second-order perturbation*
5. *Asymptotic method*
6. *Monte Carlo Simulation (10K samples)* - can be considered as benchmark.

The percentage error:

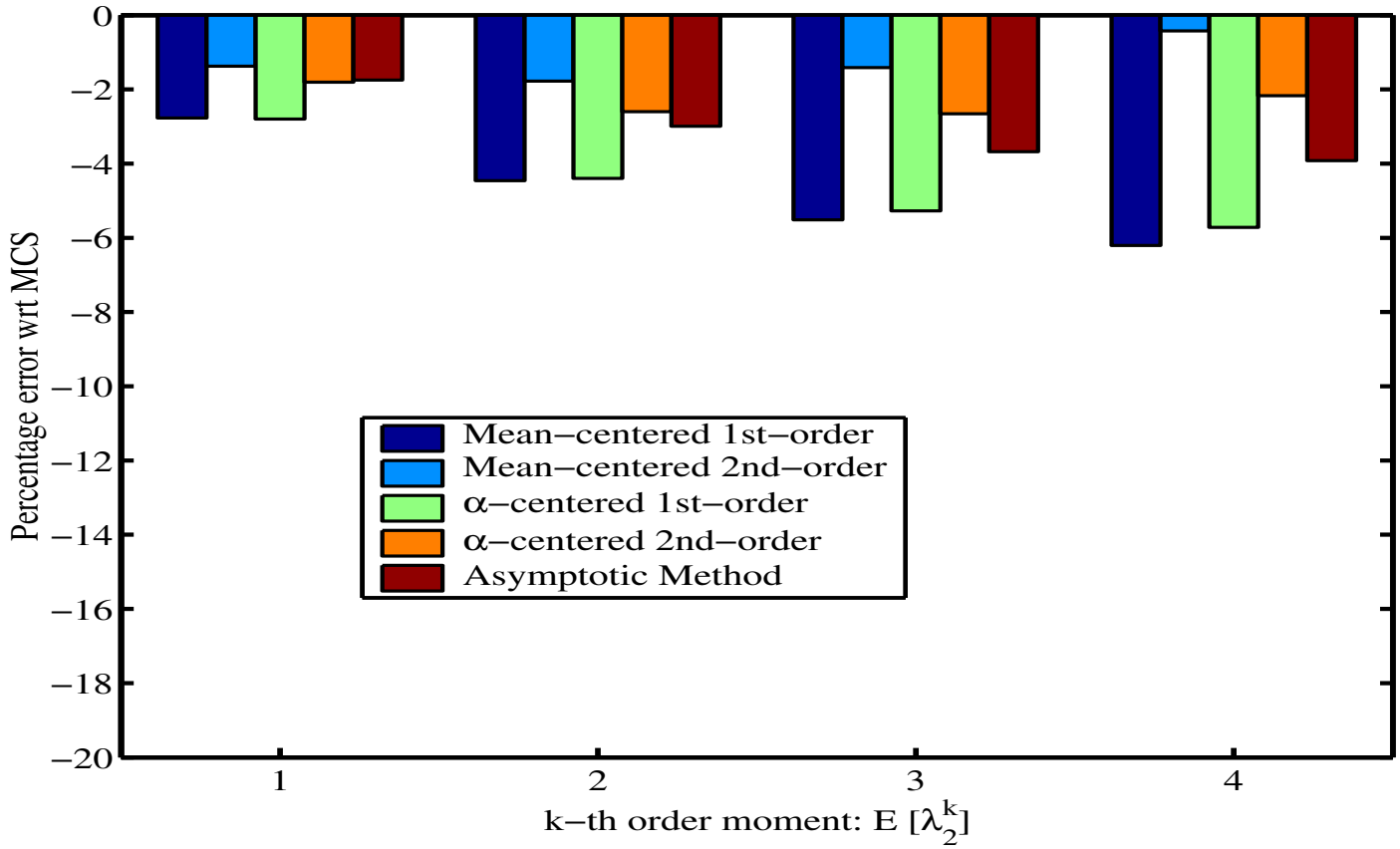
$$\text{Error}_{i\text{th method}} = \frac{\{\mu'_k\}_{i\text{th method}} - \{\mu'_k\}_{\text{MCS}}}{\{\mu'_k\}_{\text{MCS}}} \times 100$$

Numerical Example



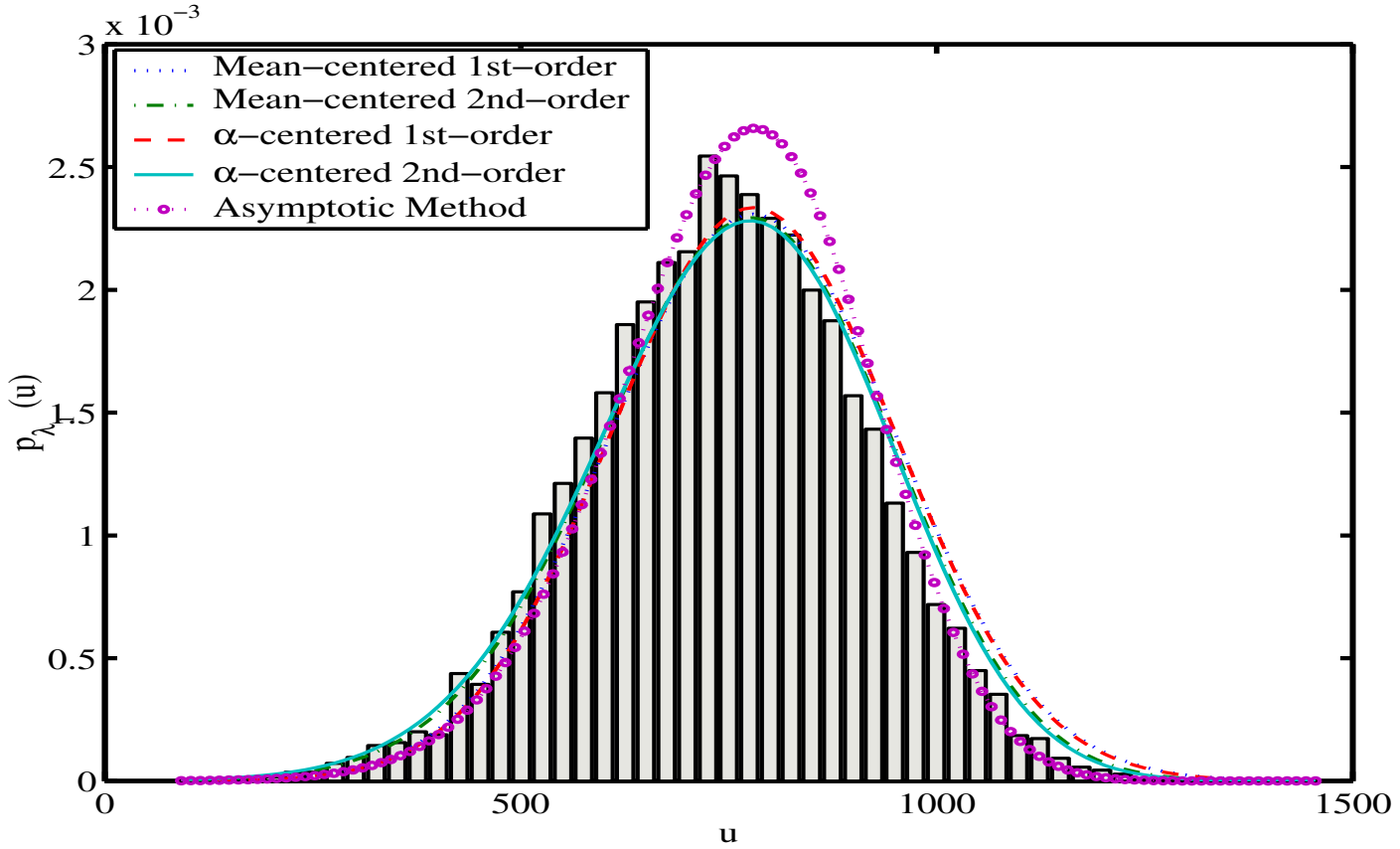
Percentage error for the first four raw moments of the first eigenvalue

Numerical Example



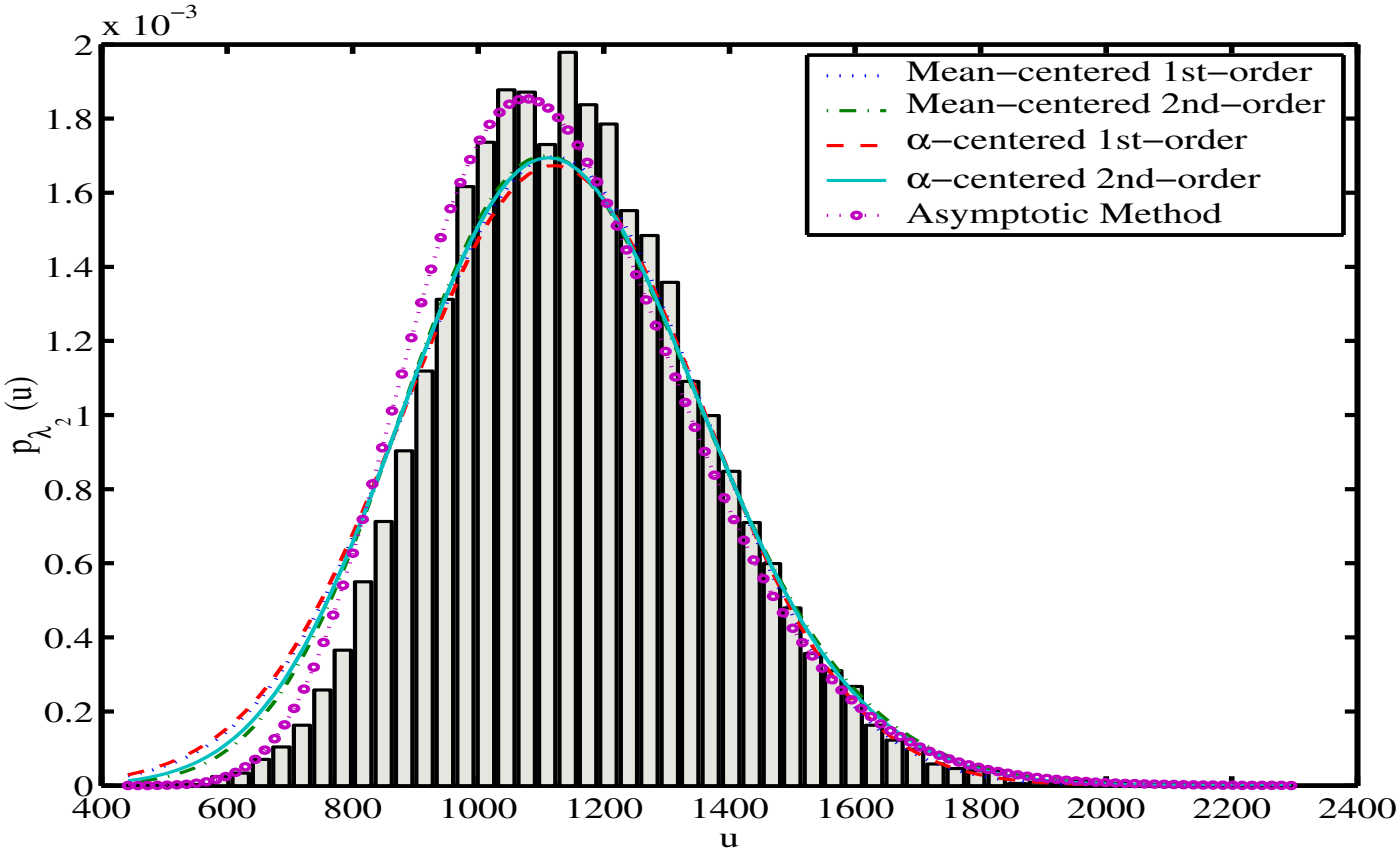
Percentage error for the first four raw moments of the second eigenvalue

Numerical Example



Probability density function of the first eigenvalue

Numerical Example



Probability density function of the second eigenvalue

Conclusions & Future Research

Two methods, namely (a) optimal point expansion method, and (b) asymptotic moment method, are proposed

The optimal point is obtained so that the mean of the eigenvalues are estimated most accurately.

The asymptotic method assumes that the eigenvalues are large compared to their 3rd order or higher derivatives.

Pdf of the eigenvalues are obtained in terms of central and non-central χ^2 densities.