Beyond the stochastic finite element method: hybrid uncertainty quantification using random PDEs

S Adhikari[1] B Pascual[1]

[1]School of Engineering, Swansea University, Singleton Park, Swansea SA2 8PP, UK

ECCM 2010: IV European Conference on Computational Mechanics, Paris

Outline

Parametric uncertainty

There many uncertain parameters in the computational model of an automobile. These include thickness, Young's modulus, Poisson's ratio, damping coefficients.

Models of parametric uncertainty

Parametric uncertainty can be modeled using either random variables or random fields.

- A random variable $\xi(\omega)$ is a measurable function defined on a probability space $(\Omega, \mathcal{F}, P)$, $\xi : \Omega \to V$ with V a measure space. It induces a probability measure on V.
- A random variable can be used to model uncertainties in discrete parameters.
- A random field $a(x, \omega)$ is, a collection of random variables indexed by x related to the spatial domain of the system.
- A random field can be used to model uncertainties in distributed parameters of a system.
- Classical stochastic finite element method (SFEM) can be used address systems with such uncertainties.

Non-parametric uncertainty

Complex aerospace systems can have millions of degrees of freedom and significant uncertainty in its numerical (Finite Element) model

Possible sources of non-parametric uncertainty

(a) model inadequacy - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(b) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(c) computational uncertainty - e.g, machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis, and
(d) model uncertainty - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.

Models of non-parametric uncertainty

- Non-parametric uncertainties arise due to the lack of knowledge.
- This can be modeled applying the maximum entropy principle to the system matrices (such as the stiffness matrix)
- Suppose we know the mean ($\mathbf{K}_0$) and the (normalized) standard deviation or dispersion parameter (δ_G) of the system matrix:

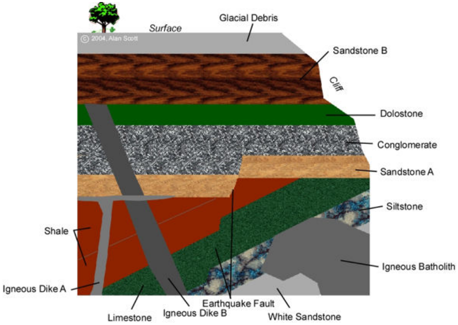
$$\delta_K^2 = \frac{\mathrm{E}\left[\|\mathbf{K} - \mathrm{E}\left[\mathbf{K}\right]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{K}\right]\|_{\mathrm{F}}^2} \tag{1}$$

- Using further constrains such as **K** is symmetric and non-negative definite, it can be shown that **K** belongs to the so called Wishart random matrix ensemble.

Hybrid uncertainty

- Although we have mentioned and made differences between the two different types of uncertainties, in practical problems it is in general very difficult, if not impossible, to distinguish them.
- From both qualitative and quantitative point of view, a random field model cannot encompass different types of uncertainty that may exist in a computational physics problem.
- Even if uncertainty is well represented by parametric uncertainty in a subdomain of the system, uncertainty in the modeling of the remaining domain can be present.
- If parametric uncertainty is considered, uncertainty associated with the random field model can appear due to the lack of data.
- We need to quantify and model both types of uncertainties simultaneously

Hybrid uncertainty over the entire domain



Consider flow through layers of random soil stratum. One can model permeability of the soil stratum as a random field but there may be uncertainty associated with the random field model itself due to the lack of data. In this case parametric and non-parametric uncertainties cover the entire domain.

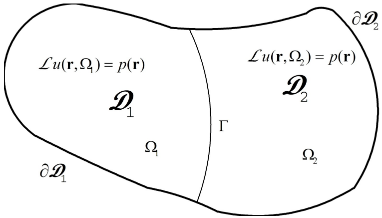
Hybrid uncertainty over non-overlapping subdomains

For the wing and engine problem, the scenario can be somewhat different from the previous case. Here one may have a reasonable random field model for the wing, but uncertainty in the modeling of the engine arise due to its sheer complexity and multiphysics nature. In this case parametric and non-parametric uncertainties cover two non-overlapping subdomains.

Elliptic PDE over non-overlapping subdomains

- We consider two subdomains $\mathscr{D}_1, \mathscr{D}_2 \in \mathscr{D}$ such that $\mathscr{D}_1 \bigcap \mathscr{D}_2 = \emptyset$. For the case of hybrid uncertainty over two non-overlapping subdomains, consider the following two problems for $j = 1, 2$:

$$-\nabla \left[a(\mathbf{r}, \omega_j) \nabla u(\mathbf{r}, \omega_j) \right] = f(\mathbf{r}); \ \mathbf{r} \, \text{in} \, \mathscr{D}_j; \ u(\mathbf{r}, \omega_j) = 0 \text{ on } \partial \mathscr{D}_j$$



- The main idea here is that the subdomain $\mathscr{D}_1$ with parametric uncertainty is expressed using the Karhunen-Loève expansion of the underlying random field, while the discreized matrix corresponding to subdomain $\mathscr{D}_2$ with non-parametric uncertainty is expressed using Wishart distribution.

Hybrid uncertainty over two non-overlapping subdomains

- An algebraic equation can be obtained after applying the finite element method, where the random field used to model parametric uncertainty has been expanded using the Karhunen-Loève expansion.
- When non-parametric uncertainty affects $\mathbf{K}_{22}(\Omega_2)$, it can be modelled with a Wishart random matrix.
- We use Polynomial Chaos expansion with parametric random variables.

Overall, the discretized equation can be expressed as:

$$\begin{bmatrix} [\mathbf{K}_{11_0} + \epsilon \sum_{i=1}^{M} \xi_i(\Omega_1)\mathbf{K}_{11_i}] & [\mathbf{K}_{12}] \\ [\mathbf{K}_{21}] & [\mathbf{K}_{22}(\Omega_2)] \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{P} \mathbf{u}_{1j}(\Omega_2)\Psi_j(\Omega_1) \\ \sum_{j=1}^{P} \mathbf{u}_{2j}(\Omega_2)\Psi_j(\Omega_1) \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \tag{2}$$

Hybrid uncertainty over two non-overlapping subdomains

- Both uncertainties are solved separately, as they are assumed independent. After some algebraic manipulations:

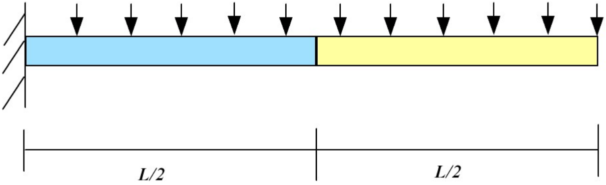
$$\hat{\mathbf{K}} \begin{bmatrix} \mathbf{u}_{11} \\ \vdots \\ \mathbf{u}_{1P} \end{bmatrix} = \begin{bmatrix} < \Psi_1, \mathbf{f}_1 > - \mathbf{K}_{12}\mathbf{K}_{22}(\Omega_2)^{-1} < \Psi_1, \mathbf{f}_2 > \\ \vdots \\ < \Psi_P, \mathbf{f}_1 > - \mathbf{K}_{12}\mathbf{K}_{22}(\Omega_2)^{-1} < \Psi_P, \mathbf{f}_2 > \end{bmatrix} \tag{3}$$

with

$$\hat{\mathbf{K}} = (\mathbf{K}_{11_0} - \mathbf{K}_{12}\mathbf{K}_{22}(\Omega_2)^{-1}\mathbf{K}_{21}) \otimes \mathbf{D} + \sum_{i=1}^{k} \mathbf{c}_i \otimes \mathbf{K}_{11_i} \tag{4}$$

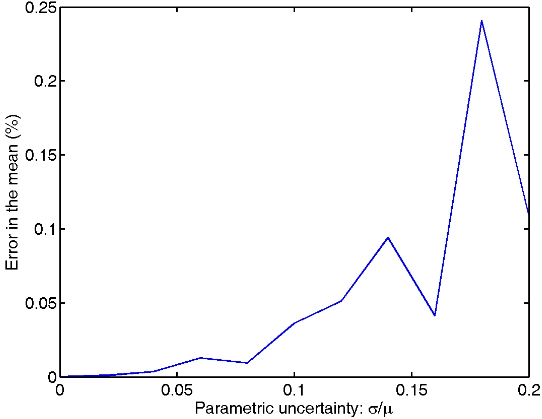
- The entries of the diagonal matrix $\mathbf{D}$ are $\mathrm{E}\left[\Psi_i^2\right]$ and the elements of matrices $\mathbf{c}_i$ are given by $\mathrm{E}\left[\xi_i \Psi_j \Psi_k\right]$. First and second moments of $\mathbf{u}$ are obtained through MCS, where $\mathbf{K}_{22} \backsim W_n(p, \mathbf{K}_{22}/p)$ is simulated.

An Euler-Bernoulli beam with hybrid uncertainty



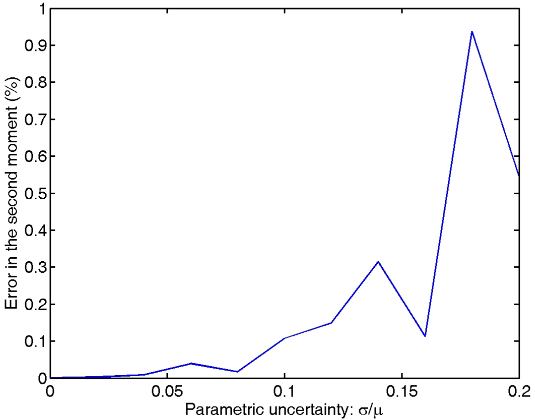
- A clamped-free beam with 20 elements is considered. Uniformly distributed load is assumed. The Karhunen-Loève expansion has 2 terms and fourth-order Polynomial Chaos is considered.
- The bending rigidity of the first part of the beam is assumed to be a Gaussian random field. The second part of the beam is assumed to have non-parametric uncertainty.
- Different values of the standard deviation of the random field (σ) and dispersion parameter of the Wishart matrix (δ) are considered.

Hybrid uncertainty over non-overlapping subdomains



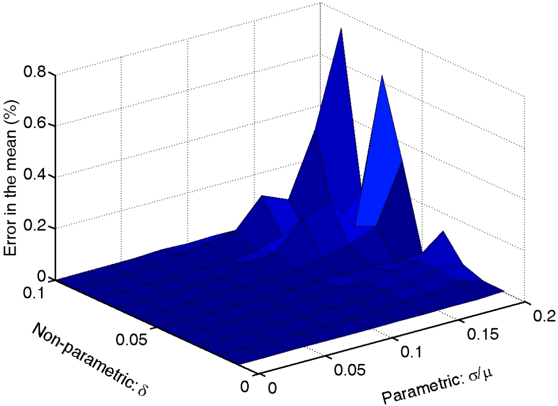
Error in the mean of the tip deflection where only $\mathbf{K}_{11}$ is random (that is the beam with the first half of the length has parametric uncertainty).

Hybrid uncertainty over non-overlapping subdomains



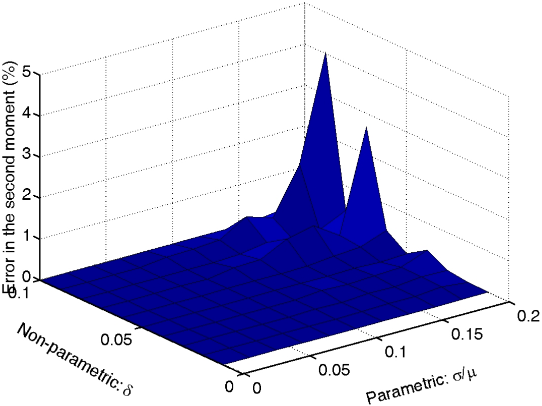
Error in the second-moment of the tip deflection where only $\mathbf{K}_{11}$ is random (that is the beam with the first half of the length has parametric uncertainty).

Hybrid uncertainty over non-overlapping subdomains



Error in the mean of the tip deflection for different values of the standard deviation of the random field (σ) and dispersion parameter of the Wishart matrix (δ).

Hybrid uncertainty over non-overlapping subdomains

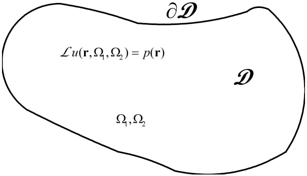


Error in the second-moment of the tip deflection for different values of the standard deviation of the random field (σ) and dispersion parameter of the Wishart matrix (δ).

Hybrid uncertainty over the entire domain

- For the case of hybrid uncertainty over the entire domain, we consider the elliptic partial differential equation with Dirichlet boundary condition

$$-\nabla\left[a(\mathbf{r},\omega_1,\omega_2)\nabla u(\mathbf{r},\omega_1,\omega_2)\right] = f(\mathbf{r}); \quad \mathbf{r} \text{ in } \mathscr{D}; \, u(\mathbf{r},\omega_1,\omega_2) = 0 \text{ on } \partial\mathscr{D} \tag{5}$$



- Discretising the equation we have

$$\mathbf{Ku} = \mathbf{f} \tag{6}$$

where from the point of non-parametric uncertainty, the 'mean' is $\mathbf{K}_0 + \sum \xi_i(\omega_i)\mathbf{K}_i$ and the dispersion parameter is δ. The mean matrix therefore contain the parametric uncertainty information.

Hybrid uncertainty over the entire domain

- Using this information and following the standard procedure of non-parametric formulation based on the maximum entropy principle, we have $\mathbf{K} \backsim W_n(p, \mathbf{K}'(\omega_1)/p)$ where $\mathbf{K}'(\omega_1) = \mathbf{K}_0 + \sum \xi_i(\omega_1)\mathbf{K}_i$
- Mean of the response:

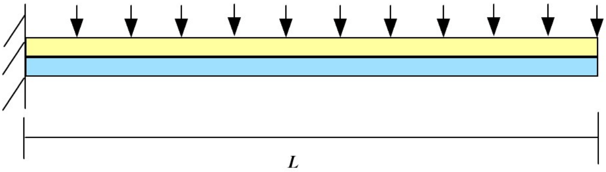
$$\mathrm{E}\left[\mathbf{u}\right] = \frac{\mathrm{E}\left[\mathbf{K}'^{-1}\right]/p}{p-n-1}\mathbf{f}$$

- Second-moment of the response:

$$\mathrm{E}\left[\mathbf{u}\mathbf{u}^T\right] = \frac{c_1+c_2}{p^2}\mathrm{E}\left[\mathbf{K}'^{-1}\mathbf{A}\mathbf{K}'^{-1}\right] + \frac{c_2}{p^2}\mathrm{E}\left[\mathrm{Trace}\left(\mathbf{A}\mathbf{K}'^{-1}\right)\mathbf{K}'^{-1}\right]$$

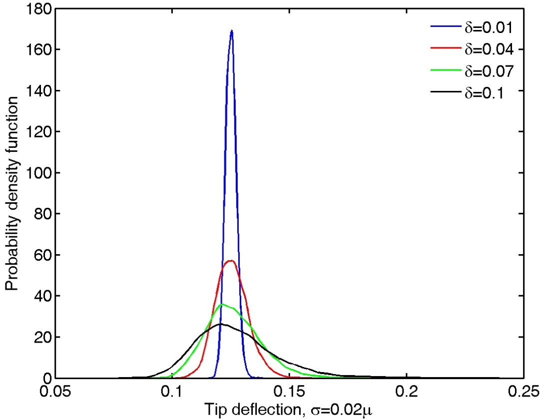
  with $\mathbf{A} = \mathbf{f}\mathbf{f}^T$.
- Here $\mathrm{E}\left[\mathbf{K}'^{-1}\right]$, $\mathrm{E}\left[\mathbf{K}'^{-1}\mathbf{A}\mathbf{K}'^{-1}\right]$ and $\mathrm{E}\left[\mathrm{Trace}\left(\mathbf{A}\mathbf{K}'^{-1}\right)\mathbf{K}'^{-1}\right]$ are approximated using the Polynomial Chaos expansion.

An Euler-Bernoulli beam with hybrid uncertainty



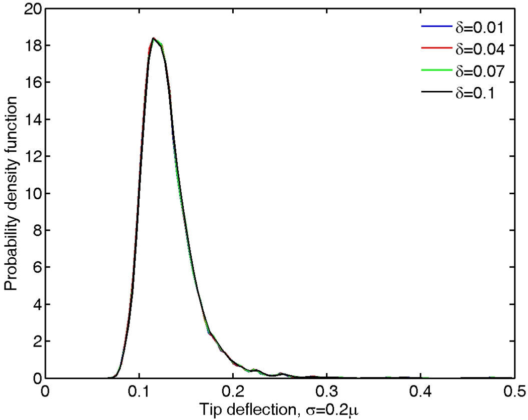
- A clamped-free beam with 20 elements is considered. Uniformly distributed load is assumed.
- The bending rigidity of the beam is assumed to be a Gaussian random field. In addition the beam has non-parametric uncertainty, characterized by the dispersion parameter δ.
- Different values of the standard deviation of the random field (σ) and dispersion parameter of the Wishart matrix (δ) are considered.

Hybrid uncertainty over the entire domain



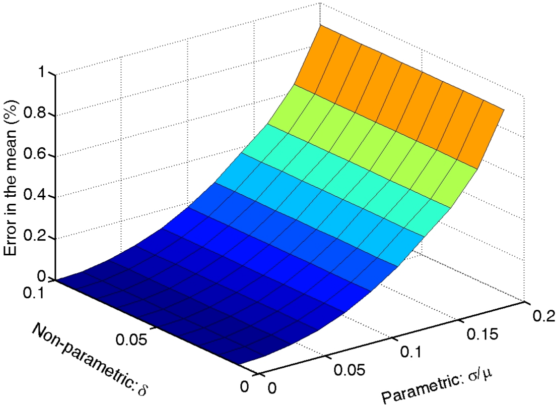
Probability density function of the tip deflection for different values of the dispersion parameter of the Wishart matrix (δ) for $\sigma = 0.02$ times the mean of the random field (μ).

Hybrid uncertainty over the entire domain



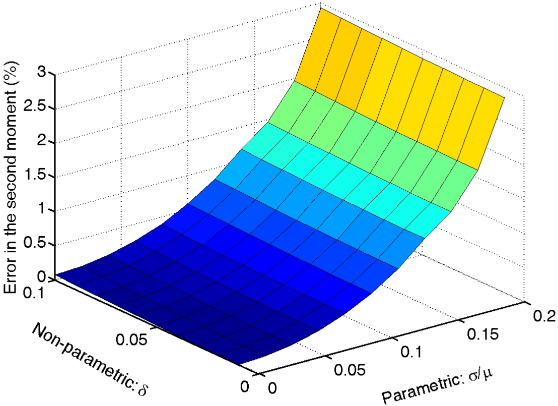
Probability density function of the tip deflection for different values of the dispersion parameter of the Wishart matrix (δ) for $\sigma = 0.2$ times the mean of the random field (μ).

Hybrid uncertainty over the entire domain



Error in the mean of the tip deflection for different values of the standard deviation of the random field (σ) and dispersion parameter of the Wishart matrix (δ).

Hybrid uncertainty over the entire domain



Error in the second-moment of the tip deflection for different values of the standard deviation of the random field (σ) and dispersion parameter of the Wishart matrix (δ).

Summary & future directions

1. Classical stochastic finite element method need to be extended to consider both parametric and non-parametric uncertainties simultaneously.
2. Two possible scenarios involving hybrid uncertainties have been considered - (a) each type uncertainty is confined within non-overlapping subdomains, and (b) both type uncertainties cover the entire domain.
3. Parametric uncertainties are modellled using random fields and non-parametric uncertainties are modeled using random matrix theory.
4. Numerical methods based on Polynomial Chaos and random matrix theory haven been proposed for both cases.

Acknowledgements

Swansea University
Prifysgol Abertawe

THE ROYAL SOCIETY
CELEBRATING 350 YEARS

The Leverhulme Trust

EPSRC
Pioneering research and skills