Reduction of Random Variables in Structural Reliability Analysis

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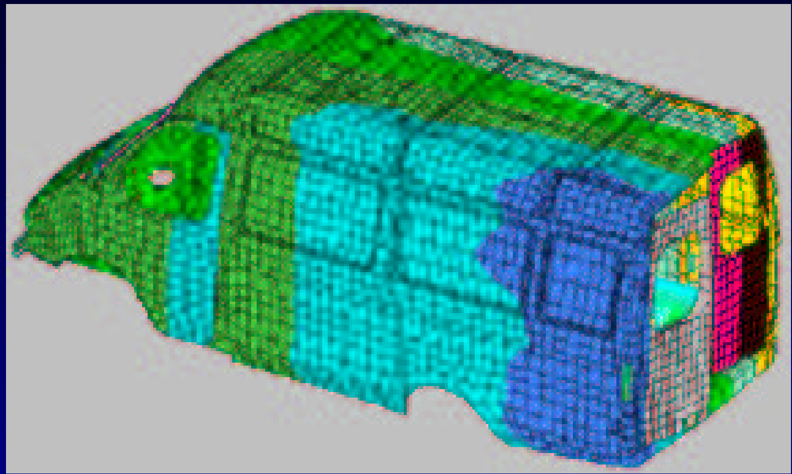
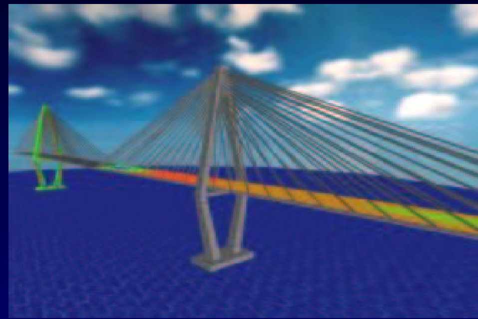
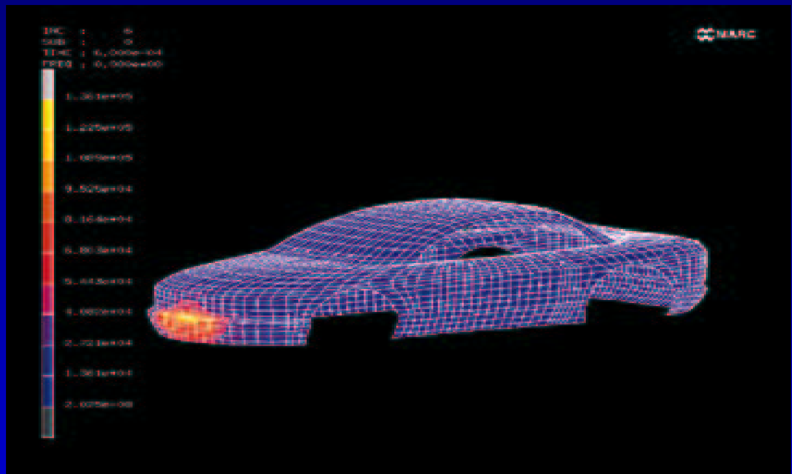
Outline of the Talk

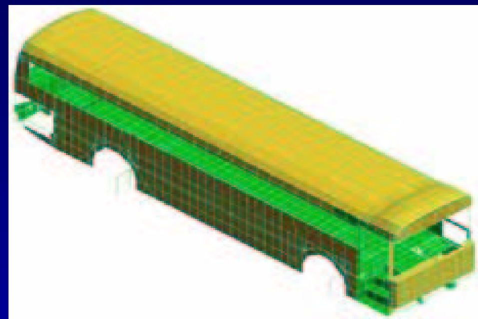
- Introduction
- Approximate Reliability Analyses: FORM and SORM
- Proposed Reduction Techniques
- Numerical examples
- Conclusions

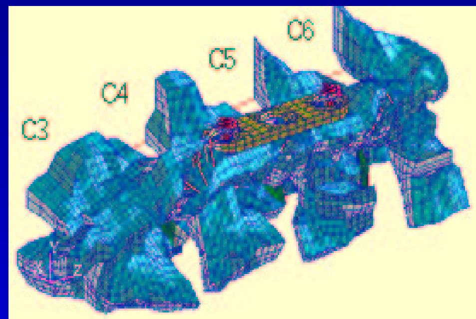
Structural Reliability Analysis

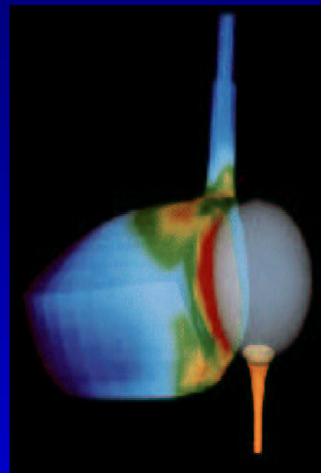












Finite Element models of some engineering structures

The Fundamental Problem

Probability of failure:

$$P_f = \int_{G(\mathbf{y})\leq 0} p(\mathbf{y}) d\mathbf{y} \tag{1}$$

- $\mathbf{y} \in \mathbb{R}^n$: vector describing the uncertainties in the structural parameters and applied loadings.
- $p(\mathbf{y})$: joint probability density function of $\mathbf{y}$
- $G(\mathbf{y})$: failure surface/limit-state function/safety margin/

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Main Difficulties

- n is large
- $p(\mathbf{y})$ is non-Gaussian
- $G(\mathbf{y})$ is a complicated nonliner function of $\mathbf{y}$

Approximate Reliability Analyses

First-Order Reliability Method (FORM):

- Requires the random variables $\mathbf{y}$ to be Gaussian.
- Approximates the failure surface by a hyperplane.

Second-Order Reliability Method (SORM):

- Requires the random variables $\mathbf{y}$ to be Gaussian.
- Approximates the failure surface by a quadratic hypersurface.

Asymptotic Reliability Analysis (ARA):

- The random variables $\mathbf{y}$ can be non-Gaussian.
- Accurate only in an asymptotic sense.

FORM

- Original non-Gaussian random variables $\mathbf{y}$ are transformed to standardized gaussian random variables $\mathbf{x}$. This transforms $G(\mathbf{y})$ to $g(\mathbf{x})$.
- The probability of failure is given by

$$P_f = \Phi(-\beta) \quad \text{with} \quad \beta = (\mathbf{x}^{*^T}\mathbf{x}^*)^{1/2} \qquad (2)$$

where $\mathbf{x}^*$, the design point is the solution of following optimization problem:

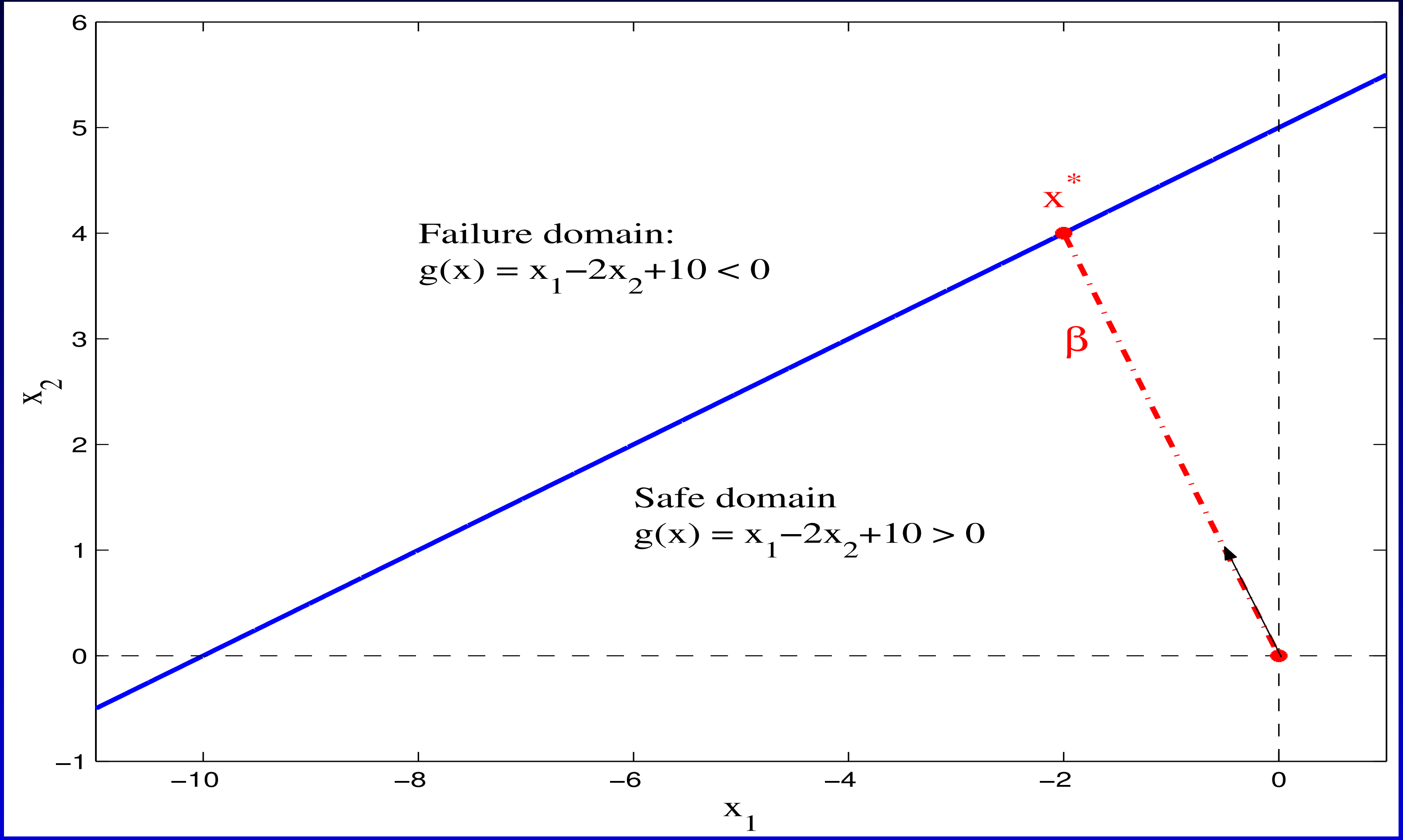
$$\min \left\{ (\mathbf{x}^T\mathbf{x})^{1/2} \right\} \quad \text{subject to} \quad g(\mathbf{x}) = 0. \qquad (3)$$

Gradient Projection Method

- Uses the gradient of $g(\mathbf{x})$ noting that ∇g is independent of $\mathbf{x}$ for linear $g(\mathbf{x})$.
- For nonlinear $g(\mathbf{x})$, the design point is obtained by an iterative method.
- Reduces the number of variables to 1 in the constrained optimization problem.
- Is expected to work well when the failure surface is 'fairly' linear.

Example 1

Linear failure surface in $\mathbb{R}^2$: $g(\mathbf{x}) = x_1 - 2x_2 + 10$



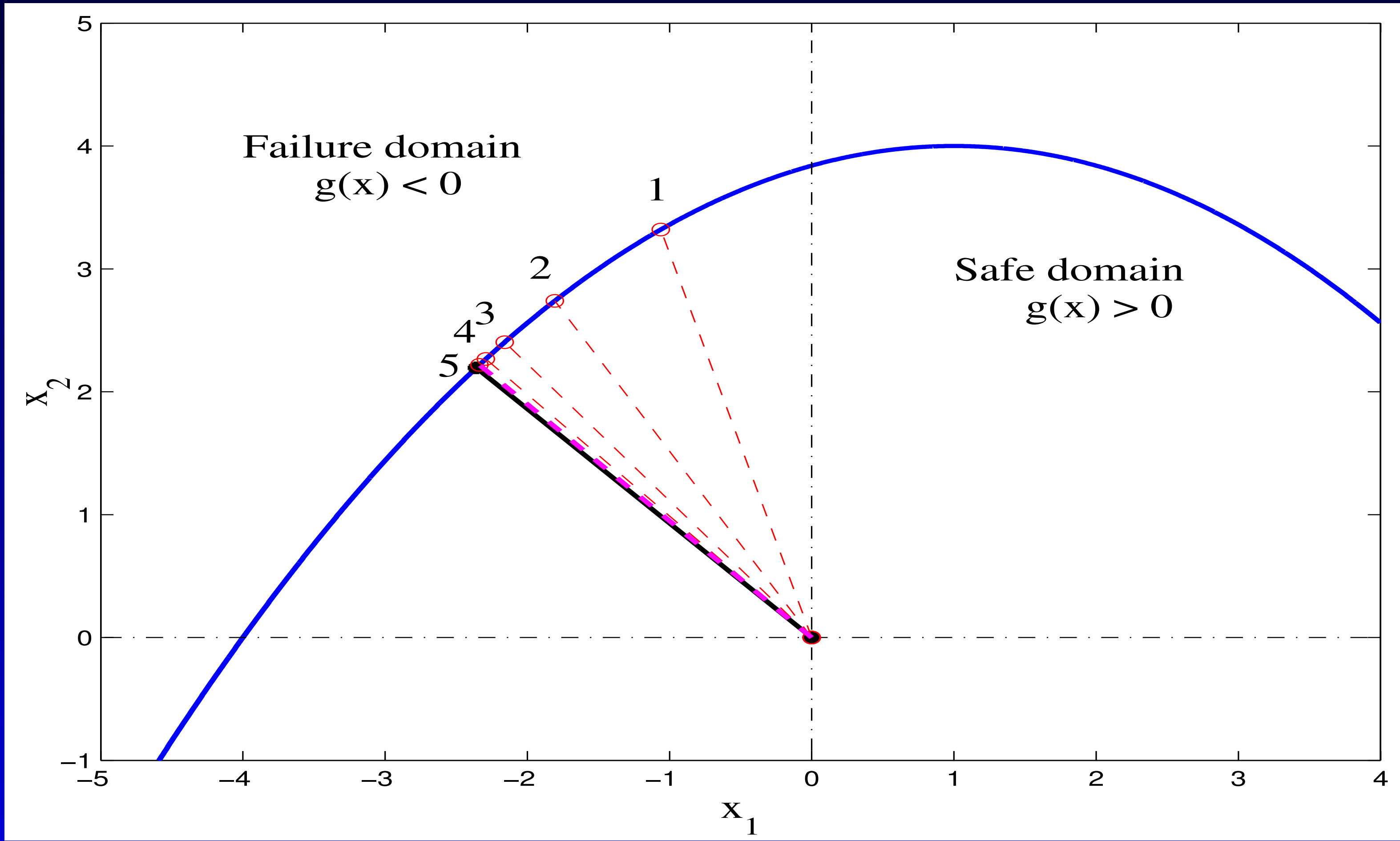
$\mathbf{x}^* = \{-2, 4\}^T$ and $\beta = 4.472$.

Main Steps

1. For $k = 0$, select $\mathbf{x}^{(k)} = \mathbf{0}$, a small value of ϵ, (say 0.001) and a large value of $\beta^{(k)}$ (say 10).
2. Construct the normalized vector $\boldsymbol{\nabla} g^{(k)} = \left\{ \frac{\partial g(\mathbf{X})}{\partial x_i} |_{\mathbf{X}=\mathbf{X}^{(k)}} \right\}, \forall i = 1, .., n$ so that $|\boldsymbol{\nabla} g^{(k)}| = 1$.
3. Solve $g(v \boldsymbol{\nabla} g^{(k)}) = 0$ for v.
4. Increase the index: $k = k + 1$; denote $\beta^{(k)} = -v$ and $\mathbf{x}^{(k)} = v \boldsymbol{\nabla} g^{(k)}$.
5. Denote $\delta\beta = \beta^{(k-1)} - \beta^{(k)}$.
6. (a) If $\delta\beta < 0$ then the iteration is going in the wrong direction. Terminate the iteration procedure and select $\beta = \beta^{(k)}$ and $\mathbf{x}^* = \mathbf{x}^{(k)}$ as the best values of these quantities.
   (b) If $\delta\beta < \epsilon$ then the iterative procedure has converged. Terminate the iteration procedure and select $\beta = \beta^{(k)}$ and $\mathbf{x}^* = \mathbf{x}^{(k)}$ as the final values of these quantities.
   (c) If $\delta\beta > \epsilon$ then go back to step 2.

Example 2

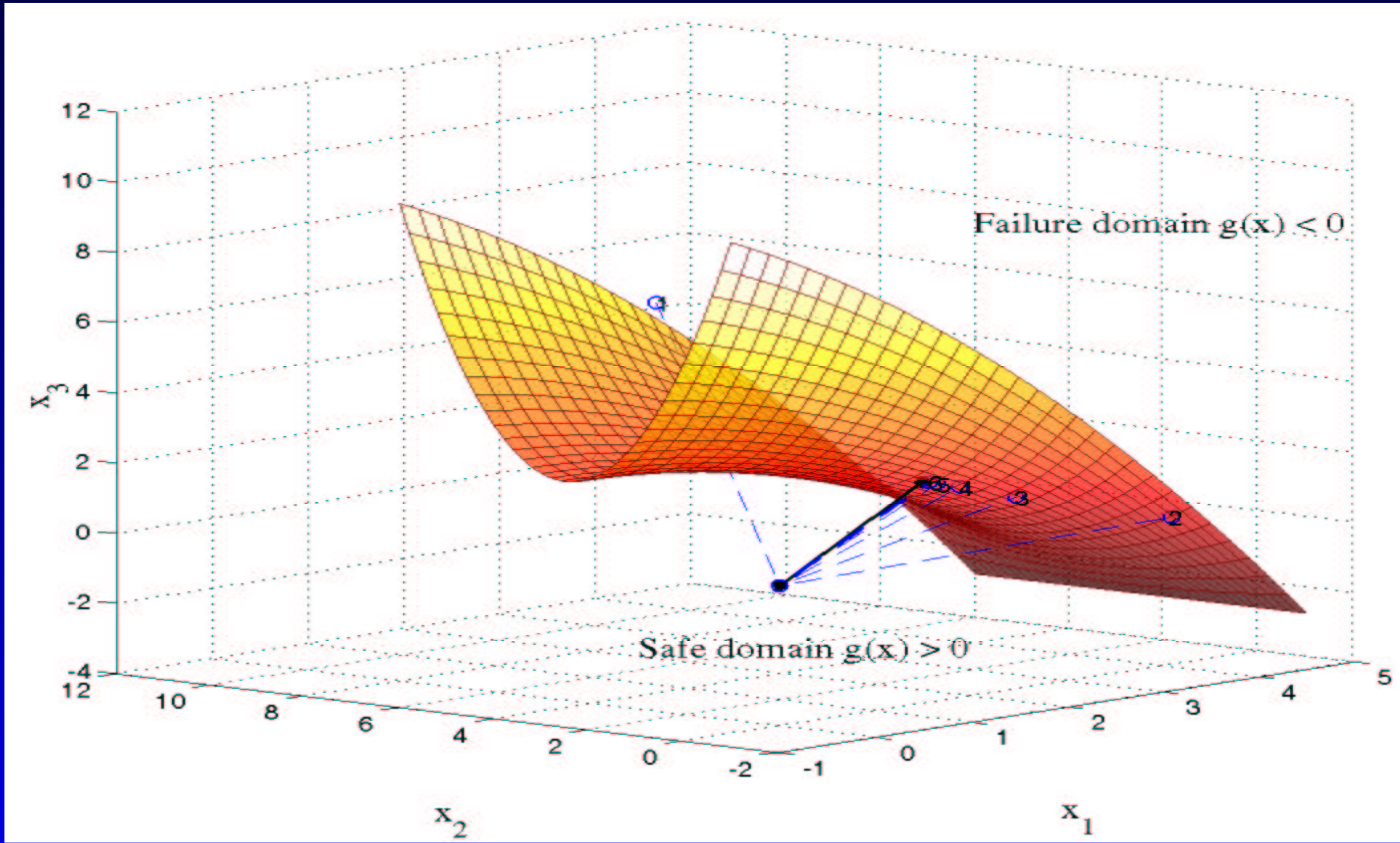
$$g(\mathbf{x}) = -\frac{4}{25}(x_1 - 1)^2 - x_2 + 4$$



$$\mathbf{x}^* = \{-2.34, 2.21\}^T \text{ and } \beta = 3.22.$$

Example 3

$$g(\mathbf{x}) = -\frac{4}{25}(x_1+1)^2 - \frac{(x_2-5/2)^2(x_1-5)}{10} - x_3 + 3$$



$\mathbf{x}^* = \{2.1286, 1.2895, 1.8547\}^T$ and $\beta = 3.104$.

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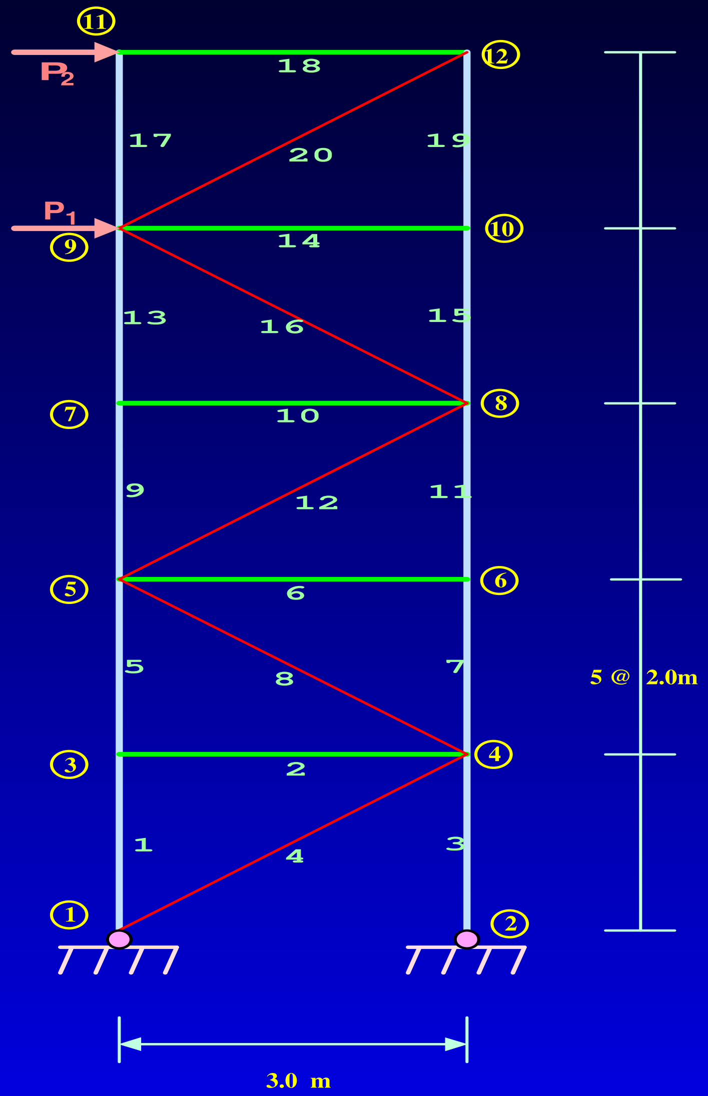
Dominant Gradient Method

- More than one random variable is kept in the constrained optimization problem.
- Dominant random variables are those for which the failure surface is most sensitive.
- Variables for which the failure surface is less sensitive is *removed* in the constrained optimization problem.
- Is expected to work well for near-linear failure surface.

Relative Importance Variable Method

- Based on the entries of ∇g the random variables are grouped into 'important' and 'unimportant' random variables.
- Unimportant random variables are not completely neglected but represented by a single random variable.
- Is expected to work well for near-linear failure surface.

Multistoried Portal Frame



Nel=20, Nnode=12

$P_1 = 4.0 \times 10^5$KN, $P_2 = 5.0 \times 10^5$KN

Random Variables:

Axial stiffness (EA) and the bending stiffness (EI) of each member are uncorrelated Gaussian random variables (Total $2 \times 20 = 40$ random variables: $\mathbf{x} \in \mathbb{R}^{40}$).

Element Type	EA (KN) Mean	EA (KN) Standard Deviation	EI (KNm^2) Mean	EI (KNm^2) Standard Deviation
1	5.0×10^9	7.0%	6.0×10^4	5.0%
2	3.0×10^9	3.0%	4.0×10^4	10.0%
3	1.0×10^9	10.0%	2.0×10^4	9.0%

Failure surface:

$g(\mathbf{x}) = d_{max} - |\delta h_{11}(\mathbf{x})|$

δh_{11}: the horizontal displacement at node 11

$d_{max} = 0.184 \times 10^{-2}$m

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Multistoried Portal Frame

Results (with one iteration)

	Method 1	Method 2	Method 3	FORM	MCS‡
	$(n_{\text{reduced}} = 1)$	$n_d = 5$	$n_d = 5$	$n = 40$	(exact)
β	3.399	3.397	3.397	3.397	–
$P_f \times 10^3$	0.338	0.340	0.340	0.340	0.345

‡ with 11600 samples (considered as benchmark)

Conclusions & Future Research

- Three iterative methods, namely (a) gradient projection method, (b) dominant gradient method, and (c) relative importance variable method, have been proposed to reduce the number of random variables in structural reliability problems involving a large number of random variables.
- All the three methods are based on the sensitivity vector of the failure surface.
- Initial numerical results show that there is a possibility to put these methods into real-life problems involving a large number of random variables.

Conclusions & Future Research

- Future research will address reliability analysis of more complicated and large systems using the proposed methods. This would be achieved by using currently existing commercial Finite Element softwares.
- Applicability and/or efficiency of the proposed methods to problems with highly non-linear failure surfaces, for example, those arising in structural dynamic problems will be investigated.