On the Validity of Random Matrix Models in Probabilistic Structural Dynamics

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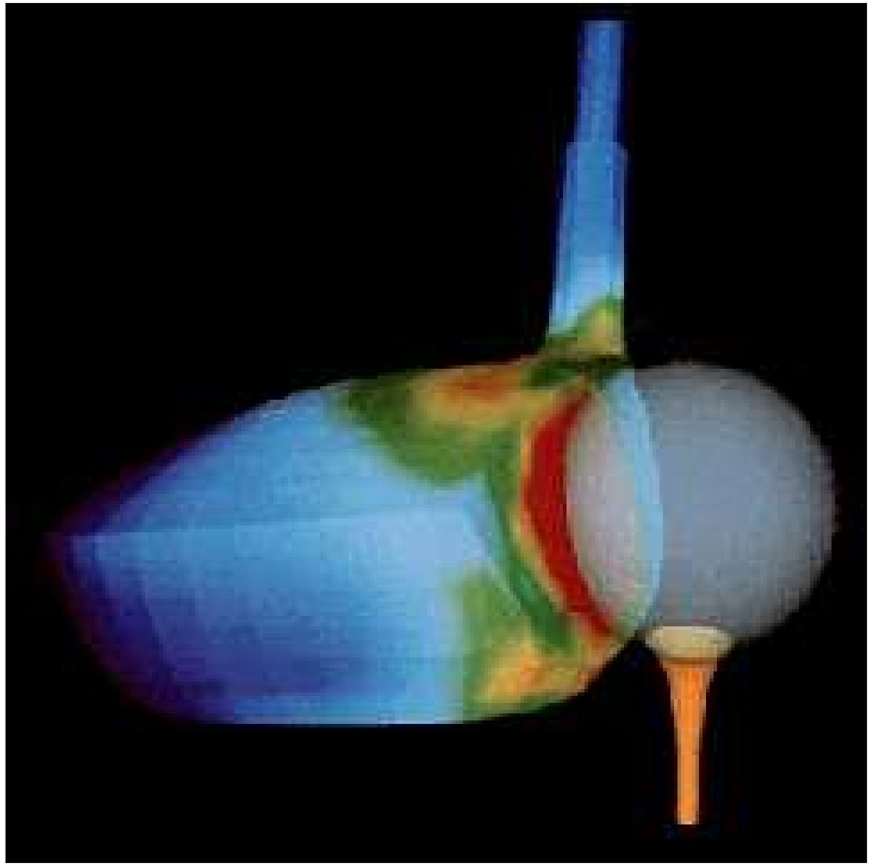
Outline of the presentation

- Introduction: current status and challenges
- Uncertainty Propagation (UP) in structural dynamics
- Wishart random matrices
  - Analytical derivation
  - Parameter selection
- Computational results
- Experimental results
- Conclusions & future directions

Ensembles of structural dynamical systems



Many structural dynamic systems are manufactured in a production line (nominally identical systems)

A complex structural dynamical system

Complex aerospace system can have millions of degrees of freedom and significant 'errors' and/or 'lack of knowledge' in its numerical (Finite Element) model

Sources of uncertainty

(a) parametric uncertainty - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) model inadequacy - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(d) computational uncertainty - e.g, machine precession, error tolerance and the so called 'h' and 'p' refinements in finite element analysis, and
(e) model uncertainty - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.

UP approaches: key challenges

The main difficulties are:

- the computational time can be prohibitively high compared to a deterministic analysis for real problems,
- the volume of input data can be unrealistic to obtain for a credible probabilistic analysis,
- the predictive accuracy can be poor if considerable resources are not spend on the previous two items, and
- the need for general purpose software tools: as the state-of-the art methodology stands now (such as the Stochastic Finite Element Method), only very few highly trained professionals (such as those with PhDs) can even attempt to apply the complex concepts (e.g., random fields) and methodologies to real-life problems.

The 10-10-10 challenge

Can we develop methodologies which will:

(a) not take more than 10 times the computational time required for the corresponding deterministic approach;

(b) result a predictive accuracy within 10% of direct Monte Carlo Simulation (MCS);

(c) use no more than 10 times of input data needed for the corresponding deterministic approach; and

(d) enable ‘normal’ engineering graduates to perform probabilistic structural dynamic analyses with a reasonable amount of training.

Current UP approaches - 1

Two different approaches are currently available

- Parametric approaches : Such as the Stochastic Finite Element Method (SFEM):
  - aim to characterize parametric uncertainty (type 'a')
  - assumes that stochastic fields describing parametric uncertainties are known in details
  - suitable for low-frequency dynamic applications (building under earthquake load, steering column vibration in cars)

Current UP approaches - 2

- Nonparametric approaches : Such as the Statistical Energy Analysis (SEA):
  - aim to characterize nonparametric uncertainty (types ‘b’ - ‘e’)
  - does not consider parametric uncertainties in details
  - suitable for high/mid-frequency dynamic applications (eg, noise propagation in vehicles)

Dynamics of a general linear system

The equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objectives in the 'forward problem' are:
  - to quantify uncertainties in the system matrices (and consequently in the eigensolutions)
  - to predict the variability in the response vector $\mathbf{q}$
- Probabilistic solution of this problem is expected to have more credibility compared to a deterministic solution

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} \left|\mathbf{\Sigma}\right|^{-p/2} \left|\mathbf{\Psi}\right|^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (2)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}\left(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi}\right)$.

Matrix variate Gamma distribution

A $n \times n$ symmetric positive definite matrix random $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a) \left|\mathbf{\Psi}\right|^{-a}\right\}^{-1} \left|\mathbf{W}\right|^{a-\frac{1}{2}(n+1)} \operatorname{etr}\left\{-\mathbf{\Psi}\mathbf{W}\right\};\ \Re(a) > \frac{1}{2}(n-1)$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right];\ \text{for}\ \Re(a) > (n-1)/2$$

Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}\left(\mathbf{S}\right) = \left\{2^{\frac{1}{2}np}\,\Gamma_n\left(\frac{1}{2}p\right)\left|\mathbf{\Sigma}\right|^{\frac{1}{2}p}\right\}^{-1}\left|\mathbf{S}\right|^{\frac{1}{2}(p-n-1)}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S}\right\} \quad (3)$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$.

Note: If $p = n + 1$, then the matrix is non-negative definite.

Some books

- Muirhead, Aspects of Multivariate Statistical Theory, John Wiely and Sons, 1982.
- Mehta, Random Matrices, Academic Press, 1991.
- Gupta and Nagar, Chapman & Hall/CRC, 2000.

Stochastic dynamic response

- Taking the Laplace transform of the equation of motion:

$$\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right] \bar{\mathbf{q}}(s) = \bar{\mathbf{f}}(s) \tag{4}$$

  The aim here is to obtain the statistical properties of $\bar{\mathbf{q}}(s) \in \mathbb{C}^n$ when the system matrices are random matrices.

- The system eigenvalue problem is given by

$$\mathbf{K}\boldsymbol{\phi}_j = \omega_j^2 \mathbf{M}\boldsymbol{\phi}_j, \quad j = 1, 2, \ldots, n \tag{5}$$

  where ω_j^2 and $\boldsymbol{\phi}_j$ are respectively the eigenvalues and mass-normalized eigenvectors of the system.

- We define the matrices

$$\boldsymbol{\Omega} = \mathrm{diag}\left[\omega_1, \omega_2, \ldots, \omega_n\right] \quad \text{and} \quad \boldsymbol{\Phi} = \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n\right]. \tag{6}$$

$$\text{so that} \quad \boldsymbol{\Phi}^T \mathbf{K}_e \boldsymbol{\Phi} = \boldsymbol{\Omega}^2 \quad \text{and} \quad \boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} = \mathbf{I}_n \tag{7}$$

Stochastic dynamic response

- Transforming it into the modal coordinates:

$$\left[s^2\mathbf{I}_n + s\mathbf{C}' + \mathbf{\Omega}^2\right]\bar{\mathbf{q}}' = \bar{\mathbf{f}}' \tag{8}$$

- Here

$$\mathbf{C}' = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = 2\zeta\mathbf{\Omega}, \quad \bar{\mathbf{q}} = \mathbf{\Phi}\bar{\mathbf{q}}' \quad \text{and} \quad \bar{\mathbf{f}}' = \mathbf{\Phi}^T\bar{\mathbf{f}} \tag{9}$$

- When we consider random systems, the matrix of eigenvalues $\mathbf{\Omega}^2$ will be a random matrix of dimension n. Suppose this random matrix is denoted by $\mathbf{\Xi} \in \mathbb{R}^{n\times n}$:

$$\mathbf{\Omega}^2 \sim \mathbf{\Xi} \tag{10}$$

Stochastic dynamic response

- Since Ξ is a symmetric and positive definite matrix, it can be diagonalized by a orthogonal matrix Ψ_r such that

$$\Psi_r^T \Xi \Psi_r = \Omega_r^2 \tag{11}$$

Here the subscript r denotes the random nature of the eigenvalues and eigenvectors of the random matrix Ξ.

- Recalling that $\Psi_r^T \Psi_r = \mathbf{I}_n$ we obtain

$$\bar{\mathbf{q}}' = \left[s^2 \mathbf{I}_n + s\mathbf{C}' + \Omega^2 \right]^{-1} \bar{\mathbf{f}}' \tag{12}$$

$$= \Psi_r \left[s^2 \mathbf{I}_n + 2s\boldsymbol{\zeta}\Omega_r + \Omega_r^2 \right]^{-1} \Psi_r^T \bar{\mathbf{f}}' \tag{13}$$

Stochastic dynamic response

- The response in the original coordinate can be obtained as

$$\bar{\mathbf{q}}(s) = \boldsymbol{\Phi}\bar{\mathbf{q}}'(s) = \boldsymbol{\Phi}\boldsymbol{\Psi}_r \left[s^2\mathbf{I}_n + 2s\boldsymbol{\zeta}\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_r^2\right]^{-1} (\boldsymbol{\Phi}\boldsymbol{\Psi}_r)^T\bar{\mathbf{f}}(s)$$

$$= \sum_{j=1}^{n} \frac{\mathbf{x}_{r_j}^T \bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j\omega_{r_j} + \omega_{r_j}^2}\mathbf{x}_{r_j}.$$

Here

$$\boldsymbol{\Omega}_r = \mathrm{diag}\left[\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_n}\right], \quad \mathbf{X}_r = \boldsymbol{\Phi}\boldsymbol{\Psi}_r = \left[\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_n}\right]$$

are respectively the matrices containing random eigenvalues and eigenvectors of the system.

Wishart random matrix approach

- Suppose we 'know' (e.g, by measurements or stochastic finite element modeling) the mean ($\mathbf{G}_0$) and the (normalized) variance (dispersion parameter) (δ_G) of the system matrices:

$$\delta_G = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2}. \tag{14}$$

- It can be proved that a positive definite symmetric matrix can e expressed by a Wishart matrix $\mathbf{G} \sim W_n(p, \mathbf{\Sigma})$ with

$$p = n + 1 + \theta \quad \text{and} \quad \mathbf{\Sigma} = \mathbf{G}_0/\theta \tag{15}$$

where

$$\theta = \frac{1}{\delta_G^2}\left\{1 + \gamma_G\right\} - (n+1) \tag{16}$$

and

$$\gamma_G = \frac{\left\{\mathrm{Trace}\left(\mathbf{G}_0\right)\right\}^2}{\mathrm{Trace}\left({\mathbf{G}_0}^2\right)} \tag{17}$$

Parameter-selection for structural dynamics

Approach 1: M and K are fully correlated Wishart (most complex). For this case $\mathbf{M} \sim W_n(p_1, \mathbf{\Sigma}_1)$, $\mathbf{K} \sim W_n(p_1, \mathbf{\Sigma}_1)$ with $\mathrm{E}\left[\mathbf{M}\right] = \mathbf{M}_0$ and $\mathrm{E}\left[\mathbf{M}\right] = \mathbf{M}_0$. This method requires the simulation of two $n \times n$ fully correlated Wishart matrices and the solution of a $n \times n$ generalized eigenvalue problem with two fully populated matrices. Here

$$\mathbf{\Sigma}_1 = \mathbf{M}_0/p_1, p_1 = \frac{\gamma_M + 1}{\delta_M} \tag{18}$$

$$\text{and} \quad \mathbf{\Sigma}_2 = \mathbf{K}_0/p_2, p_2 = \frac{\gamma_K + 1}{\delta_K} \tag{19}$$

$$\gamma_G = \left\{\mathrm{Trace}\left(\mathbf{G}_0\right)\right\}^2 / \mathrm{Trace}\left({\mathbf{G}_0}^2\right) \tag{20}$$

Parameter-selection for structural dynamics

Approach 2: Scalar Wishart (most simple) In this case it is assumed that

$$\mathbf{\Xi} \sim W_n\left(p, \frac{a^2}{n}\mathbf{I}_n\right) \tag{21}$$

Considering $\mathrm{E}\left[\mathbf{\Xi}\right] = \mathbf{\Omega}_0^2$ and $\delta_{\mathbf{\Xi}} = \delta_H$ the values of the unknown parameters can be obtained as

$$p = \frac{1+\gamma_H}{\delta_H^2} \quad \text{and} \quad a^2 = \mathrm{Trace}\left(\mathbf{\Omega}_0^2\right)/p \tag{22}$$

Parameter-selection for structural dynamics

Approach 3: Diagonal Wishart with different entries (something in the middle). For this case $\mathbf{\Xi} \sim W_n\left(p, \mathbf{\Omega}_0^2/\theta\right)$ with $\mathrm{E}\left[\mathbf{\Xi}^{-1}\right] = \mathbf{\Omega}_0^{-2}$ and $\delta_{\mathbf{\Xi}} = \delta_H$. This requires the simulation of one $n \times n$ uncorrelated Wishart matrix and the solution of an $n \times n$ standard eigenvalue problem.

The parameters can be obtained as

$$p = n + 1 + \theta \quad \text{and} \quad \theta = \frac{(1+\gamma_H)}{\delta_H^2} - (n+1) \tag{23}$$

Parameter-selection for structural dynamics

- Defining $\mathbf{H}_0 = {\mathbf{M}_0}^{-1}\mathbf{K}_0$, the constant γ_H:

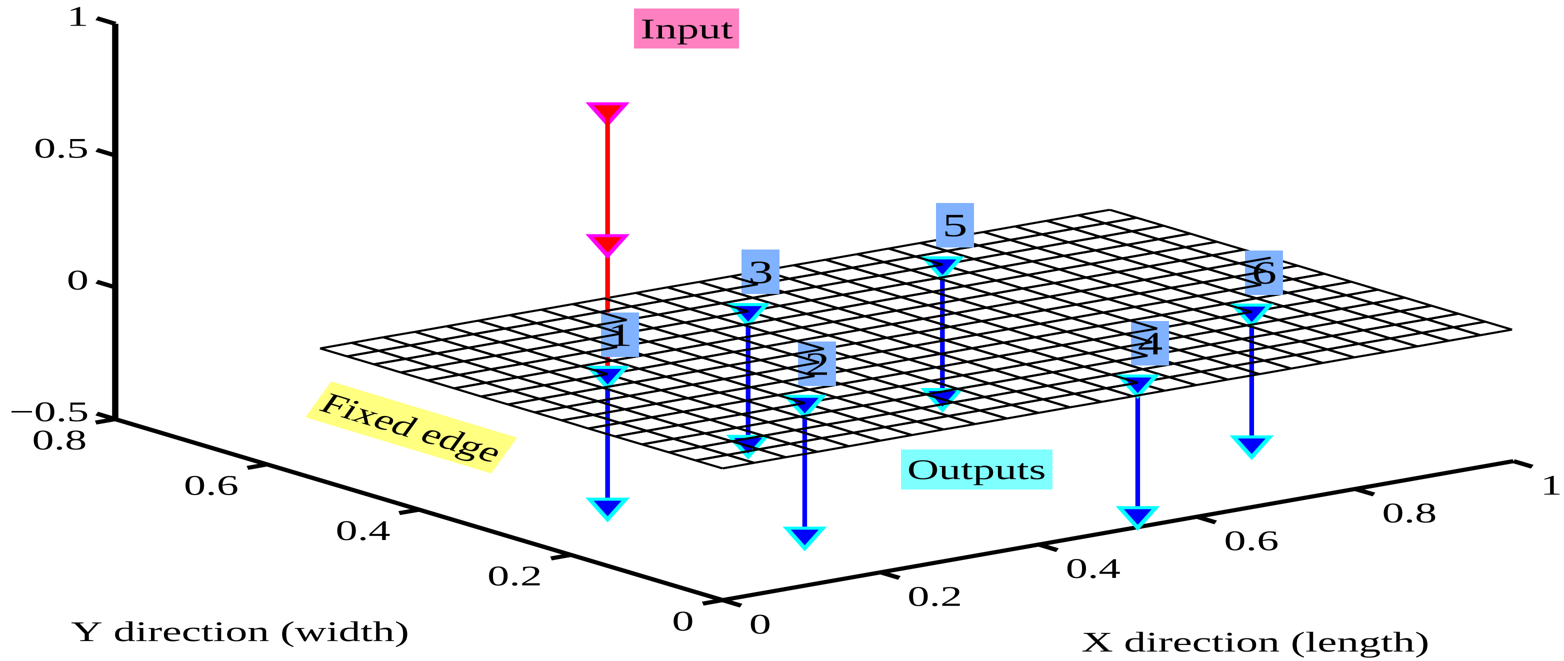
$$\gamma_H = \frac{\{\text{Trace}\,(\mathbf{H}_0)\}^2}{\text{Trace}\,({\mathbf{H}_0}^2)} = \frac{\left\{\text{Trace}\,(\mathbf{\Omega}_0^2)\right\}^2}{\text{Trace}\,(\mathbf{\Omega}_0^4)} = \frac{\left(\sum_j \omega_{0_j}^2\right)^2}{\sum_j \omega_{0_j}^4} \tag{24}$$

- Obtain the dispersion parameter of the generalized Wishart matrix

$$\delta_H = \frac{\left({p_1}^2 + (p_2 - 2 - 2\,n)\,p_1 + (-n-1)\,p_2 + n^2 + 1 + 2\,n\right)\gamma_H}{p_2\,(-p_1+n)\,(-p_1+n+3)} + \frac{{p_1}^2 + (p_2 - 2\,n)\,p_1 + (1-n)\,p_2 - 1 + n^2}{p_2\,(-p_1+n)\,(-p_1+n+3)} \tag{25}$$

Numerical Examples

A vibrating cantilever plate



Baseline Model: Thin plate elements with 0.7% modal damping assumed for all the modes.

Physical properties

Plate Properties	Numerical values
Length (L_x)	998 mm
Width (L_y)	530 mm
Thickness (t_h)	3.0 mm
Mass density (ρ)	7860 kg/m^3
Young's modulus (E)	2.0×10^5 MPa
Poisson's ratio (μ)	0.3
Total weight	12.47 kg

Material and geometric properties of the cantilever plate considered for the experiment. The data presented here are available from http://engweb.swan.ac.uk/∼adhikaris/uq/.

Uncertainty type 1: random fields

The Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\left(1 + \epsilon_E f_1(\mathbf{x})\right) \tag{26}$$

$$\mu(\mathbf{x}) = \bar{\mu}\left(1 + \epsilon_\mu f_2(\mathbf{x})\right) \tag{27}$$

$$\rho(\mathbf{x}) = \bar{\rho}\left(1 + \epsilon_\rho f_3(\mathbf{x})\right) \tag{28}$$

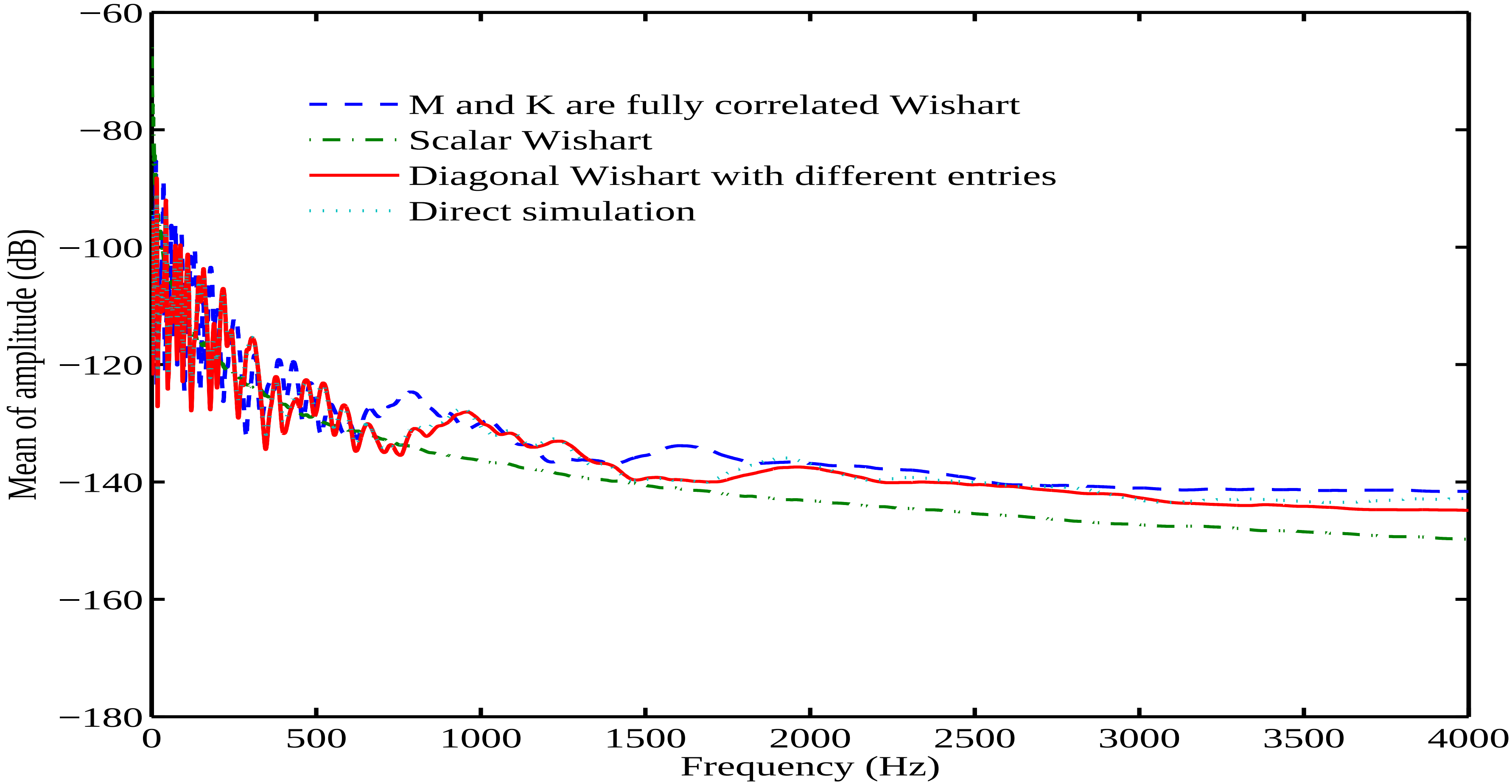
$$\text{and} \quad t(\mathbf{x}) = \bar{t}\left(1 + \epsilon_t f_4(\mathbf{x})\right) \tag{29}$$

- The strength parameters: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.15$, $\epsilon_\rho = 0.10$ and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x}), i = 1, \cdots, 4$ are delta-correlated homogenous Gaussian random fields.

Uncertainty type 2: random attached oscillators

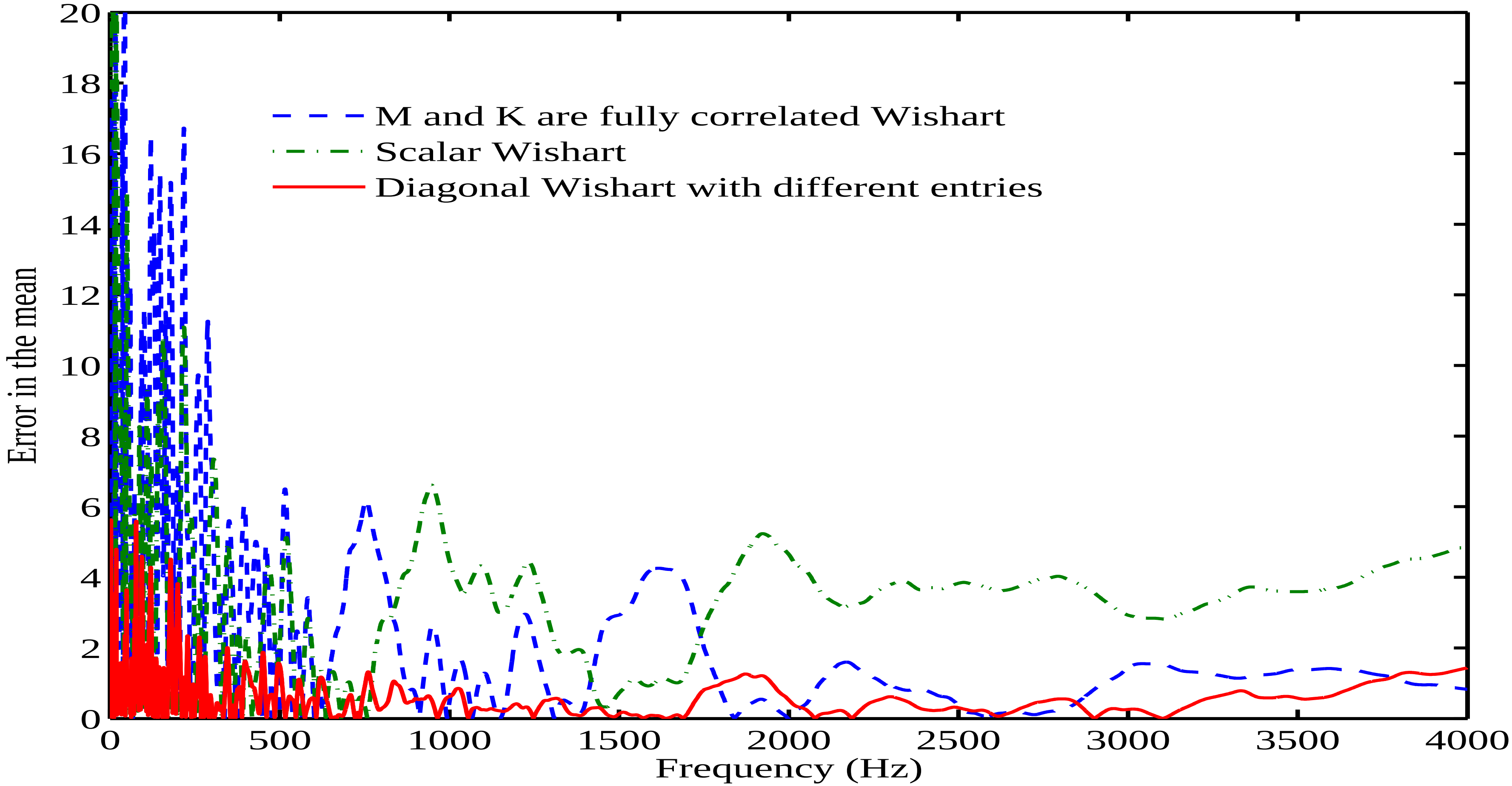
- Here we consider that the baseline plate is ‘perturbed’ by attaching 10 oscillators with random spring stiffnesses at random locations
- This is aimed at modeling non-parametric uncertainty.
- This case will be investigated experimentally later.

Mean of cross-FRF: Utype 1



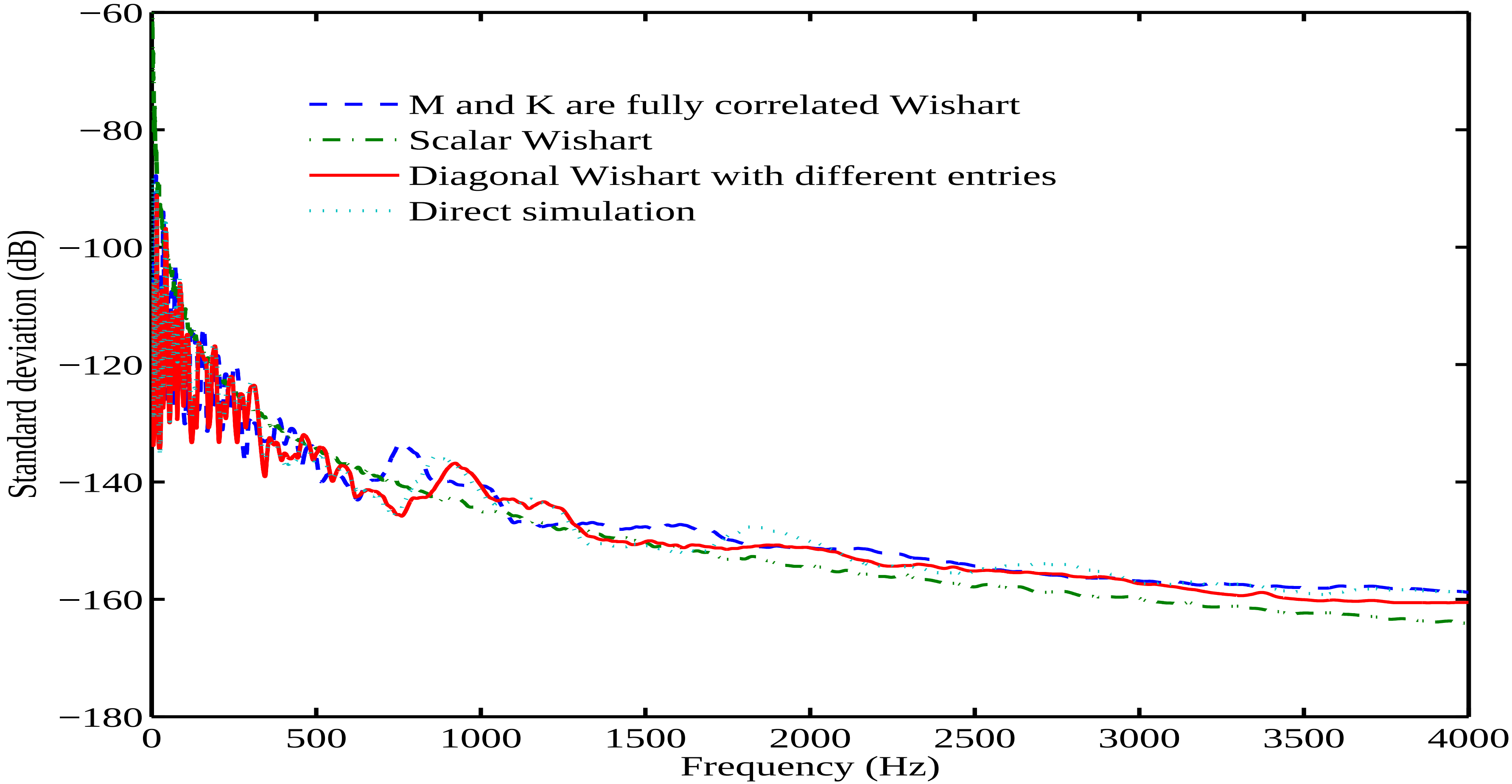
Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Error in the mean of cross-FRF: Utype 1



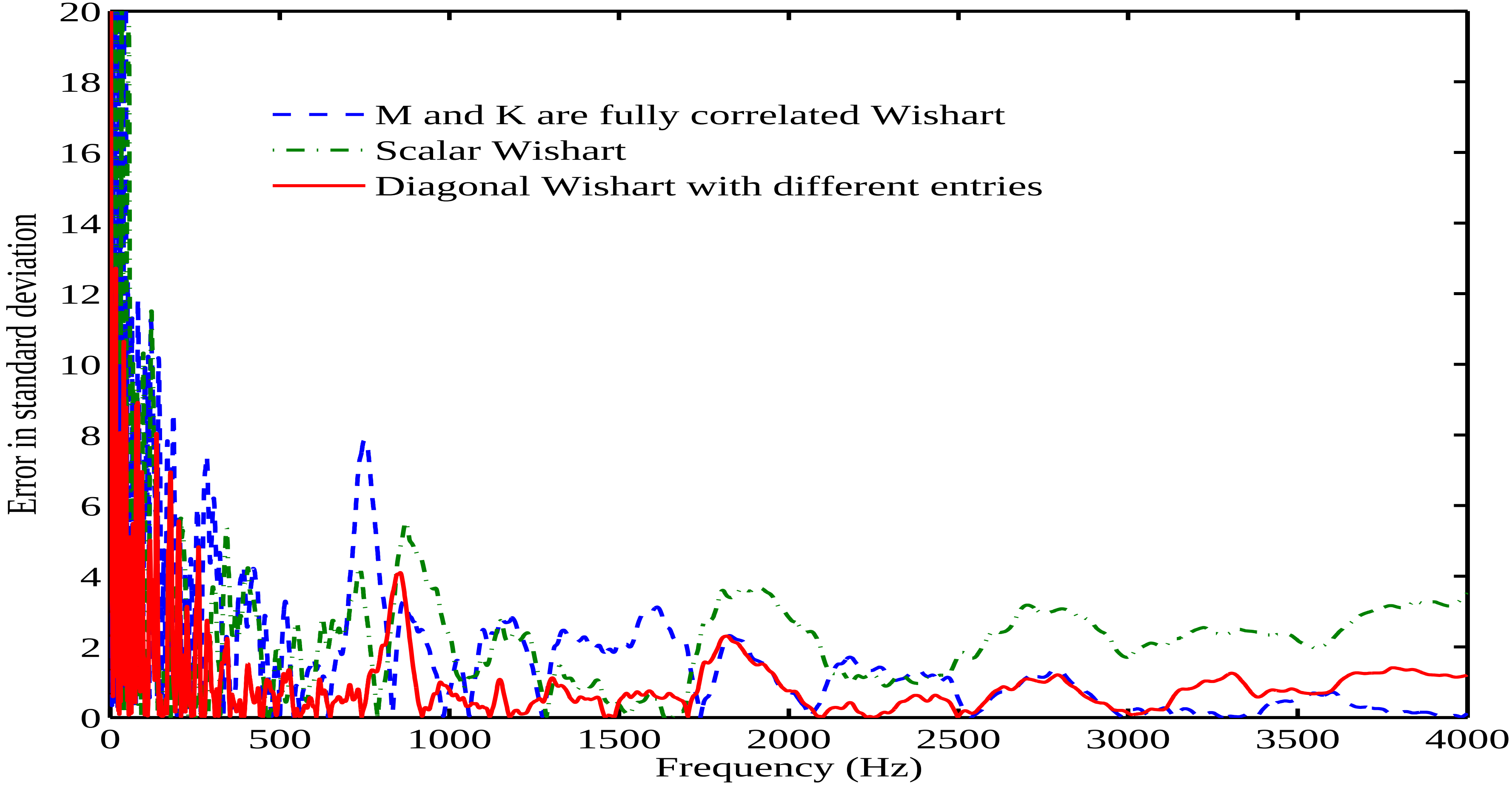
Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Standard deviation of driving-point-FRF: Utype 1



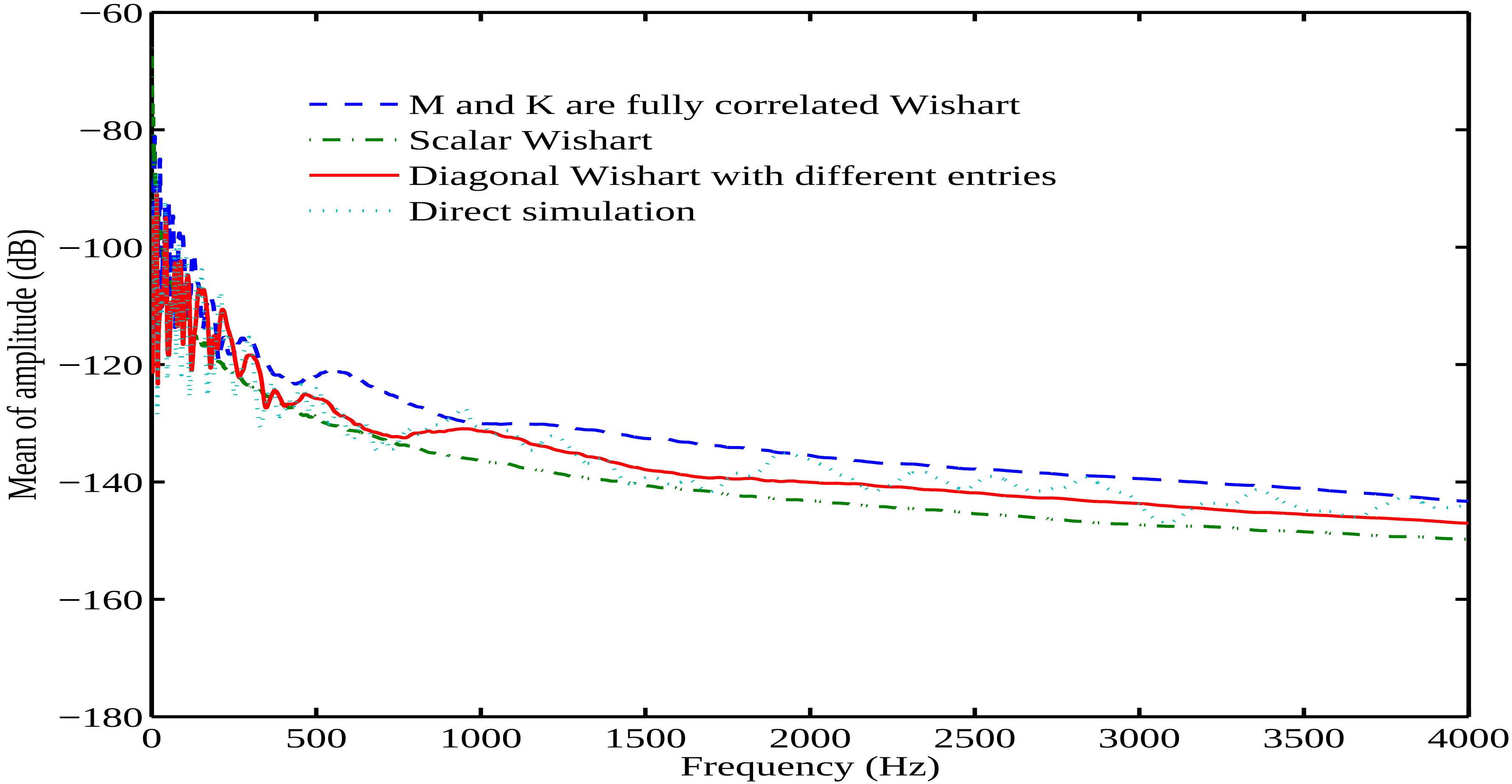
Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Error in the standard deviation of driving-point-FRF: Utype 1



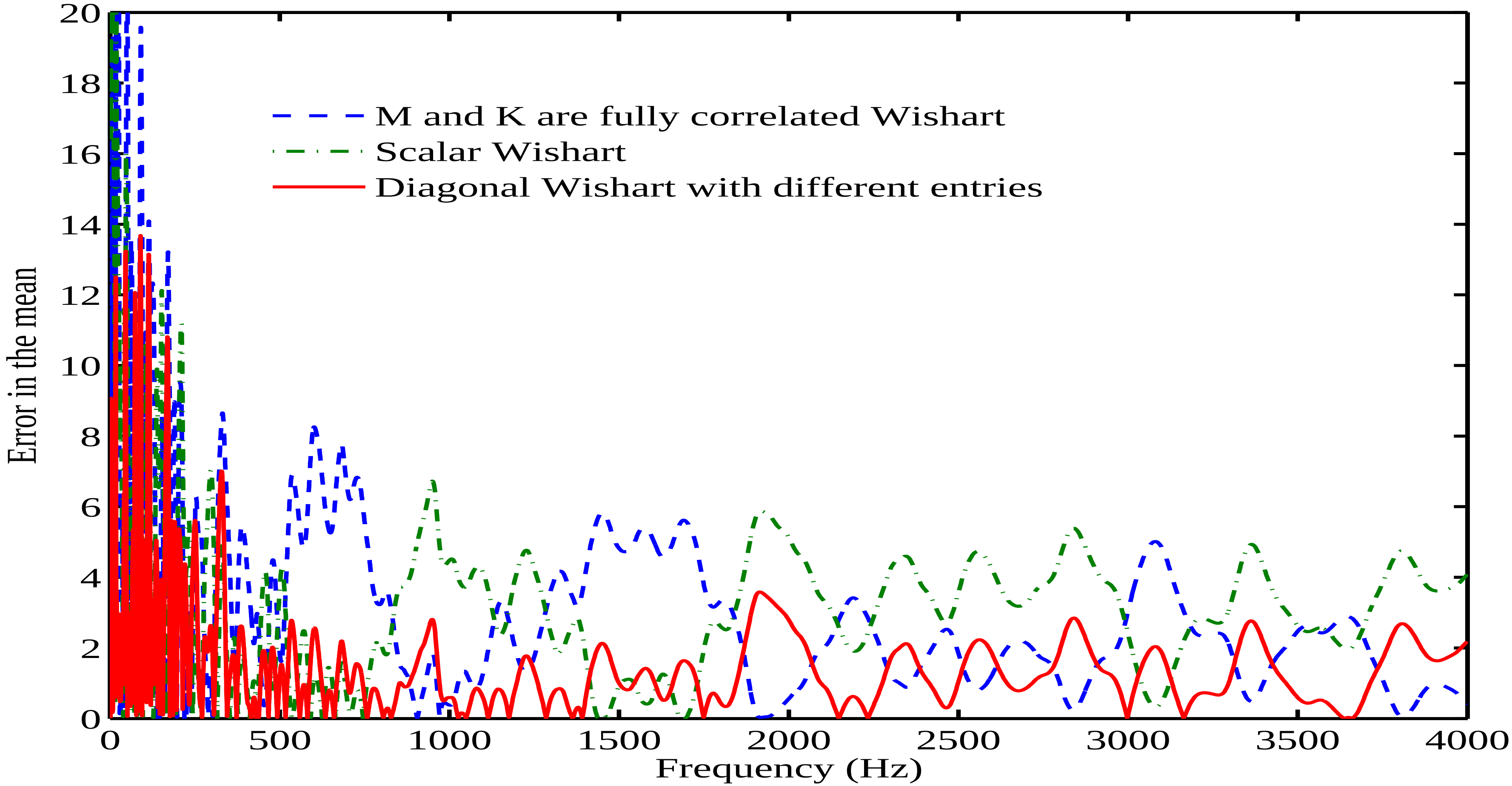
Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$.

Mean of cross-FRF: Utype 2



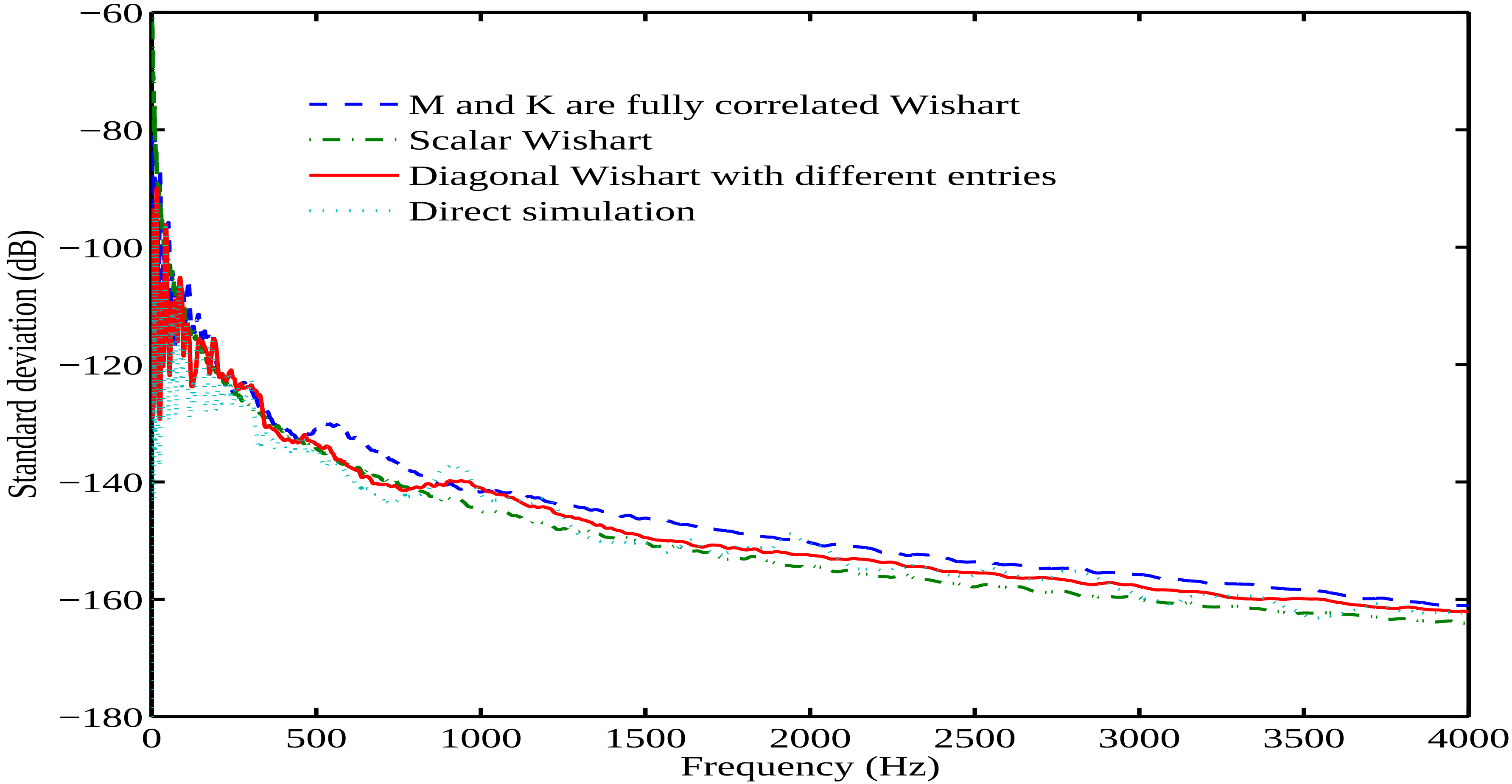
Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

Error in the mean of cross-FRF: Utype 2



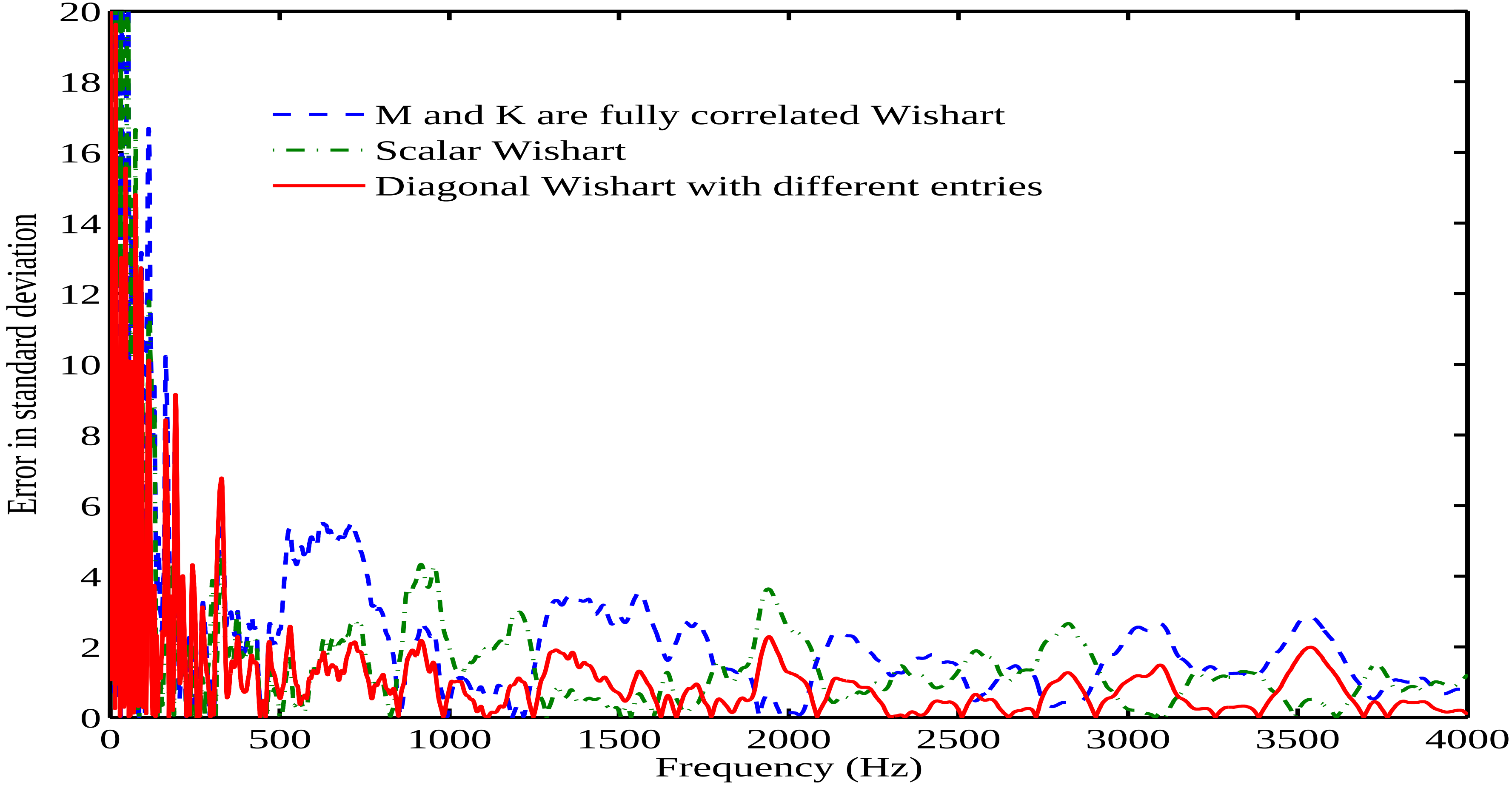
Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

Standard deviation of driving-point-FRF: Utype 2



Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

Error in the standard deviation of driving-point-FRF: Utype 2



Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.133$ and $\sigma_K = 0.420$.

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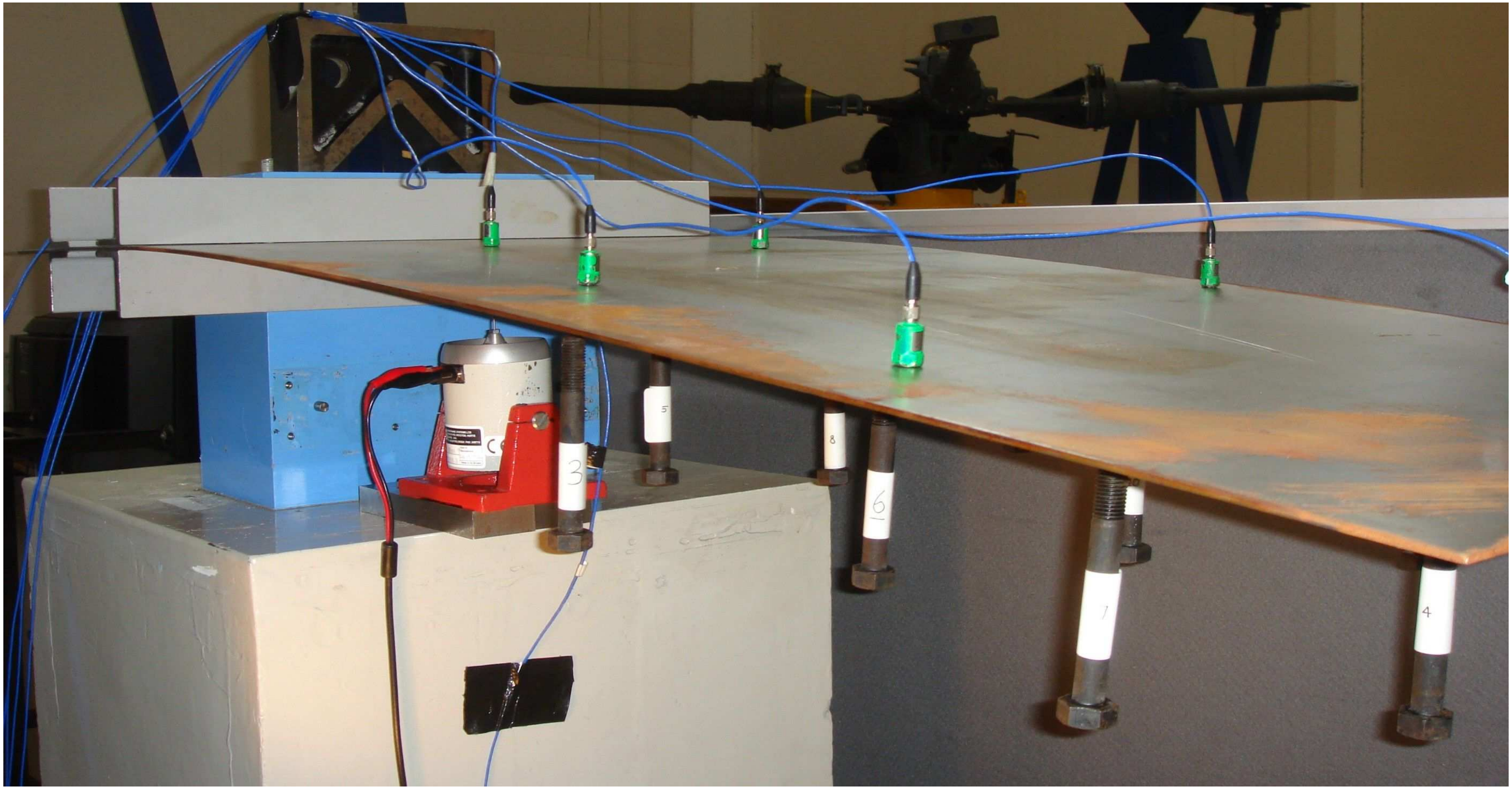
Experimental investigation for uncertainty type 2 (randomly attached oscillators)

A cantilever plate: front view



The test rig for the cantilever plate; front view (to appear in Probabilistic Engineering Mechanics).

A cantilever plate: side view

The test rig for the cantilever plate; side view.

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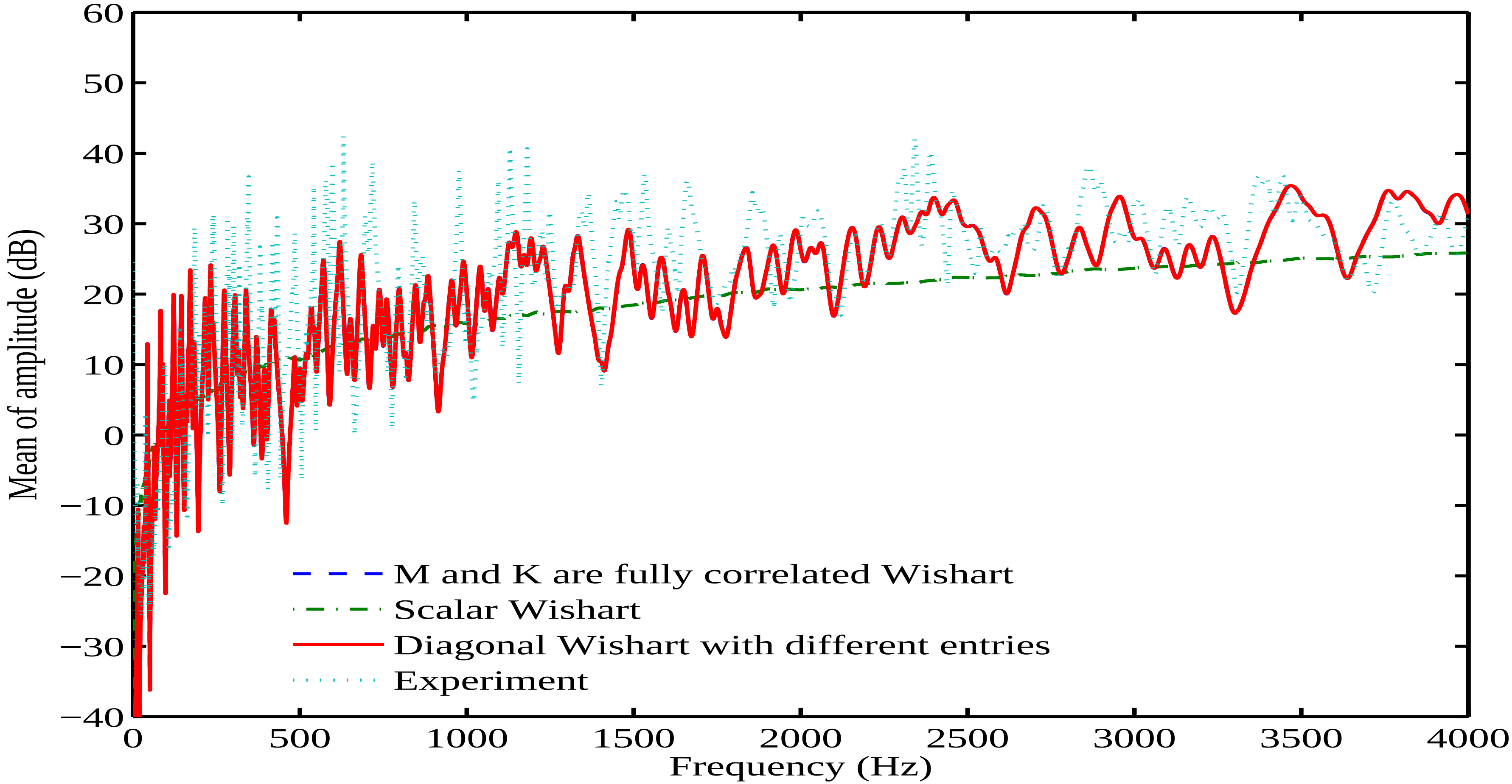
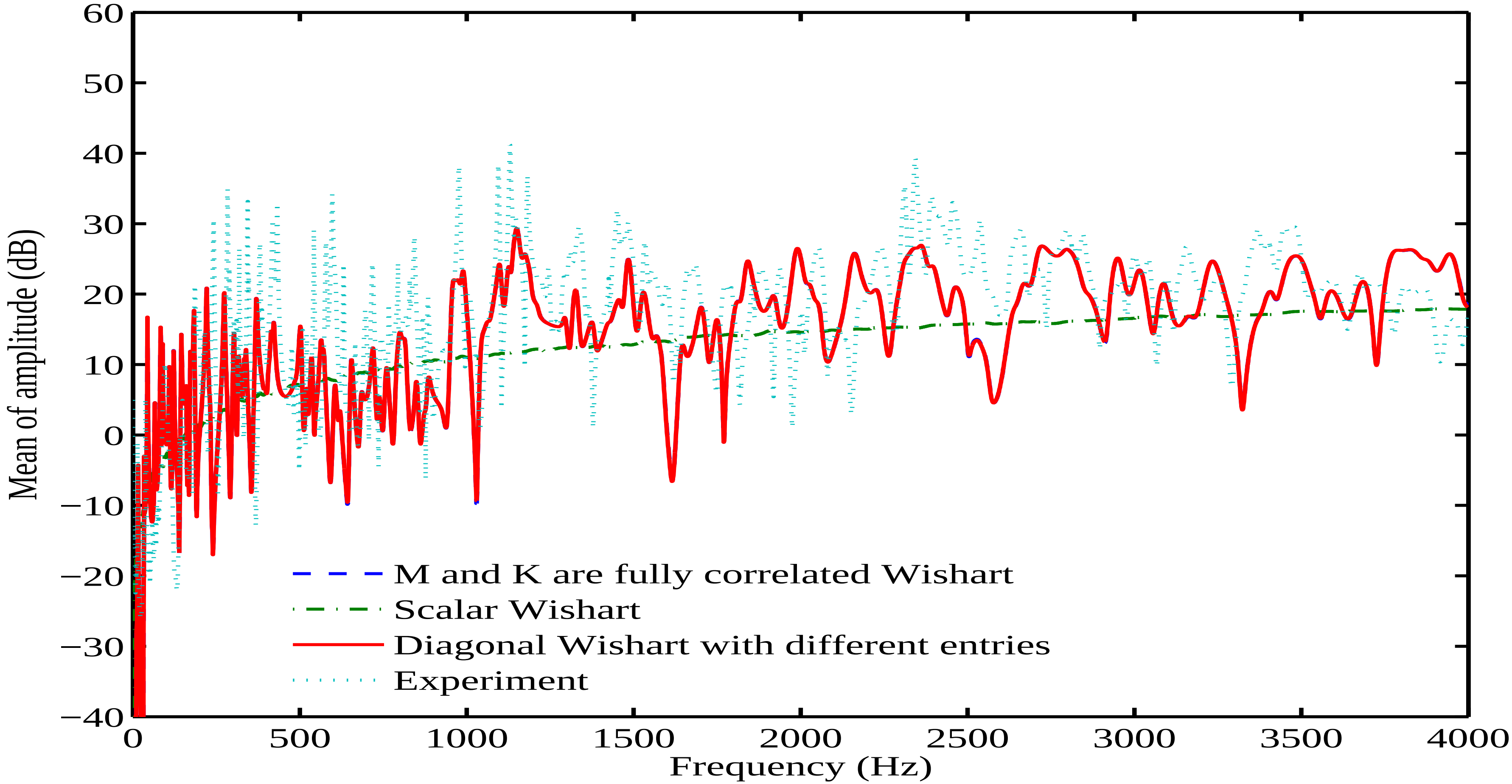
Comparison of driving-point-FRF



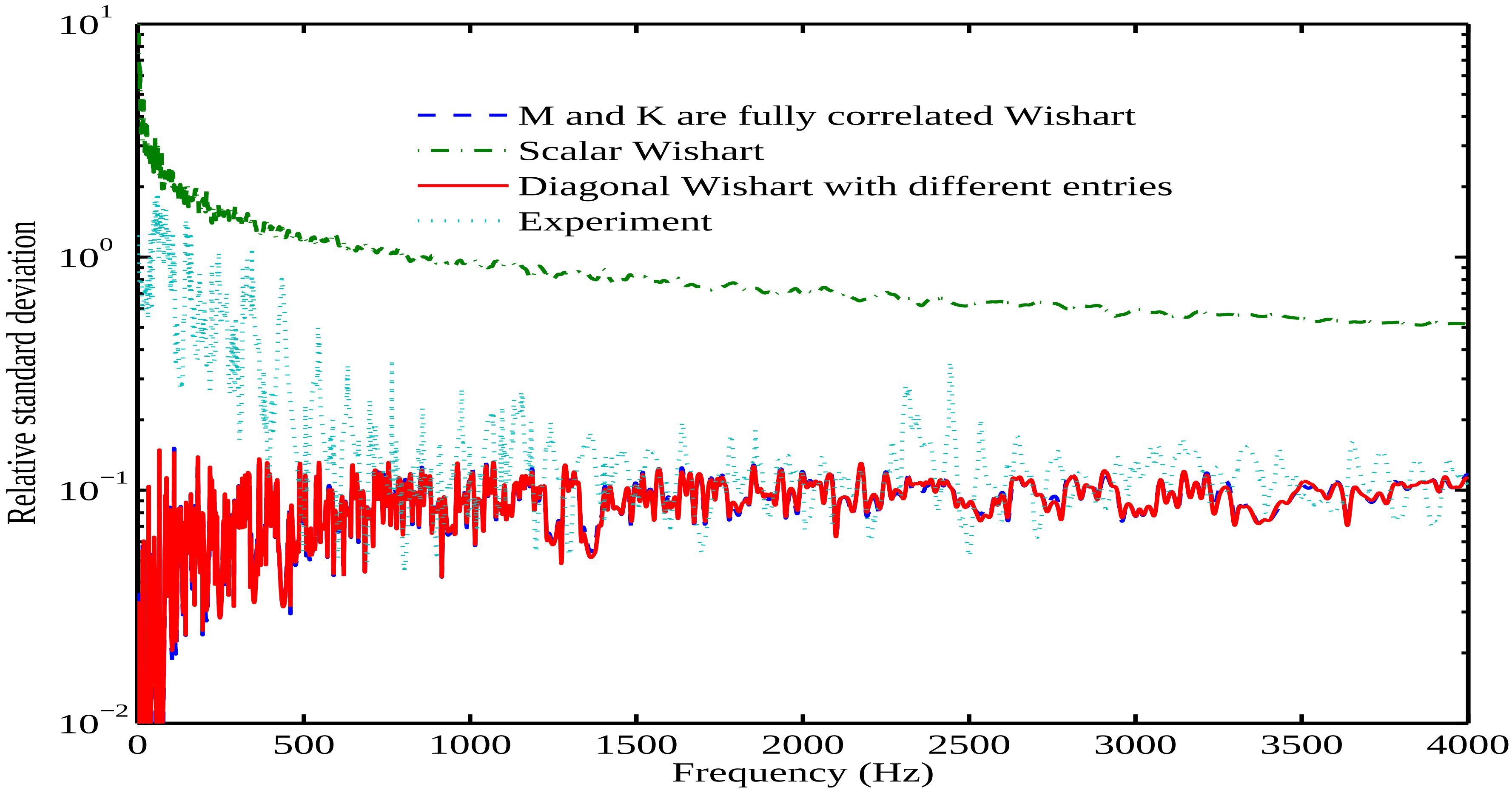
Comparison of the mean of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Comparison of Cross-FRF



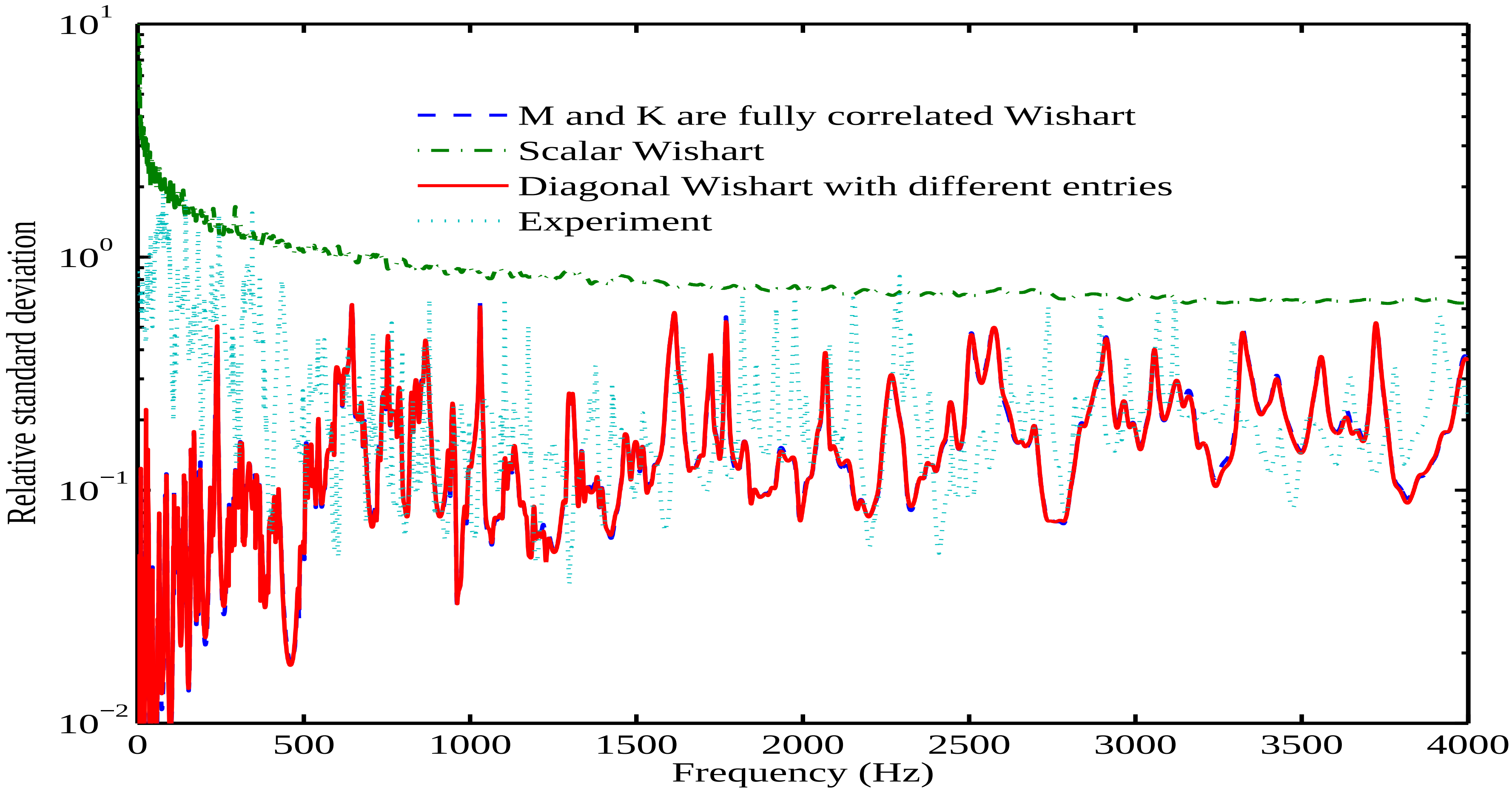
Comparison of the mean of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Comparison of driving-point-FRF



Comparison of relative standard deviation of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Comparison of Cross-FRF



Comparison of relative standard deviation of the amplitude obtained using the experiment and three Wishart matrix approaches for the plate with randomly attached oscillators

Conclusions

- Uncertainties need to be taken into account for credible predictions using computational methods.
- This talk concentrated on Uncertainty Propagation (UP) in linear structural dynamic problems.
- A general UP approach based on Wishart random matrix is discussed and the results are compared with experimental results.
- Based on numerical and experimental studies, a suitable simple Wishart random matrix model has been identified.

Future directions

- Efficient computational methods based on analytical approaches involving random eigenvalue problems
- Model calibration/updating: from experimental measurements (with uncertainties) how to identify/update the model (ie, the system matrices) and its associated uncertainty.
- High performance computing software for uncertain systems: how the UP approaches can be integrated with high performance computing and general purpose commercial software? This is becoming a very important issue due the availability of relatively inexpensive 'clusters'.