Response statistics of linear stochastic systems: A simultaneous diagonalisation approach

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Outline

- Background and Motivation
- Discretisation of stochastic material parameters
- Current methods for response-statistics calculation
- Joint diagonalisation approach
- Numerical examples
- Conclusions

Background

- In many stochastic mechanics problems we need to solve a system of linear stochastic equations:

$$\mathbf{K}\mathbf{u} = \mathbf{f}. \tag{1}$$

  $\mathbf{K} \in \mathbb{R}^{n\times n}$ is a $n \times n$ real non-negative definite random matrix, $\mathbf{f} \in \mathbb{R}^{n}$ is a n-dimensional real deterministic input vector and $\mathbf{u} \in \mathbb{R}^{n}$ is a n-dimensional real uncertain output vector which we want to determine.

- This typically arise due to the discretisation of stochastic partial differential equations (eg. in the stochastic finite element method)

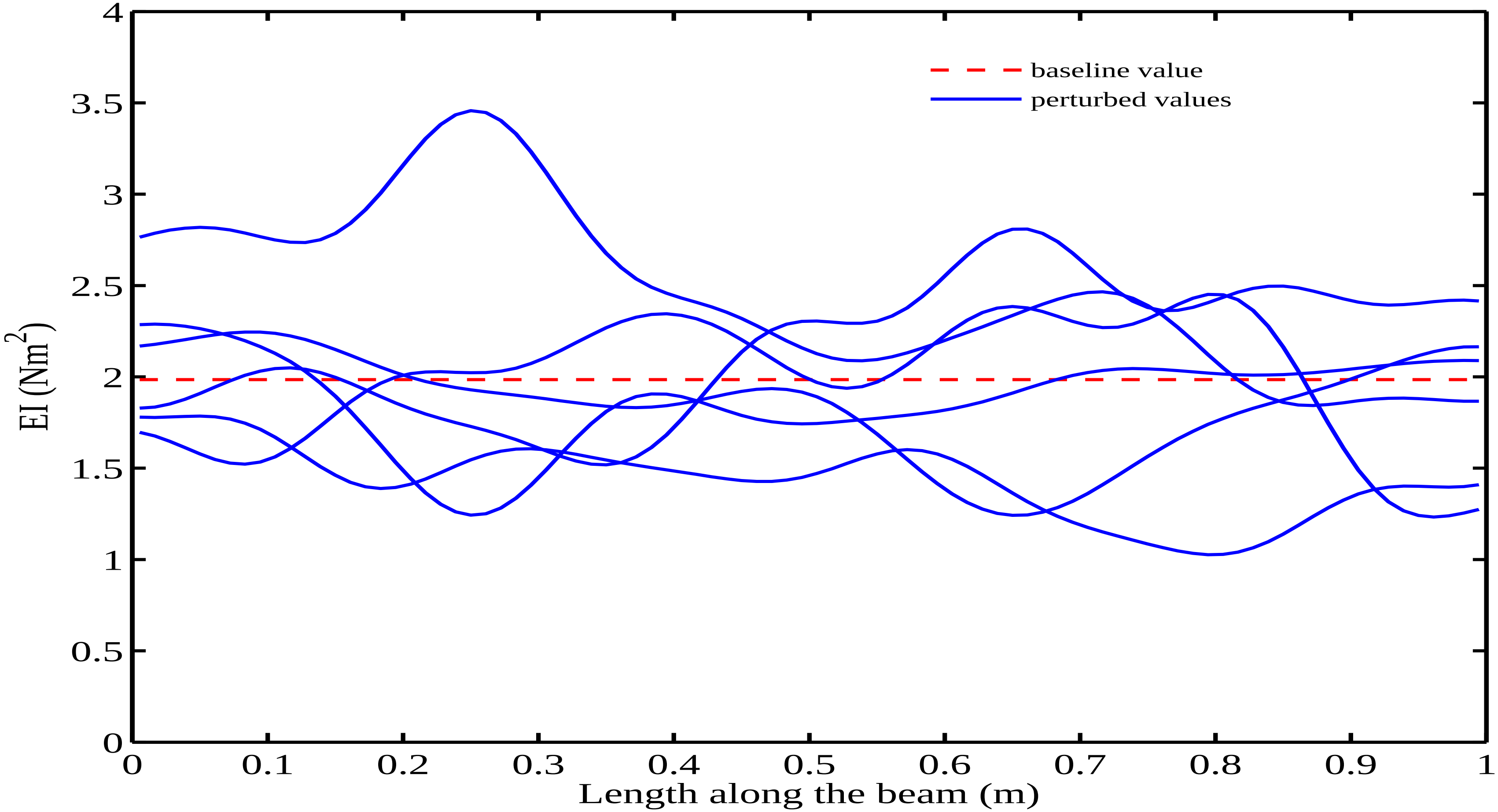
Background

- In the context of linear structural mechanics, $\mathbf{K}$ is known as the stiffness matrix, $\mathbf{f}$ is the forcing vector and $\mathbf{u}$ is the vector of structural displacements.
- Often, the objective is to determine the probability density function (pdf) and consequently the cumulative distribution function (cdf) of $\mathbf{u}$. This will allow one to calculate the reliability of the system.
- It is generally difficult to obtain the probably density function (pdf) of the response. As a consequence, engineers often intend to obtain only the fist few moments (typically the fist two) of the response quantity.

Objectives

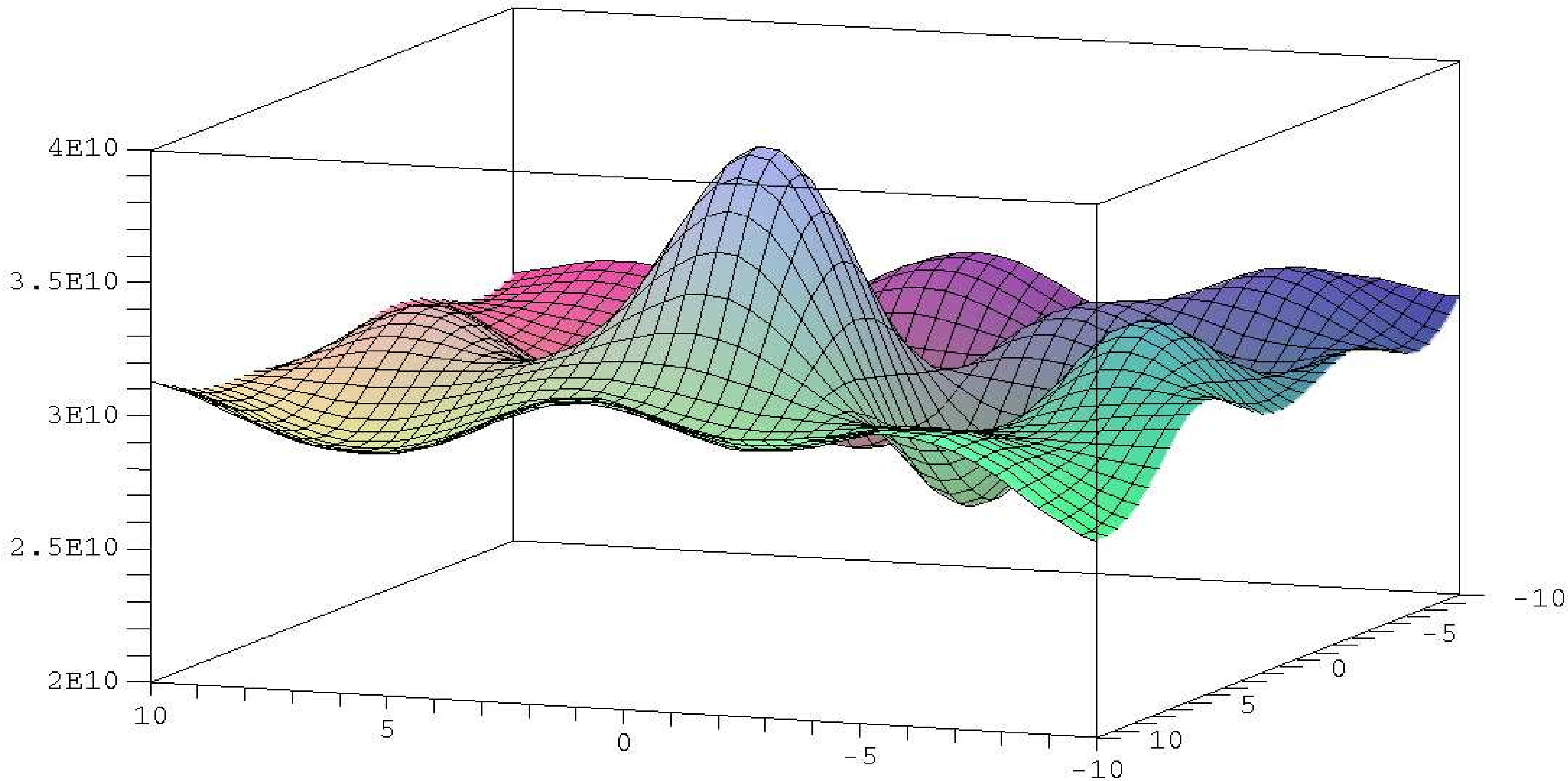
- We propose a joint diagonalisation method for the solution of stochastic linear systems.
- The method is based on the recently developed joint diagonalisation solution strategy and the Neumann expansion of inverse matrix.
- The joint diagonalisation method is applicable to stochastic linear systems with arbitrary number of random variables and has no registration on the type of probability distribution.

Example of heterogeneous materials: 1D



An example of heterogeneous property in one dimension

Example of heterogeneous materials: 2D



An example of heterogeneous property in two dimensions

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Discretisation of stochastic material parameters

- middle point method
- local averaging method
- shape function method
- least-squares discretization method
- trigonometric series approximation
- K-L expansion method
- F-K-L expansion method

Karhunen-Loève expansion

- A second-order stochastic field can be represented as:

$$b(\mathbf{x},\omega) = E\left(b(\mathbf{x},\omega)\right) + \sum_{i=1}^{+\infty} \sqrt{\lambda_i}\xi_i(\omega)\psi_i(\mathbf{x}) \qquad (2)$$

- The required eigen-structure is obtained through solving a generalized eigen-value problem

$$\int_D \mathrm{Cov}\left(b(\mathbf{x}_1,\omega), b(\mathbf{x}_2,\omega)\right)\psi(\mathbf{x}_1)\,d\mathbf{x}_1 = \lambda\psi(\mathbf{x}_2) \quad \Rightarrow \quad \mathbf{A}\mathbf{v} = \lambda\mathbf{B}\mathbf{v} \qquad (3)$$

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Karhunen-Loève expansion

- After sorting from high to low the eigen-values of stochastic field, the K-L expansion is optimal in terms of approximation of the total variance of the random material parameter

$$\sum_{i=1}^{+\infty} \lambda_i = \int_D \mathrm{Cov}\left(b\left(\mathbf{x},\omega\right), b\left(\mathbf{x},\omega\right)\right) d\mathbf{x} \tag{4}$$

Fourier-Karhunen-Loève expansion

- For stationary stochastic fields in regular domains, the Fourier-Karhunen-Loève expansion is developed by Li et al.[13,14] and by combining the representation theory of stationary stochastic fields, the F-K-L result is much more accurate and is obtained explicitly without solving any equation.

$$a\left(\mathbf{x},\omega\right) = a_o\left(\mathbf{x}\right) + \int_{R^n} e^{\sqrt{-1}\mathbf{x}\mathbf{y}}\, dZ\left(\mathbf{y},\omega\right) \tag{5}$$

$$f\left(\mathbf{y}\right) = \frac{1}{\left(2\pi\right)^n} \int_n R\left(\tau\right) e^{-\sqrt{-1}\tau\mathbf{y}}\, d\tau \tag{6}$$

Current Approaches

The random matrix can be represented as

$$\mathbf{K} = \mathbf{K}^0 + \mathbf{\Delta K} \tag{7}$$

$\mathbf{K}^0 \in \mathbb{R}^{n\times n}$ is the deterministic part and the random part:

$$\mathbf{\Delta K} = \sum_{j=1}^{m} \xi_j \mathbf{K}_j^I + \sum_{j=1}^{m} \sum_{l=1}^{j} \xi_j \xi_l \mathbf{K}_{jl}^{II} + \cdots \tag{8}$$

m is the number of random variables, $\mathbf{K}_j^I, \mathbf{K}_{jl}^{II} \in \mathbb{R}^{n\times n}, \forall\, j, l$ are deterministic matrices and $\xi_j, \forall\, j$ are real random variables.

Perturbation based approach

Represent the response as

$$\mathbf{u} = \mathbf{u}^0 + \xi_j \mathbf{u}_j^I + \sum_{j=1}^{m} \sum_{l=1}^{j} \xi_j \xi_l \mathbf{u}_{jl}^{II} + \cdots . \tag{9}$$

where

$$\mathbf{u}^0 = \mathbf{K}^{0^{-1}} \mathbf{f} \tag{10}$$

$$\mathbf{u}_j^I = -\mathbf{K}^{0^{-1}} \mathbf{K}_j^I \mathbf{u}^0, \quad \forall\, j \tag{11}$$

and $$\mathbf{u}_{jl}^{II} = -\mathbf{K}^{0^{-1}} [\mathbf{K}_{jl}^{II} \mathbf{u}^0 + \mathbf{K}_j^I \mathbf{u}_l^I + \mathbf{K}_l^I \mathbf{u}_j^I], \quad \forall\, j, l. \tag{12}$$

Neumann expansion

Provided $\left\| \mathbf{K}^{0^{-1}} \boldsymbol{\Delta}\mathbf{K} \right\|_{\mathrm{F}} < 1$,

$$\mathbf{K}^{-1} = \left[\mathbf{K}_0 (\mathbf{I}_n + \mathbf{K}^{0^{-1}} \boldsymbol{\Delta}\mathbf{K}) \right]^{-1}$$
$$= \mathbf{K}^{0^{-1}} - \mathbf{K}^{0^{-1}} \boldsymbol{\Delta}\mathbf{K}\mathbf{K}^{0^{-1}} + \mathbf{K}^{0^{-1}} \boldsymbol{\Delta}\mathbf{K}\mathbf{K}^{0^{-1}} \boldsymbol{\Delta}\mathbf{K}\mathbf{K}^{0^{-1}} + \cdots .$$

Therefore,

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{f} = \mathbf{u}^0 - \mathbf{T}\mathbf{u}_0 + \mathbf{T}^2\mathbf{u}_0 + \cdots \tag{13}$$

where $\mathbf{T} = \mathbf{K}^{0^{-1}} \boldsymbol{\Delta}\mathbf{K} \in \mathbb{R}^{n \times n}$ is a random matrix.

Projection methods

Here one 'projects' the solution vector onto a complete stochastic basis. Depending on how the basis is selected, several methods are proposed.

Using the classical Polynomial Chaos (PC) projection scheme

$$\mathbf{u} = \sum_{j=0}^{P-1} \mathbf{u}_j \Psi_j(\boldsymbol{\xi}) \tag{14}$$

where $\mathbf{u}_j \in \mathbb{R}^n, \forall j$ are unknown vectors and $\Psi_j(\boldsymbol{\xi})$ are multidimensional Hermite polynomials in ξ_r.

A partial summary

	Methods	Sub-methods
1.	Perturbation based methods	First and second order perturbation[11,22], Neumann expansion[1,26], improved perturbation method[3].
2.	Projection methods	Polynomial chaos expansion[7], random eigenfunction expansion[1], stochastic reduced basis method[16,18,19], Wiener−Askey chaos expansion[23–25], domain decomposition method[20,21].
3.	Monte carlo simulation and other methods	Simulation methods[8,17], Analytical method in references[5,6,9,10,15], Exact solutions for beams[2,4].

Joint diagonalisation approach

- We consider a stochastic linear system:

$$(\mathbf{K}_0 + \xi_1(\omega)\mathbf{K}_1 + \xi_2(\omega)\mathbf{K}_2 + \cdots + \xi_m(\omega)\mathbf{K}_m)\mathbf{u} = \mathbf{f}, \qquad (15)$$

  where $\mathbf{K}_j, \forall\, j$ are real symmetric deterministic matrices and $\xi_j(\omega), \forall\, j$ are real random variables.
- The above stochastic linear system is commonly obtained in a SFEM formulation after discretising the random material parameters with the K-L (or F-K-L) expansion method.

Joint diagonalisation approach

By using a sequence of Givens transformations, stiffness matrices $\mathbf{K}_j$ can be simultaneously diagonalised such that

$$\mathbf{Q}^{-1}\mathbf{K}_j\mathbf{Q} = \mathbf{\Lambda}_j + \mathbf{\Delta}_j \approx \mathbf{\Lambda}_j, \quad j = 1, 2, \cdots, m \tag{16}$$

where $\mathbf{\Lambda}_j, \forall\, j$ are diagonal matrices and $\mathbf{\Delta}_j, \forall\, j$ are matrices with zero diagonal entries and small magnitude off-diagonal entries. The transform matrix $\mathbf{Q}$ is explicitly obtained as the product of the Givens rotation matrices

$$\mathbf{Q} = \mathbf{G}_1^T\mathbf{G}_2^T \cdots \mathbf{G}_k^T \tag{17}$$

where k is the total number of Givens transformations and $\mathbf{G}_j, \forall\, j$ are Givens rotation matrices

Givens rotation matrix

$$\mathbf{G} = \mathbf{G}\left(p, q, \theta\right) \triangleq \begin{array}{c} \\ \\ \\ \\ p \\ \\ \\ \\ q \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccccc} & & & p & & & & q & & & \end{array} \\ \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \cos\theta & & & & \sin\theta & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & -\sin\theta & & & & \cos\theta & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix} \end{array} . \qquad (18)$$

Optimal Givens rotation angle

For $n \times n$ real symmetric matrices $\mathbf{K}_l$ let

$$\text{off}(\mathbf{K}_l) \triangleq \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} (\mathbf{K}_l)_{ij}^2 \qquad (l = 1, \cdots, m) \tag{19}$$

denote the quadratic sums of off-diagonal elements in $\mathbf{K}_l$. The aim here is to gradually reduce $\sum_{l=1}^{m} \text{off}(\mathbf{K}_l)$ through a sequence of orthogonal similarity transformations that have no effect on $\|\mathbf{K}_l\|_F$, the Frobenius norm of $\mathbf{K}_l$.

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Optimal Givens rotation angle

The optimal Givens rotation angle θ_{opt} that in the current iteration maximizes the diagonal entries and minimizes the off-diagonal entries can be accurately obtained by solving the following characteristic equation

$$\left(\cos 2\theta_{opt} \quad \sin 2\theta_{opt}\right)^T \mathbf{J} = \lambda_J \left(\cos 2\theta_{opt} \quad \sin 2\theta_{opt}\right)^T \tag{20}$$

where $\mathbf{J}$ is a two by two matrix given by

$$\mathbf{J} = \sum_{j=1}^{m} \begin{pmatrix} 2\left(\mathbf{K}_j\right)_{pq}^2 & \left(\mathbf{K}_j\right)_{pq}\left(\left(\mathbf{K}_j\right)_{qq} - \left(\mathbf{K}_j\right)_{pp}\right) \\ \left(\mathbf{K}_j\right)_{pq}\left(\left(\mathbf{K}_j\right)_{qq} - \left(\mathbf{K}_j\right)_{pp}\right) & \frac{1}{2}\left(\left(\mathbf{K}_j\right)_{qq} - \left(\mathbf{K}_j\right)_{pp}\right)^2 \end{pmatrix} \tag{21}$$

and λ_J is the smallest eigen-value of $\mathbf{J}$.

Joint diagonalisation solution

In the original joint diagonalisation solution, the off-diagonal entries $\mathbf{\Delta}_j, \forall j$ are completely ignored and this significantly simplifies the random equation system which in turn leads to an explicit solution

$$\mathbf{u} \approx \mathbf{Q}(\mathbf{\Lambda}_0 + \xi_1(\omega)\mathbf{\Lambda}_1 + \cdots + \xi_m(\omega)\mathbf{\Lambda}_m)^{-1}\mathbf{Q}^{-1}\mathbf{f} \qquad (22)$$

This work propose to take into consideration the contribution from the off-diagonal matrices $\mathbf{\Delta}_j, \forall j$. Specifically

$$\mathbf{Q}\left[(\mathbf{\Lambda}_0 + \sum_{j=1}^{m} \xi_j(\omega)\mathbf{\Lambda}_j) + (\mathbf{\Delta}_0 + \sum_{j=1}^{m} \xi_j(\omega)\mathbf{\Delta}_j)\right]\mathbf{Q}^{-1}\mathbf{u} = \mathbf{f}. \qquad (23)$$

Improved joint diagonalisation solution

Let

$$\mathbf{V} = \mathbf{\Lambda}_0 + \sum_{j=1}^{m} \xi_j(\omega)\mathbf{\Lambda}_j \quad \text{and} \quad \mathbf{A} = \mathbf{\Delta}_0 + \sum_{j=1}^{m} \xi_j(\omega)\mathbf{\Delta}_j \tag{24}$$

the solution can be expressed as

$$\mathbf{u} = \mathbf{Q}\left[\mathbf{V}(\mathbf{I}_n + \mathbf{V}^{-1}\mathbf{A})\right]^{-1}\mathbf{Q}^{-1}\mathbf{f} \tag{25}$$

Noting that matrix $\mathbf{V}$ is a diagonal matrix whose inverse can be explicitly obtained, the above expression can be further simplified by using the Neumann expansion as

$$\mathbf{u} = \mathbf{Q}\left[\mathbf{V}^{-1} - (\mathbf{V}^{-1}\mathbf{A})\mathbf{V}^{-1} + (\mathbf{V}^{-1}\mathbf{A})^2\mathbf{V}^{-1} - \cdots\right]\mathbf{Q}^{-1}\mathbf{f}. \tag{26}$$

Summary of the joint diagonalisation approach

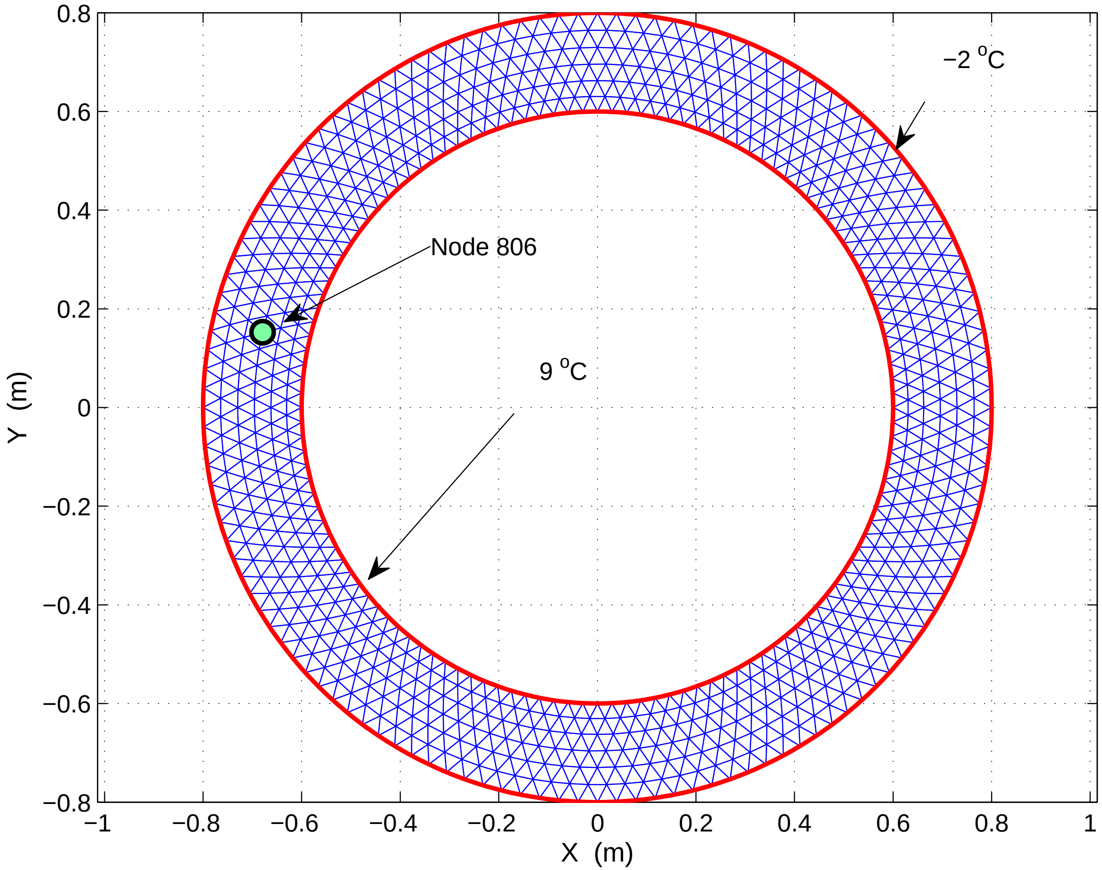
The joint diagonalisation approach is applicable to any real symmetric matrices. The response statistics for static (or steady-state) stochastic systems can be obtained by following these steps:

- Discretise the random material parameters by using the F-K-L expansion scheme
- In space dimension, discretise the unknown field with finite element mesh
- Following a standard finite element formulation procedure and taking into consideration the F-K-L expansion of random material parameters, construct the stochastic linear system

Summary of the joint diagonalisation approach

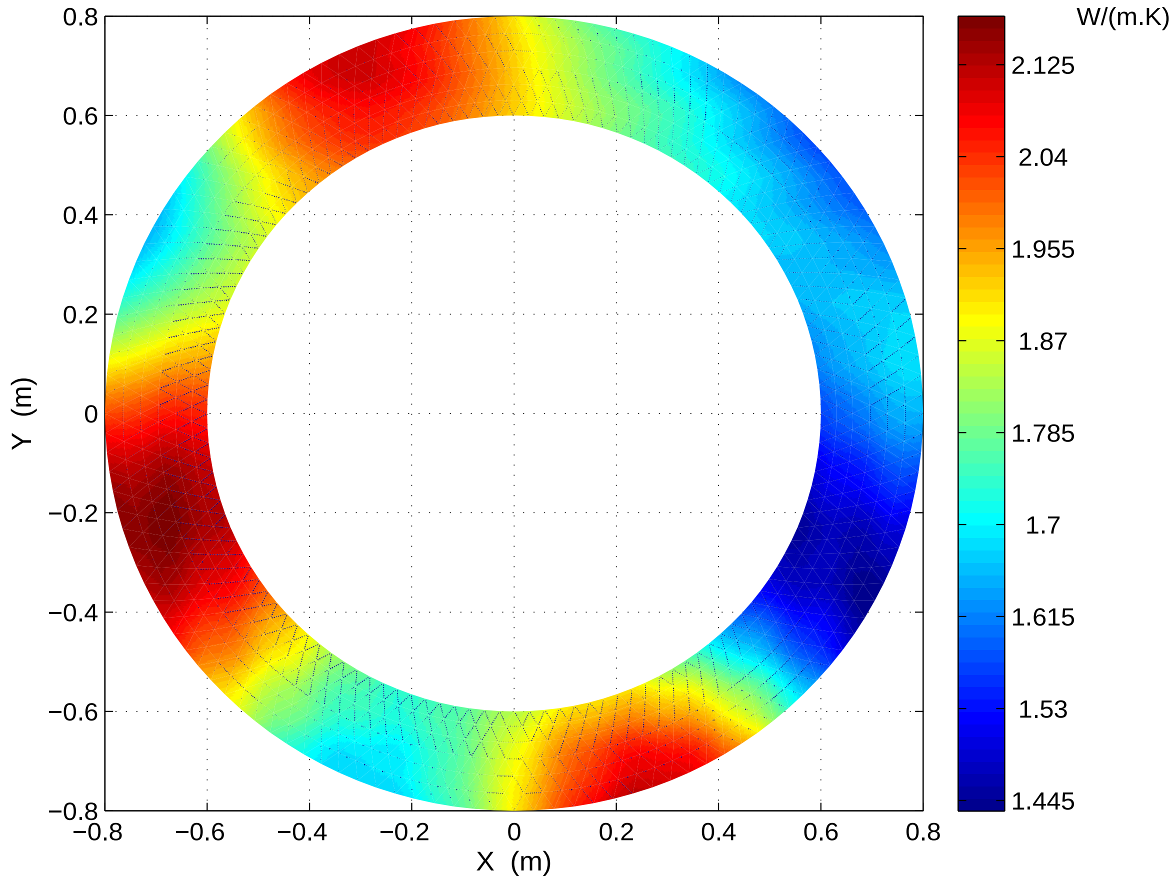
- Approximate joint diagonalisation of all matrices in the stochastic linear system
  - Check in turn all the off-diagonal entries in matrices $\mathbf{K}_j$ and find an entry (p,q) , $p \neq q$ such that $\sum_{j=1}^{m} (\mathbf{K}_j)_{pq}^2 \neq 0$.
  - For every entry (p,q) satisfying the above condition, compute the optimal Givens rotation angle θ_{opt}
  - Apply Givens similarity transformation to all the matrices
  - Repeat the above procedure until the process converges below the given threshold
- For a specific realization of random variables $\xi_j(\omega)$, $\forall\, j$ and by using Neumann expansion, the response vector $\mathbf{u}$ is obtained

Heat conduction of a concrete pipe



Finite element mesh and boundary

Heat conduction of a concrete pipe



Thermal conductivity reconstructed from the F-K-L expansion

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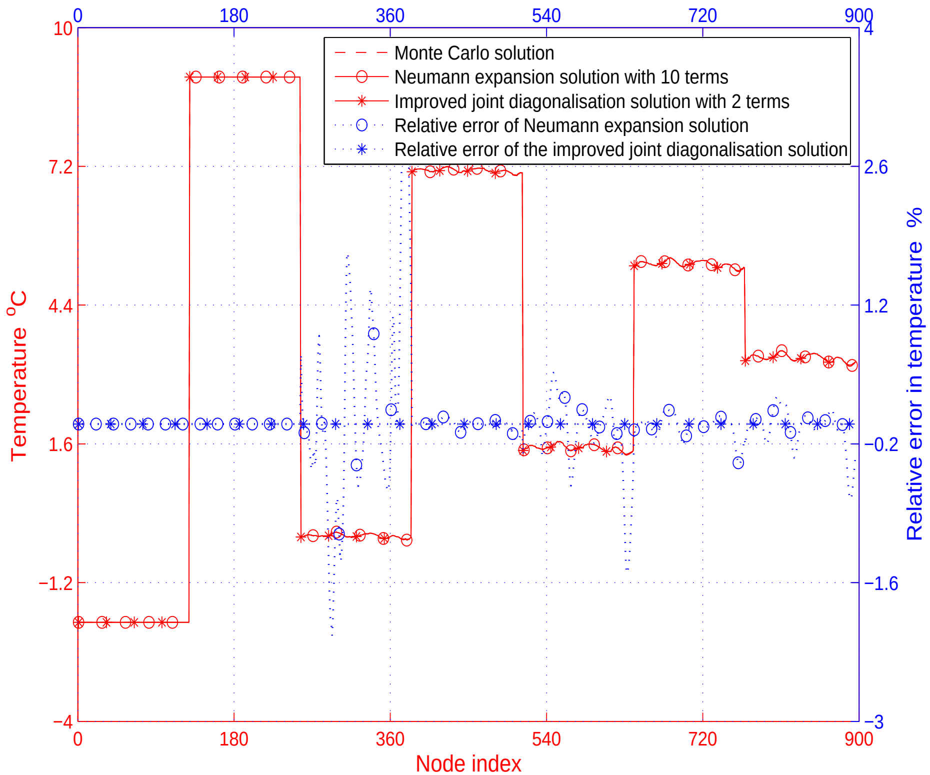
Heat conduction of a concrete pipe



Comparison of the improved joint diagonalisation solution, the Neumann expansion solution and the Monte Carlo solution

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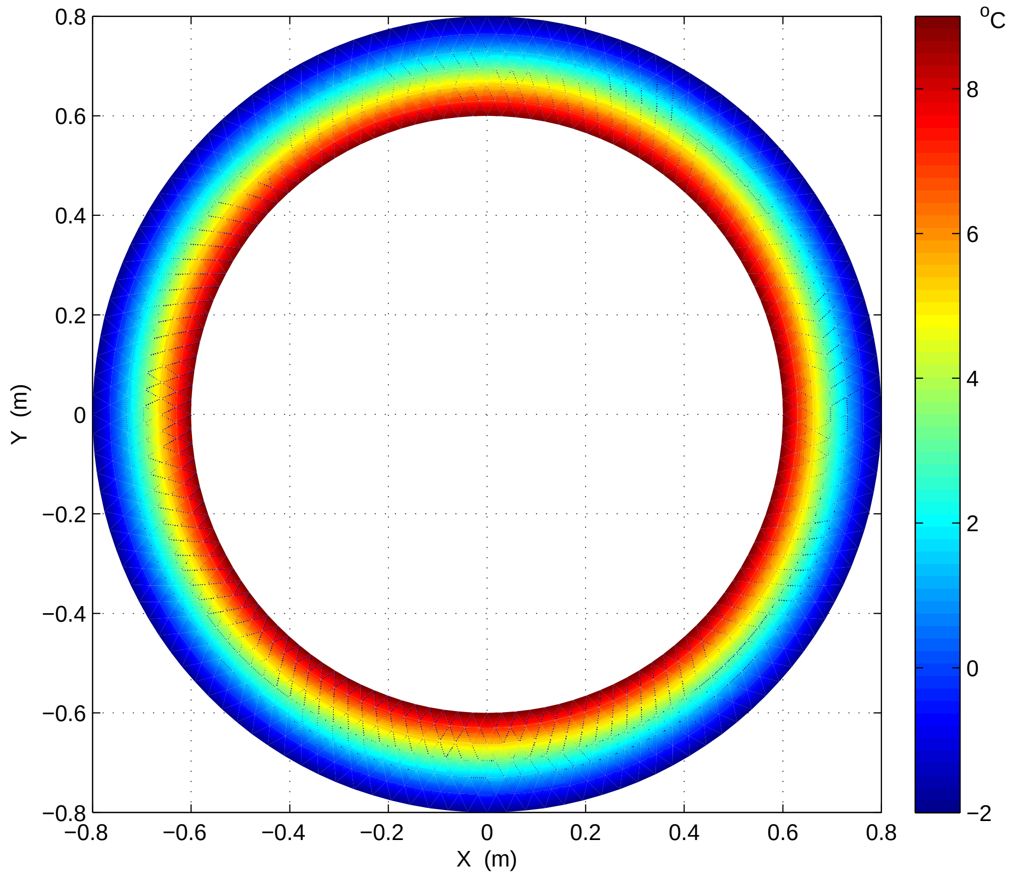
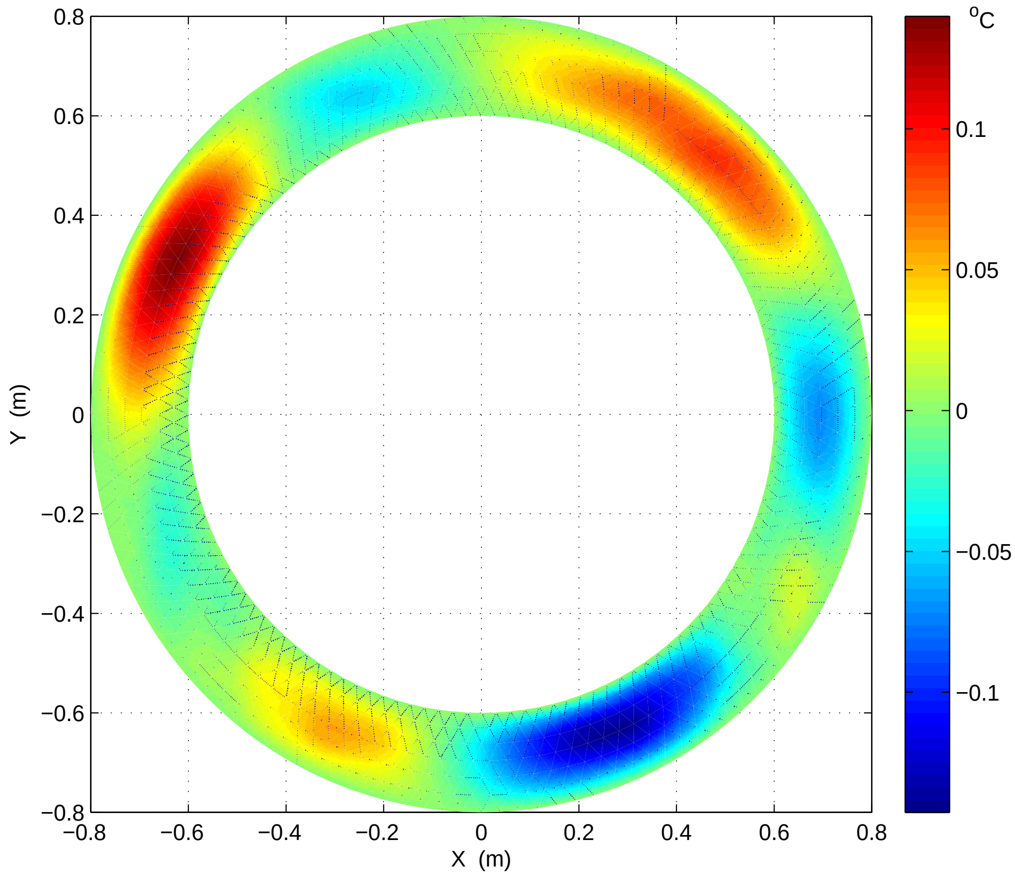
Heat conduction of a concrete pipe



A sample solution of temperature distribution

Heat conduction of a concrete pipe



Difference between the random temperature distribution and the deterministic mean-value temperature distribution

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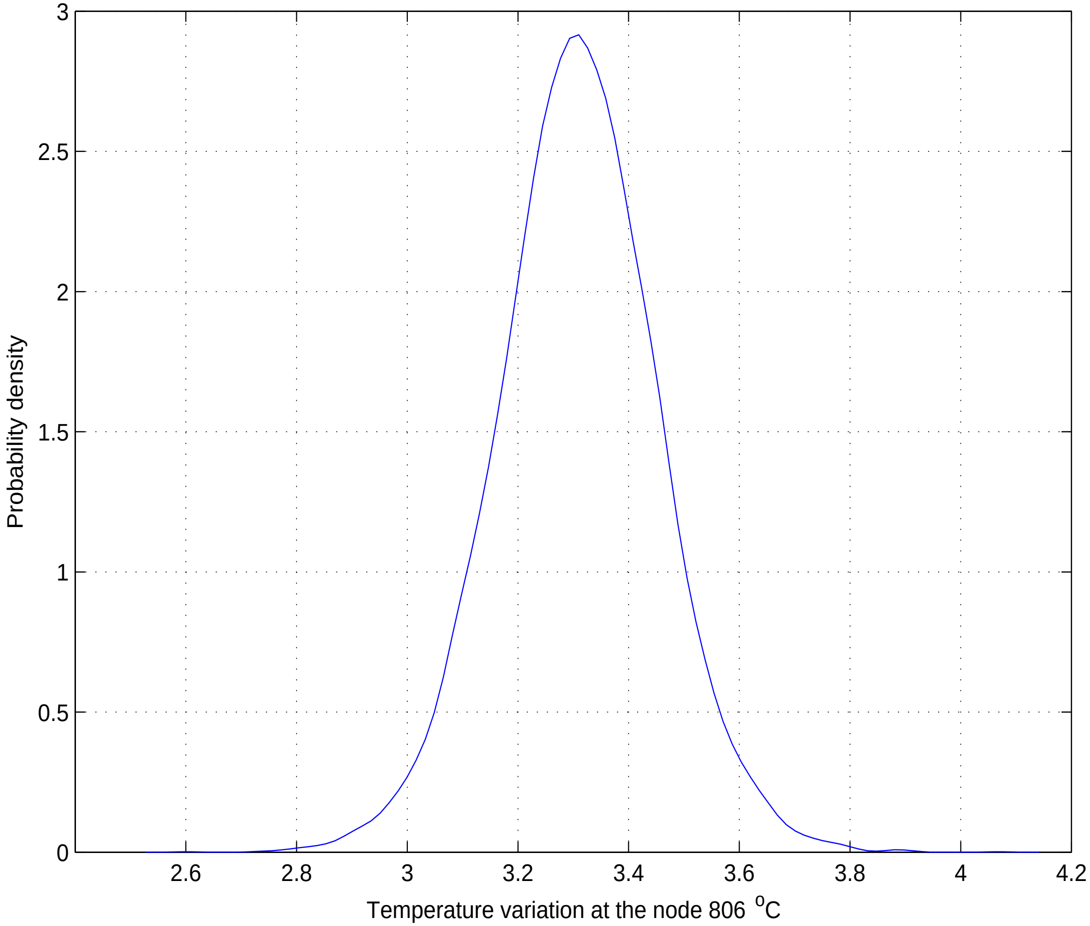
Heat conduction of a concrete pipe



Probability distribution of temperature at node 806

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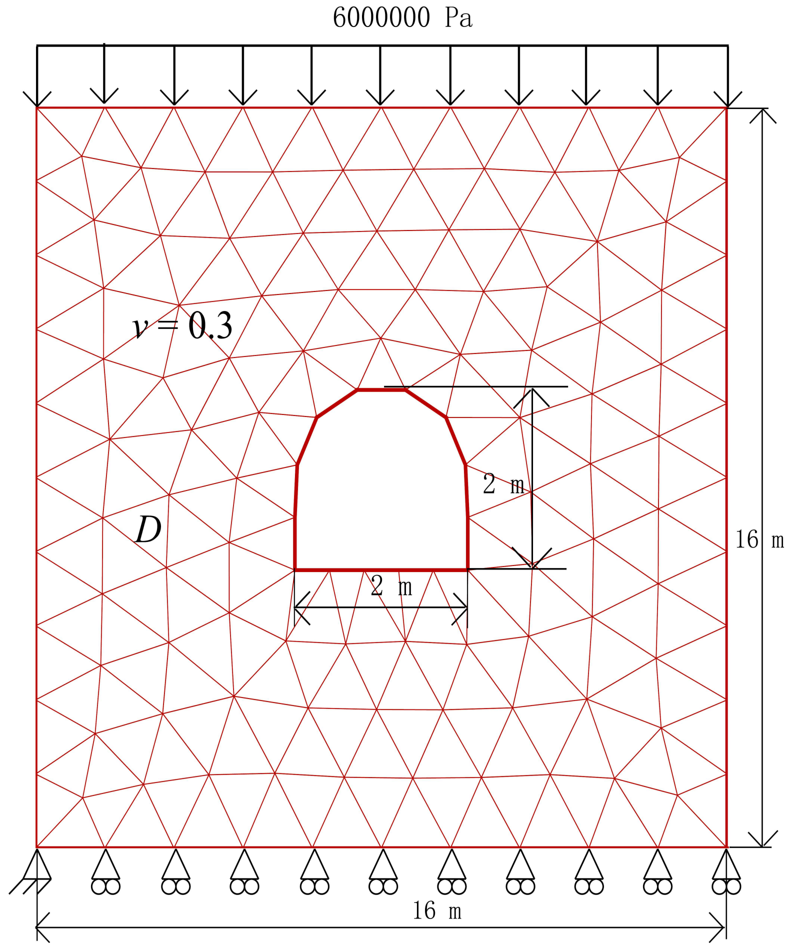
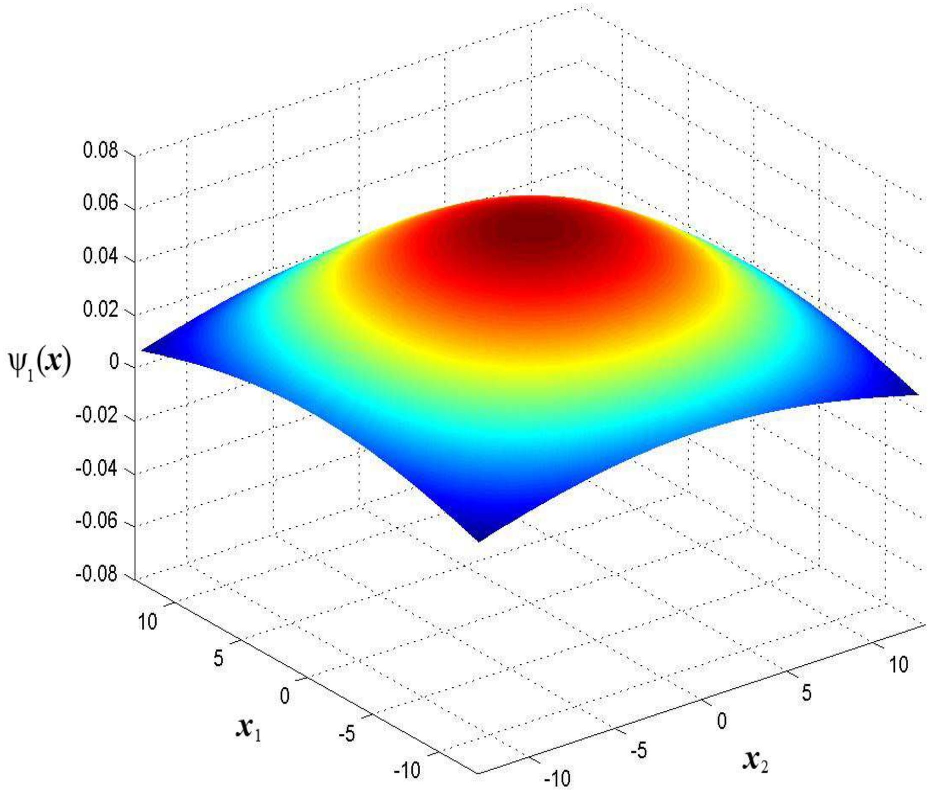
Plain strain analysis of a tunnel model



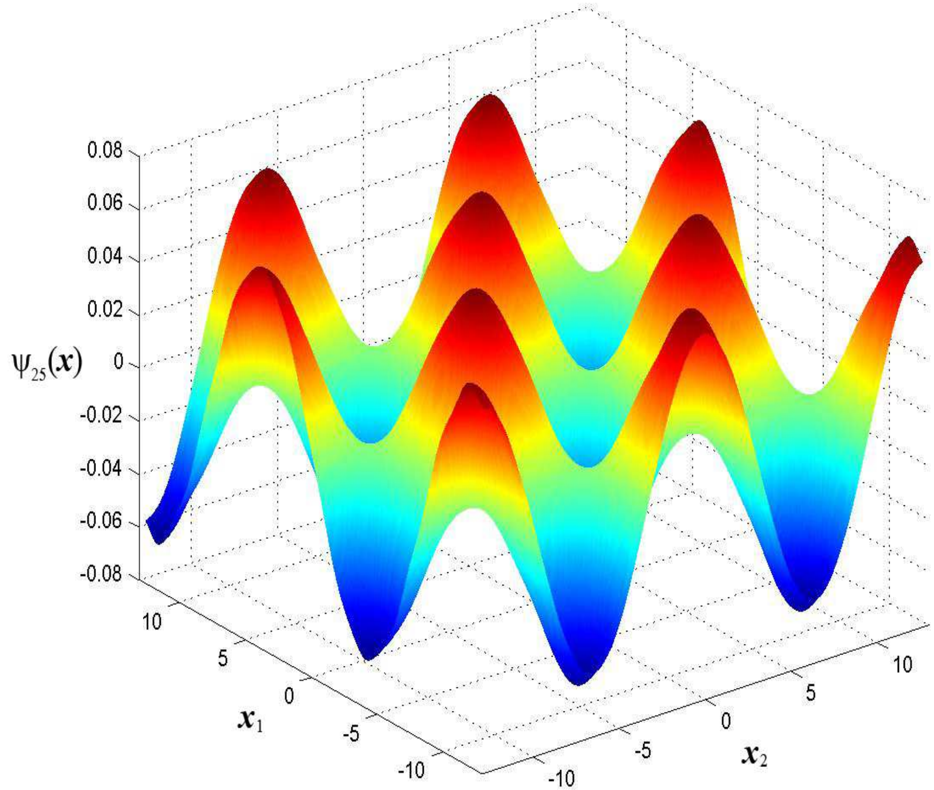
A tunnel model and its boundary conditions

Plain strain analysis of a tunnel model



The first term in the F-K-L expansion

Plain strain analysis of a tunnel model



The 25th term in the F-K-L expansion

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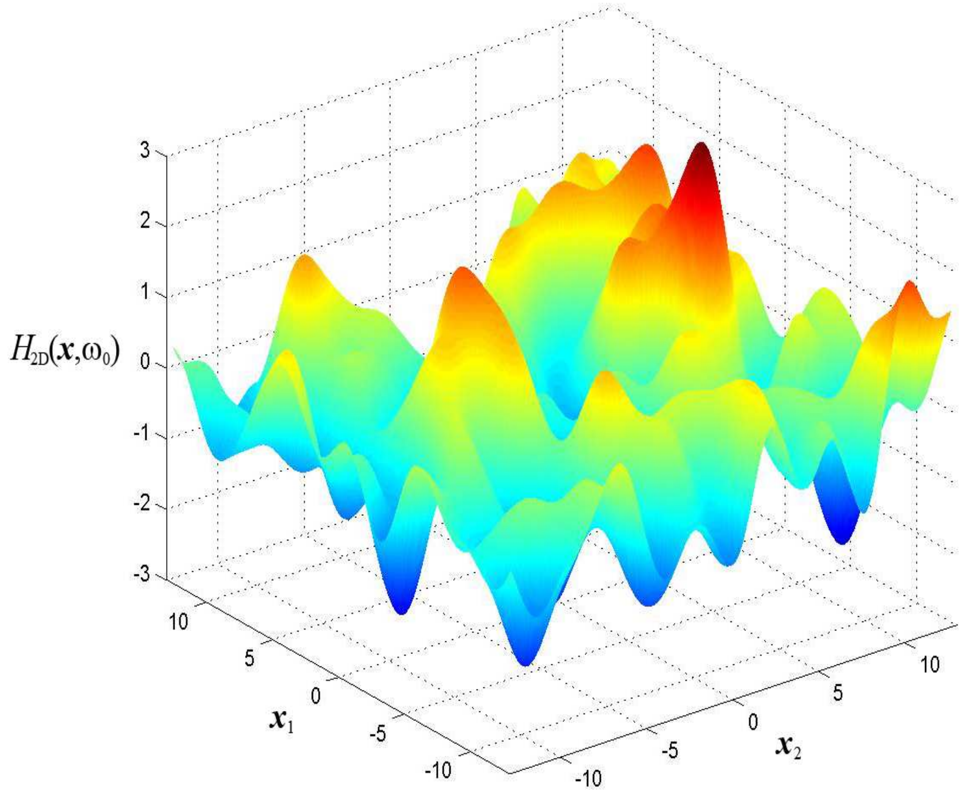
Plain strain analysis of a tunnel model



A specific realization of the random Young's modulus

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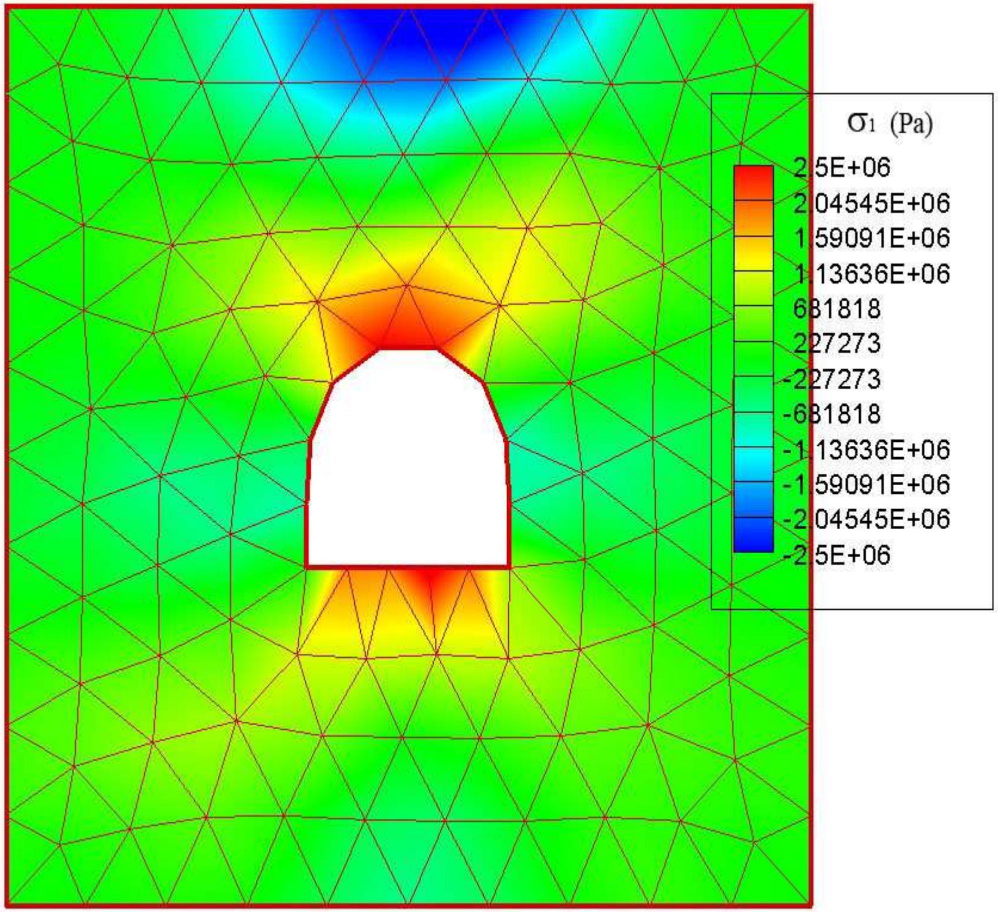
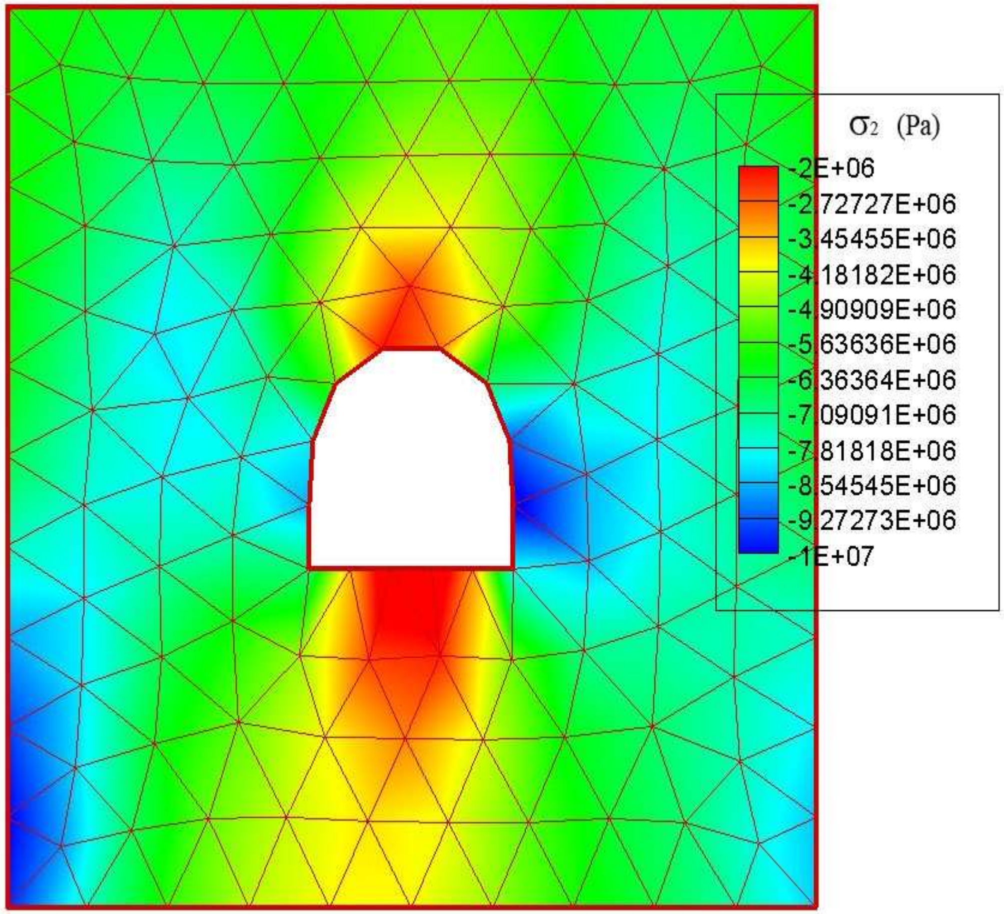
Plain strain analysis of a tunnel model



Principle stress distribution - σ_1

Plain strain analysis of a tunnel model



Principle stress distribution - σ_2

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Summary

For the solution of static and steady-state problems of random media, this paper presents an improved joint diagonalisation solution framework.

- The random medial properties are discretised by using the Fourier-Karhunen-Loève expansion scheme.
- The resulting stochastic linear system is solved by using the improved joint diagonalization method, in which
  - A Jacobi-like algorithm is developed to jointly diagonalise multiple real-symmetric matrices
  - The Neumann expansion is used to account for small off-diagonal entries and obtain accurate solutions.

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