Uncertainty quantification in computational predictions for complex aero-mechanical systems

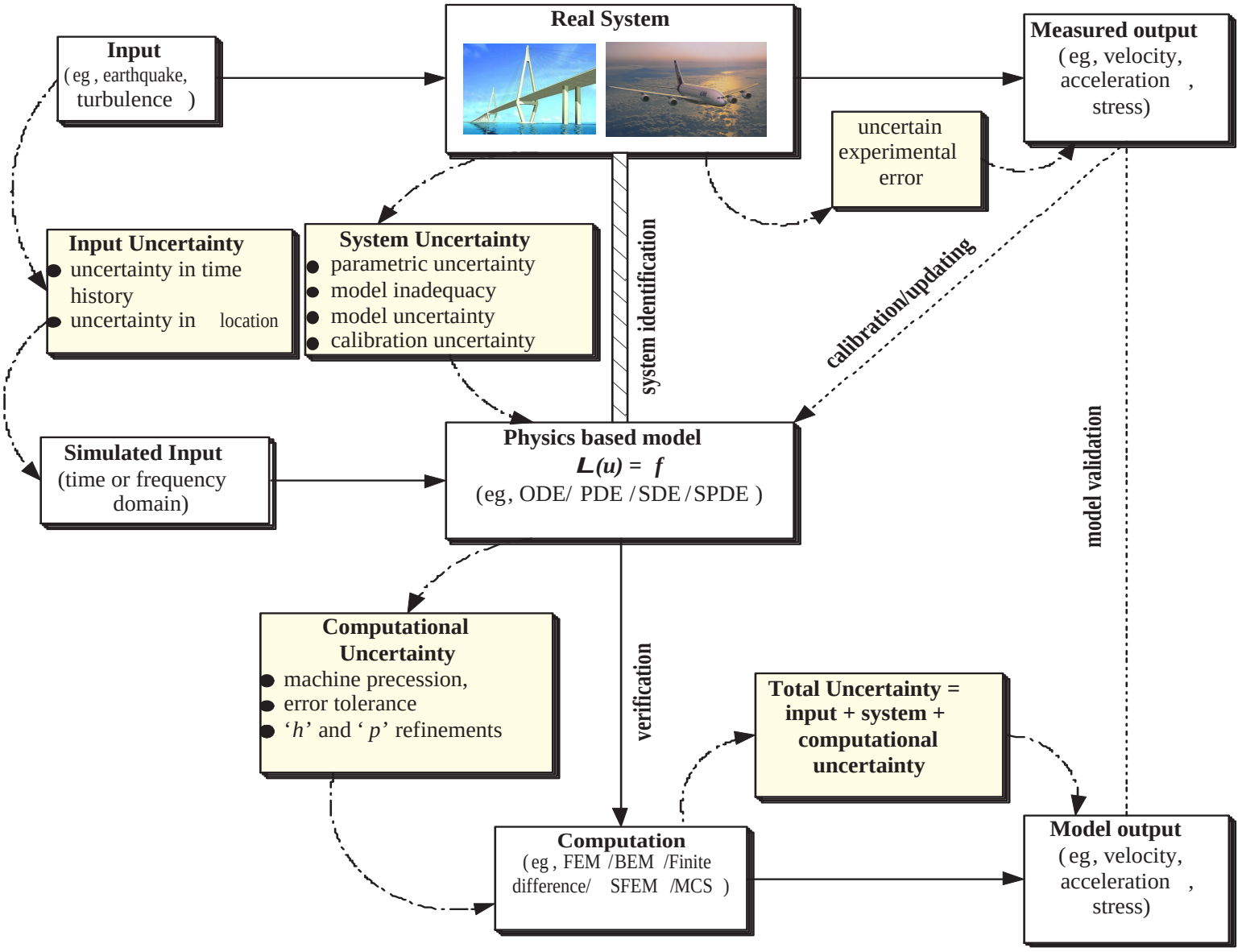
S Adhikari

School of Engineering, Swansea University, Swansea, UK

Email: S.Adhikari@swansea.ac.uk

URL: http://engweb.swan.ac.uk/~adhikaris

The role of uncertainty in science based predictions



Sources of uncertainty

(a) parametric uncertainty - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) model inadequacy - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(d) computational uncertainty - e.g, machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis, and
(e) model uncertainty - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.

Problem-types in computational science

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (deterministic)	Unknown	*Analysis (forward problem)*	FEM/BEM/Finite difference
Known (deterministic)	Incorrect (deterministic)	Known (deterministic)	*Updating/calibration*	Modal updating
Known (deterministic)	Unknown	Known (deterministic)	*System identification*	Kalman filter
Assumed (deterministic)	Unknown (deterministic)	Prescribed	*Design*	Design optimisation
Unknown	Partially Known	Known	*Structural Health Monitoring (SHM)*	SHM methods
Known (deterministic)	Known (deterministic)	Prescribed	*Control*	Modal control
Known (random)	Known (deterministic)	Unknown	*Random vibration*	Random vibration methods

Problem-types in computational science

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (random)	Unknown	*Stochastic analysis (forward problem)*	SFEM/SEA/RMT
Known (random)	Incorrect (random)	Known (random)	*Probabilistic updating/calibration*	Bayesian calibration
Assumed (random/deterministic)	Unknown (random)	Prescribed (random)	*Probabilistic design*	RBOD
Known (random/deterministic)	Partially known (random)	Partially known (random)	*Joint state and parameter estimation*	Particle Kalman Filter/Ensemble Kalman Filter
Known (random/deterministic)	Known (random)	Known from experiment and model (random)	*Model validation*	Validation methods
Known (random/deterministic)	Known (random)	Known from different computations (random)	*Model verification*	verification methods

UQ approaches: key challenges

The main difficulties are:

- the computational time can be prohibitively high compared to a deterministic analysis for real problems,
- the volume of input data can be unrealistic to obtain for a credible probabilistic analysis,
- the predictive accuracy can be poor if considerable resources are not spend on the previous two items, and
- the need for general purpose software tools: as the state-of-the art methodology stands now (such as the Stochastic Finite Element Method), only very few highly trained professionals (such as those with PhDs) can even attempt to apply the complex concepts (e.g., random fields) and methodologies to real-life problems.

Main objectives

Our work is aimed at developing methodologies [the 10-10-10 challenge] with the ambition that they should:

- not take more than 10 times the computational time required for the corresponding deterministic approach;
- result a predictive accuracy within 10% of direct Monte Carlo Simulation (MCS);
- use no more than 10 times of input data needed for the corresponding deterministic approach; and
- enable 'normal' engineering graduates to perform probabilistic structural dynamic analyses with a reasonable amount of training.

Outline of the presentation

- Uncertainty Quantification (UQ) in structural dynamics
- Review of current approaches
- Parametric approach: stochastic finite element
  - Spectral approach
  - Gaussian emulator
- Non-parametric approach: Wishart random matrices
  - Parameter selection
  - Computational method
- Experimental results
- Conclusions & future directions

Current UQ approaches - 1

Two different approaches are currently available

- Parametric approaches : Such as the Stochastic Finite Element Method (SFEM):
  - aim to characterize parametric uncertainty (type ‘a’)
  - assumes that stochastic fields describing parametric uncertainties are known in details
  - suitable for low-frequency dynamic applications

Current UQ approaches - 2

- Nonparametric approaches: Such as the Statistical Energy Analysis (SEA) and Wishart random matrix theory:
  - aim to characterize nonparametric uncertainty (types ‘b’ - ‘e’)
  - does not consider parametric uncertainties in details
  - suitable for high/mid-frequency dynamic applications

Random continuous dynamical systems

The equation of motion:

$$\rho(\mathbf{r},\theta)\frac{\partial^2 U(\mathbf{r},t)}{\partial t^2}+L_1\frac{\partial U(\mathbf{r},t)}{\partial t}+L_2U(\mathbf{r},t)=p(\mathbf{r},t);\quad \mathbf{r}\in\mathcal{D},t\in[0,T] \tag{1}$$

$U(\mathbf{r},t)$ is the displacement variable, $\mathbf{r}$ is the spatial position vector and t is time.

- $\rho(\mathbf{r},\theta)$ is the random mass distribution of the system, $p(\mathbf{r},t)$ is the distributed time-varying forcing function, L_1 is the random spatial self-adjoint damping operator, L_2 is the random spatial self-adjoint stiffness operator.
- Eq (1) is a Stochastic Partial Differential Equation (SPDE) [ie, the coefficients are random processes].

Stochastic Finite Element Method

Problems of structural dynamics in which the uncertainty in specifying mass and stiffness of the structure is modeled within the framework of random fields can be treated using the Stochastic Finite Element Method (SFEM). The application of SFEM in linear structural dynamics typically consists of the following key steps:

1. Selection of appropriate probabilistic models for parameter uncertainties and boundary conditions
2. Replacement of the element property random fields by an equivalent set of a finite number of random variables. This step, known as the 'discretisation of random fields' is a major step in the analysis.
3. Formulation of the equation of motion of the form $\mathbf{D}(\omega)\mathbf{u} = \mathbf{f}$ where $\mathbf{D}(\omega)$ is the random dynamic stiffness matrix, $\mathbf{u}$ is the vector of random nodal displacement and $\mathbf{f}$ is the applied forces. In general $\mathbf{D}(\omega)$ is a random symmetric complex matrix.
4. Calculation of the response statistics by either (a) solving the random eigenvalue problem, or (b) solving the set of complex random algebraic equations.

Spectral Decomposition of random fields-1

- Just like the displacement fields (or any other continuous state variables) in the deterministic FEM, in SFEM we need to discretise the random fields appearing in the governing SPDE.
- Various approaches (mid-point method, collocation method, weighted integral approach etc) have been proposed in literature.
- Here we use the spectral decomposition of random fields due to its useful mathematical properties (eg, orthogonal eigenfunctions, mean-square convergence etc).

Spectral Decomposition of random fields-2

Suppose $H(\mathbf{r}, \theta)$ is a random field with a covariance function $C_H(\mathbf{r}_1, \mathbf{r}_2)$ defined in a space Ω. Since the covariance function is finite, symmetric and positive definite it can be represented by a spectral decomposition. Using this spectral decomposition, the random process $H(\mathbf{r}, \theta)$ can be expressed in a generalized fourier type of series as

$$H(\mathbf{r}, \theta) = H_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \xi_i(\theta) \varphi_i(\mathbf{r}) \tag{2}$$

where $\xi_i(\theta)$ are uncorrelated random variables, λ_i and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\Omega} C_H(\mathbf{r}_1, \mathbf{r}_2) \varphi_i(\mathbf{r}_1) \mathrm{d}\mathbf{r}_1 = \lambda_i \varphi_i(\mathbf{r}_2), \quad \forall\ i = 1, 2, \cdots \tag{3}$$

The spectral decomposition in equation (2) is known as the Karhunen-Loève (KL) expansion. The series in (2) can be ordered in a decreasing series so that it can be truncated after a finite number of terms with a desired accuracy.

Exponential autocorrelation function

The autocorrelation function:

$$C(x_1, x_2) = e^{-|x_1 - x_2|/b} \tag{4}$$

The underlying random process $H(x, \theta)$ can be expanded using the Karhunen-Loève expansion in the interval $-a \leq x \leq a$ as

$$H(x, \theta) = \sum_{n=1}^{\infty} \left[\xi_n \sqrt{\lambda_n} \varphi_n(x) + \xi_n^* \sqrt{\lambda_n^*} \varphi_n^*(x) \right]. \tag{5}$$

The corresponding eigenvalues and eigenfunctions:

$$\lambda_n = \frac{2c}{\omega_n^2 + c^2}; \quad \varphi_n(x) = \frac{\cos(\omega_n x)}{\sqrt{a + \dfrac{\sin(2\omega_n a)}{2\omega_n}}} \quad \text{and} \quad \tan(\omega a) = \frac{c}{\omega}; \text{ for even } n \tag{6}$$

$$\lambda_n^* = \frac{2c}{\omega_n^{*\,2} + c^2}; \quad \varphi_n^*(x) = \frac{\sin(\omega_n^* x)}{\sqrt{a - \dfrac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \quad \text{and} \quad \tan(\omega^* a) = \frac{\omega^*}{-c}; \text{ for odd } n \tag{7}$$

Triangular autocorrelation function

The autocorrelation function:

$$C(x_1, x_2) \;=\; 1 - d|x_1 - x_2|; \quad |x_1 - x_2| \in [0, \frac{1}{d}]. \tag{8}$$

The underlying random process $H(x, \theta)$ can be expanded using the Karhunen-Loève expansion (5) and the eigenvalues and eigenfunctions in the interval $0 \leq x \leq a$:

$$\lambda_n = \frac{2d}{\omega_n^2}; \quad \varphi_n(x) = \frac{\cos(\omega_n x)}{\sqrt{\frac{a}{2} + \frac{\sin(2\omega_n a)}{2\omega_n}}} \quad \text{and} \quad \tan(\frac{\omega_n a}{2}) = \frac{2}{\omega_n(\frac{2}{d} - a)}; \text{ for even } n$$

$$\lambda_n^* = \frac{2d}{\omega_n^{*2}}; \quad \varphi_n^*(x) = \frac{\cos(\omega_n x) + \tan(\frac{\omega_n^* a}{2})\sin(\omega_n^* x)}{\sqrt{a + (\tan^2(\frac{\omega_n^* a}{2}) - 1)(\frac{a}{2} - \frac{\sin(2\omega_n^* a)}{4\omega_n^*}) + \frac{\sin^2(\omega_n^* a)}{\omega_n^*}\tan(\frac{\omega_n^* a}{2})}}$$

$$\text{and} \quad \omega_n^* = n\frac{\pi}{a}; \text{ for odd } n \tag{9}$$

Spectral Stochastic Finite Element Method (SSFEM)

Like the classical finite element method, suppose that frequency-dependent displacement within an element is interpolated from the nodal displacements as

$$u_e(\mathbf{r},\omega) = \mathbf{N}^T(\mathbf{r},\omega)\widehat{\mathbf{u}}_e(\omega) \tag{10}$$

Here $\widehat{\mathbf{u}}_e(\omega) \in \mathbb{C}^n$ is the nodal displacement vector and $\mathbf{N}(\mathbf{r},\omega) \in \mathbb{C}^n$, the vector of frequency-dependent shape functions and n is the number of the nodal degrees-of-freedom. Suppose the $s_j(\mathbf{r},\omega) \in \mathbb{C}, j = 1, 2, \cdots m$ (m is the order of the equation) are the basis functions which satisfy the governing differential equation in the frequency domain. It can be shown that the shape function vector can be expresses as

$$\mathbf{N}(\mathbf{r},\omega) = \mathbf{\Gamma}(\omega)\mathbf{s}(\mathbf{r},\omega) \tag{11}$$

where the vector $\mathbf{s}(\mathbf{r},\omega) = \{s_j(\mathbf{r},\omega)\}^T, \forall\, j \in \mathbb{C}^m$ and the complex matrix $\mathbf{\Gamma}(\omega) \in \mathbb{C}^{nm}$ depends

on the boundary conditions.

Stochastic element matrices-1

Using the weak-form, the frequency depended random stiffness, mass and damping matrices can be obtained as

$$\mathbf{K}_e(\omega,\theta) = \int_{\mathcal{D}_e} k(\mathbf{r},\theta)\mathcal{L}_2\left\{\mathbf{N}(\mathbf{r},\omega)\right\}\mathcal{L}_2\left\{\mathbf{N}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \in \mathbb{C}^{nn} \tag{12}$$

$$\mathbf{M}_e(\omega,\theta) = \int_{\mathcal{D}_e} \rho(\mathbf{r},\theta)\mathbf{N}(\mathbf{r},\omega)\mathbf{N}^T(\mathbf{r},\omega)\mathrm{d}\mathbf{r} \in \mathbb{C}^{nn} \tag{13}$$

$$\text{and} \quad \mathbf{C}_e(\omega,\theta) = \int_{\mathcal{D}_e} c(\mathbf{r},\theta)\mathcal{L}_1\left\{\mathbf{N}(\mathbf{r},\omega)\right\}\mathcal{L}_1\left\{\mathbf{N}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \in \mathbb{C}^{nn} \tag{14}$$

Where, $(\bullet)^T$ denotes Hermitian transpose, $k(\mathbf{r},\theta)$ is the random distributed stiffness parameter, $\mathcal{L}_2\{\bullet\}$ is the strain energy operator, $c(\mathbf{r},\theta)$ is the random distributed damping parameter and $\mathcal{L}_1\{\bullet\}$ is the energy dissipation operator. The random fields $k(\mathbf{r},\theta)$, $\rho(\mathbf{r},\theta)$ and $c(\mathbf{r},\theta)$ should be expanded using the Karhunen-Loève expansion.

Stochastic element matrices-2

Using finite number of terms, each of the complex element matrices can be expanded in a spectral series as

$$\mathbf{K}_e(\omega,\theta) = \mathbf{K}_{0e}(\omega) + \sum_{j=1}^{N_{\mathrm{K}}} \xi_{\mathrm{K_j}}(\theta)\mathbf{K}_{je}(\omega), \quad \mathbf{M}_e(\omega,\theta) = \mathbf{M}_{0e}(\omega) + \sum_{j=1}^{N_{\mathrm{M}}} \xi_{\mathrm{M_j}}(\theta)\mathbf{M}_{je}(\omega) \tag{15}$$

and $$\mathbf{C}_e(\omega,\theta) = \mathbf{C}_{0e}(\omega) + \sum_{j=1}^{N_{\mathrm{C}}} \xi_{\mathrm{C_j}}(\theta)\mathbf{C}_{je}(\omega) \tag{16}$$

The complex deterministic matrices can be obtained as (the stiffness matrix is shown only)

$$\mathbf{K}_{0e}(\omega) = \int_{\mathcal{D}_e} k_0(\mathbf{r})\mathcal{L}_2\left\{\mathbf{N}(\mathbf{r},\omega)\right\}\mathcal{L}_2\left\{\mathbf{N}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \in \mathbb{C}^{nn} \tag{17}$$

and $$\mathbf{K}_{je}(\omega) = \sqrt{\lambda_{\mathrm{K_j}}}\int_{\mathcal{D}_e} \varphi_{\mathrm{K_j}}(\mathbf{r})\mathcal{L}_2\left\{\mathbf{N}(\mathbf{r},\omega)\right\}\mathcal{L}_2\left\{\mathbf{N}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \in \mathbb{C}^{nn}; \quad \forall j = 1,2,\cdots,N_{\mathrm{K}} \tag{18}$$

Stochastic element matrices-3

Substituting the shape functions one obtains

$$\mathbf{K}_{0e}(\omega) = \mathbf{\Gamma}(\omega)\widetilde{\mathbf{K}}_{0e}(\omega)\mathbf{\Gamma}^T(\omega) \tag{19}$$

$$\text{and} \quad \mathbf{K}_{je}(\omega) = \sqrt{\lambda_{\mathrm{K_j}}}\mathbf{\Gamma}(\omega)\widetilde{\mathbf{K}}_{je}(\omega)\mathbf{\Gamma}^T(\omega); \quad \forall j = 1, 2, \cdots, N_{\mathrm{K}} \tag{20}$$

where [a]

$$\widetilde{\mathbf{K}}_{0e}(\omega) = \int_{\mathcal{D}_e} k_0(\mathbf{r})\mathcal{L}_2\left\{\mathbf{s}(\mathbf{r},\omega)\right\}\mathcal{L}_2\left\{\mathbf{s}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \in \mathbb{C}^{mm} \tag{21}$$

$$\text{and} \quad \widetilde{\mathbf{K}}_{je}(\omega) = \int_{\mathcal{D}_e} \varphi_{\mathrm{K_j}}(\mathbf{r})\mathcal{L}_2\left\{\mathbf{s}(\mathbf{r},\omega)\right\}\mathcal{L}_2\left\{\mathbf{s}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \in \mathbb{C}^{mm}; \quad \forall j = 1, 2, \cdots, N_{\mathrm{K}} \tag{22}$$

[a] paper under review in AIAA

Parametric uncertainly quantification-1

- Global matrices can be assembled using the element matrices following the usual method. Each global matrix has a general series form involving random variables:

$$\mathbf{G}(\theta) = \mathbf{G}_0(\omega) + \sum_{j=1}^{N_{\mathrm{G}}} \xi_{\mathrm{G_j}}(\theta)\mathbf{G}_j(\omega)$$

- If the original random fields are Gaussian, then the resulting random matrices will be Gaussian random matrices.
- Efficient computational approaches are needed for stochastic analysis of discrete dynamical systems.

Parametric uncertainly quantification-2

- Complex engineering dynamical systems with parametric uncertainty are investigated running computer codes (e.g, with Monte Carlo Simulation), also known as simulators.
- A simulator is a function $\eta(\cdot)$ that, given an input **x**, it produces an output **y**.
- Sophisticated simulators can have a high cost of execution.

Emulator - 1

- A possible solution is to build an emulator of the expensive simulator.
- An emulator is a statistical approximation to the simulator, i.e., it provides a probability distribution for $\eta(\cdot)$.
- Emulators have already been implemented in a number of fields, which include:
  - Environmental science (Challenor et al., 2006)
  - Climate modeling (Rougier, 2007)
  - Medical science (Haylock and O'Hagan, 1996)

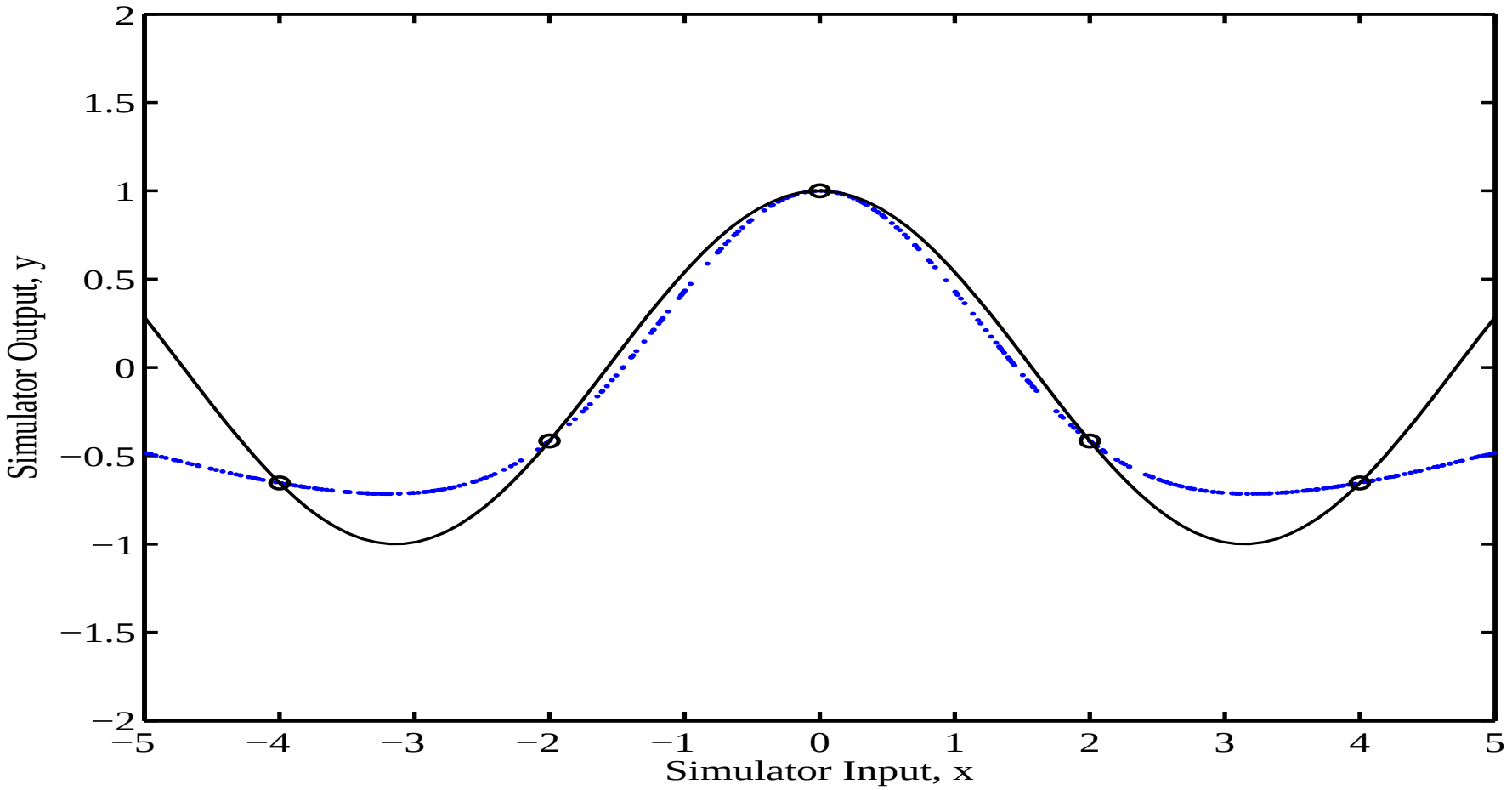
Emulator - 2

- An emulator is built by first choosing n design points in the input domain of the simulator and obtaining the training set $\{\eta(\mathbf{x}_1), \ldots, \eta(\mathbf{x}_n)\}$.
- After that initial choice is made, an emulator should:
  - Reproduce the known output at any design point.
  - At any untried input, provide a distribution whose mean value constitutes a plausible interpolation of the training data. The probability distribution around this mean value should also express the uncertainty about how the emulator might interpolate.

Emulator: simple example - 1

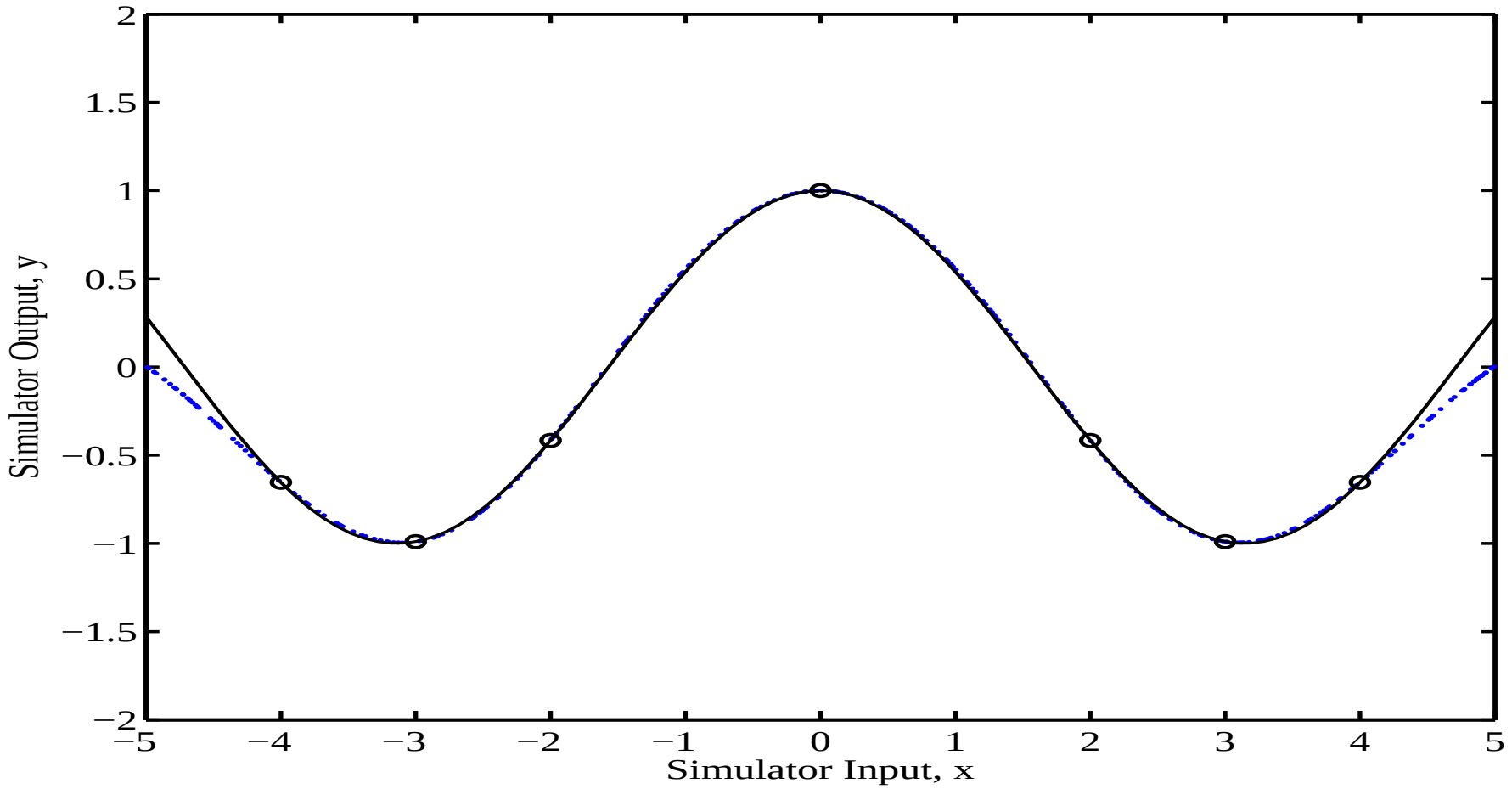
- To illustrate what do the above criteria mean, an emulator was constructed to approximate the simple simulator $\mathbf{y} = cos(\mathbf{x})$.
- In the following figures, the solid line is the true output of the simulator. The circles represent the training runs, and the dots are the mean of the distribution provided by the emulator, which is the approximation.
- Note how the approximation improves when more design points are chosen.

Emulator: simple example - 2



Approximation using 5 design points.

Emulator: simple example - 3

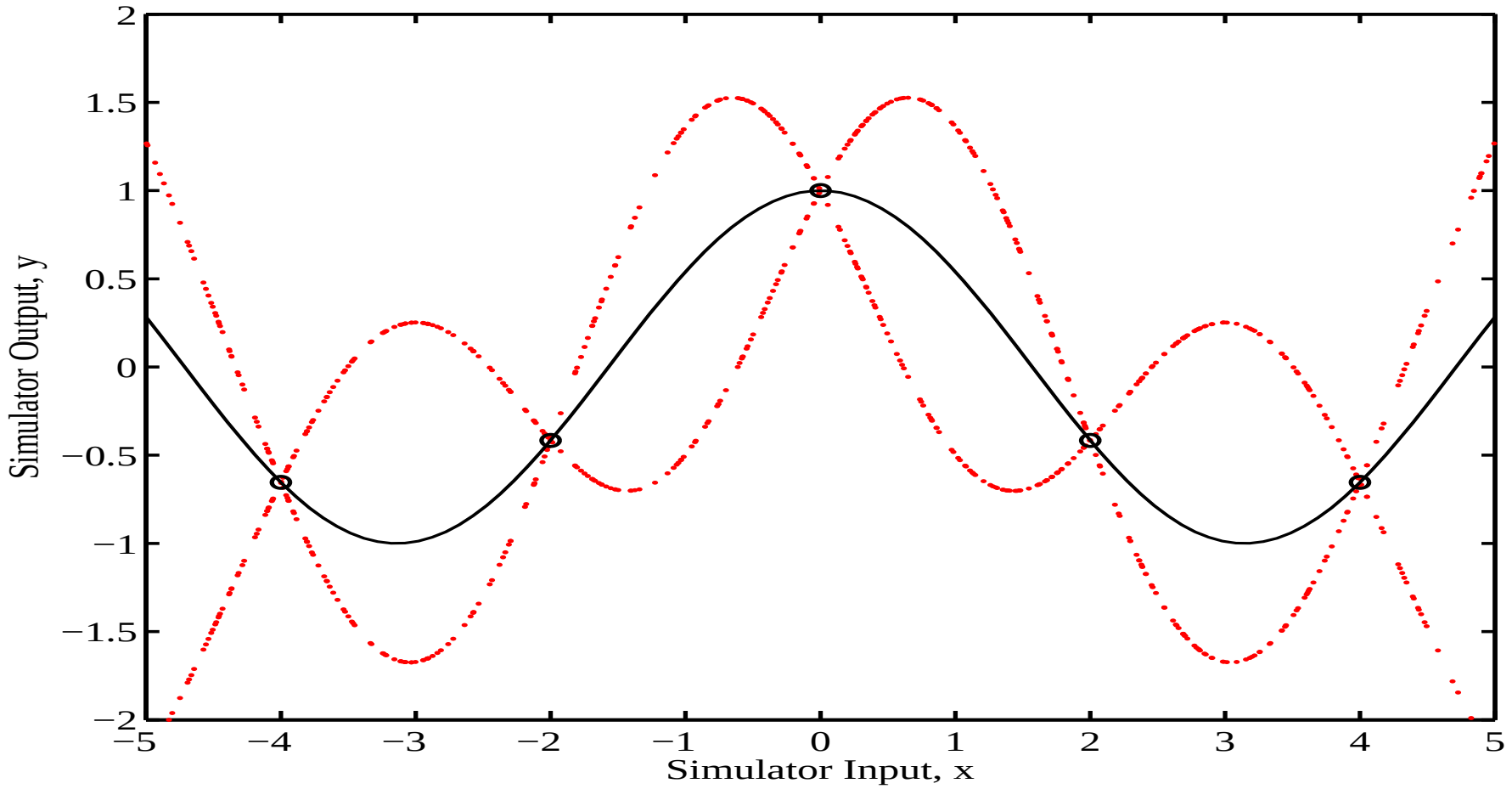


Approximation using 7 design points.

Emulator: simple example - 4

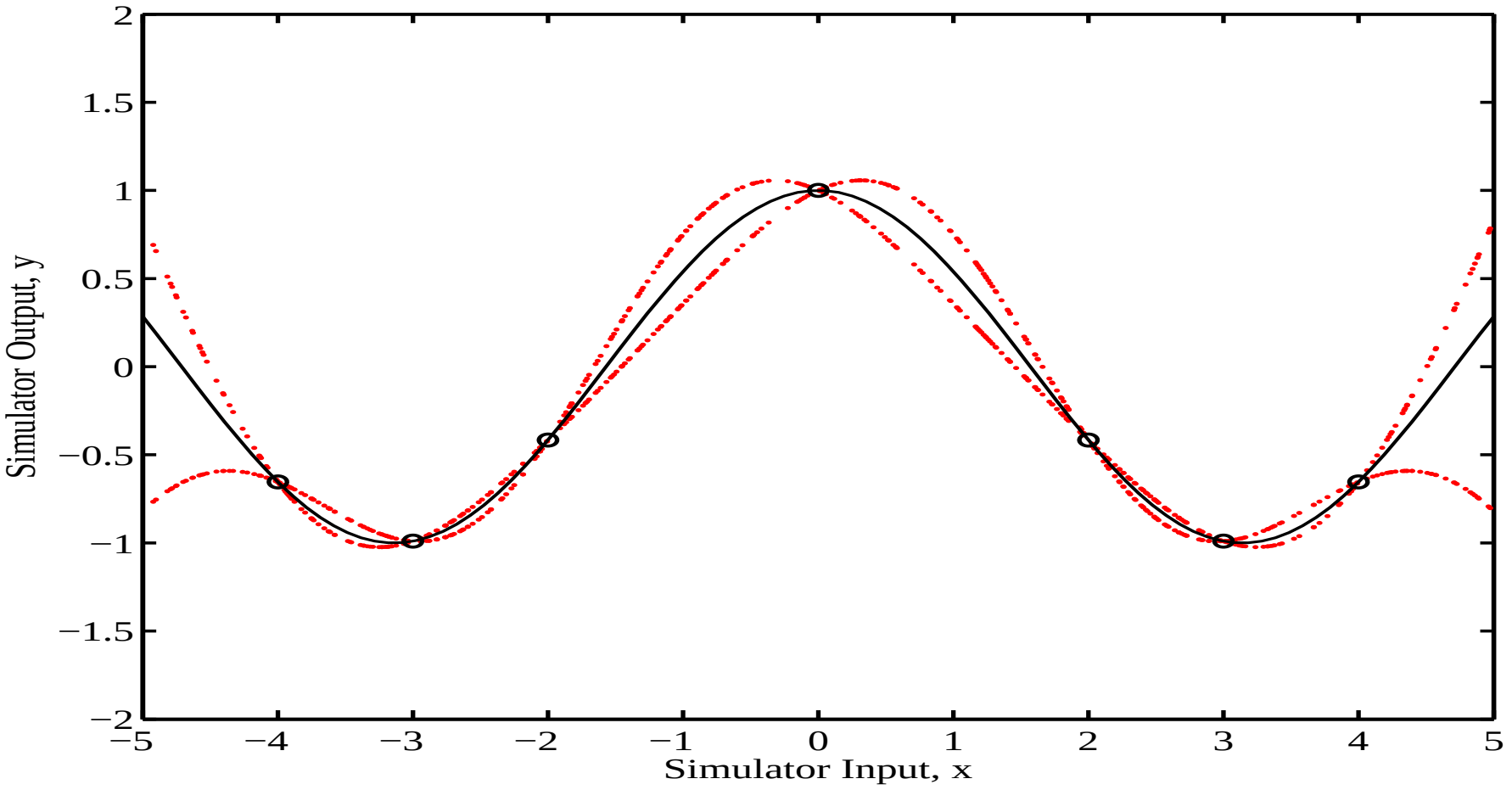
- In the same way, the following figures show upper and lower probability bounds of two standard deviations for the mean of the emulator. The solid line is the true output of the simulator. The circles represent the training runs, and the dots are the bounds.
- Note how the uncertainty about the approximation is reduces as more design points are chosen.

Emulator: simple example - 5



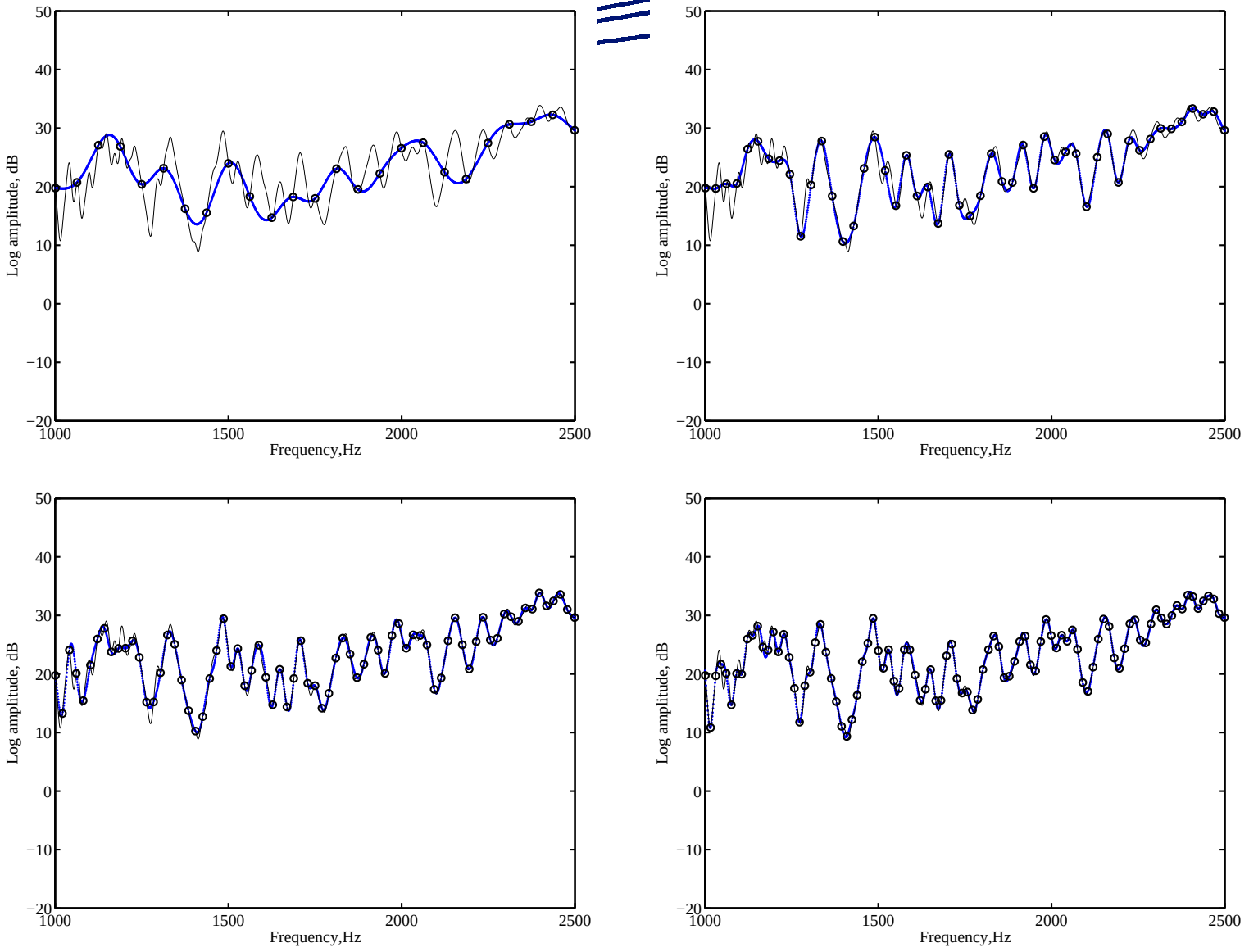
Uncertainty using 5 design points.

Emulator: simple example - 6



Uncertainty using 7 design points.

Application: experimentally measured FRF of a plate



Emulation with 25, 50, 75 and 100 design points, mid-freq range.

Stochastic Finite Element (SFE) problems

- A random field $\mathcal{H}(\mathbf{x}, \theta)$ can be discretized using the Karhunen-Loève (KL) expansion as

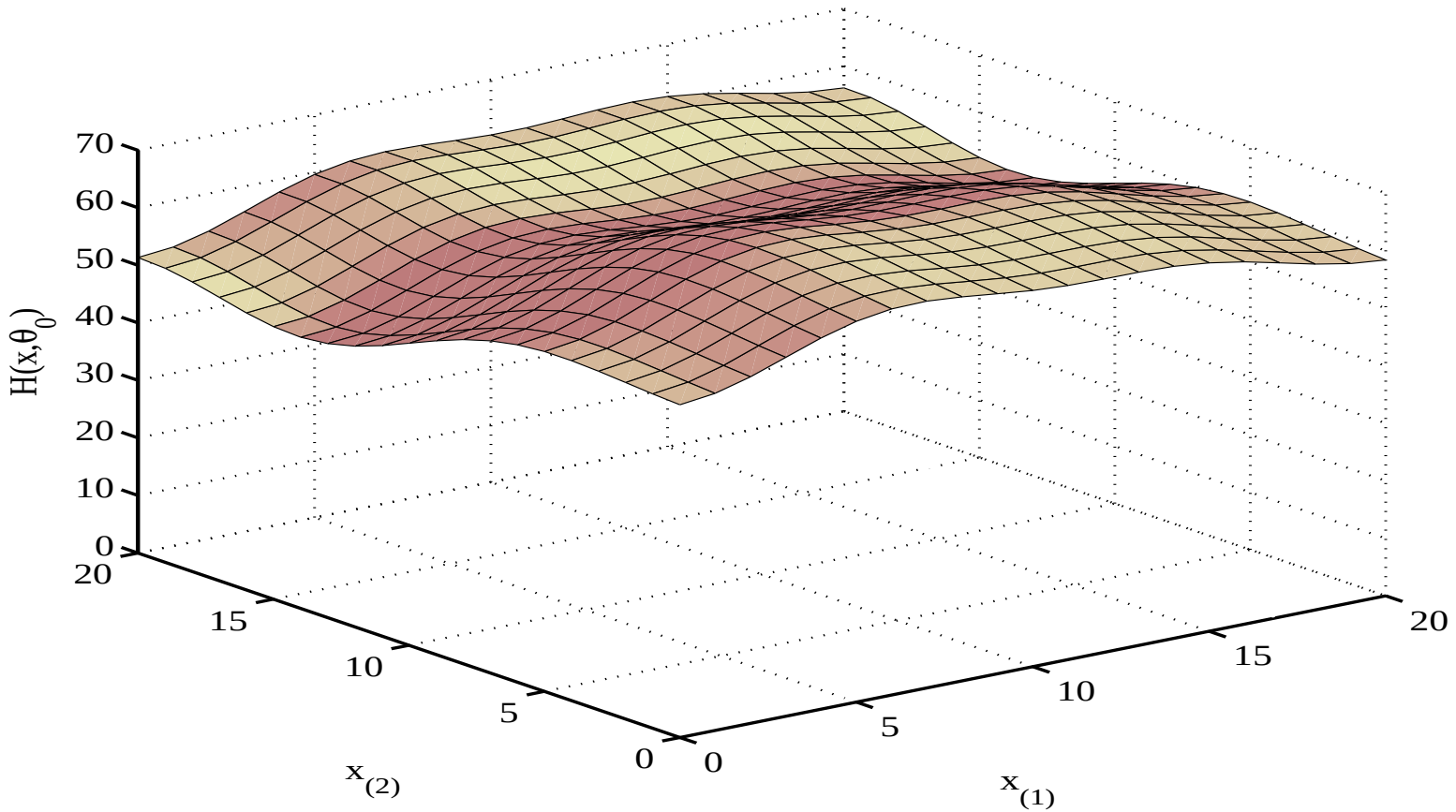
$$\mathcal{H}(\mathbf{x}, \theta) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i} \xi_i(\theta) \phi_i(\mathbf{x}) \tag{23}$$

- Using this, the system equation can be represented as

$$[\mathbf{K}_0 + \sum_{i=0}^{M} \mathbf{K}_i \xi(\theta)] \mathbf{u} = \mathbf{f} \tag{24}$$

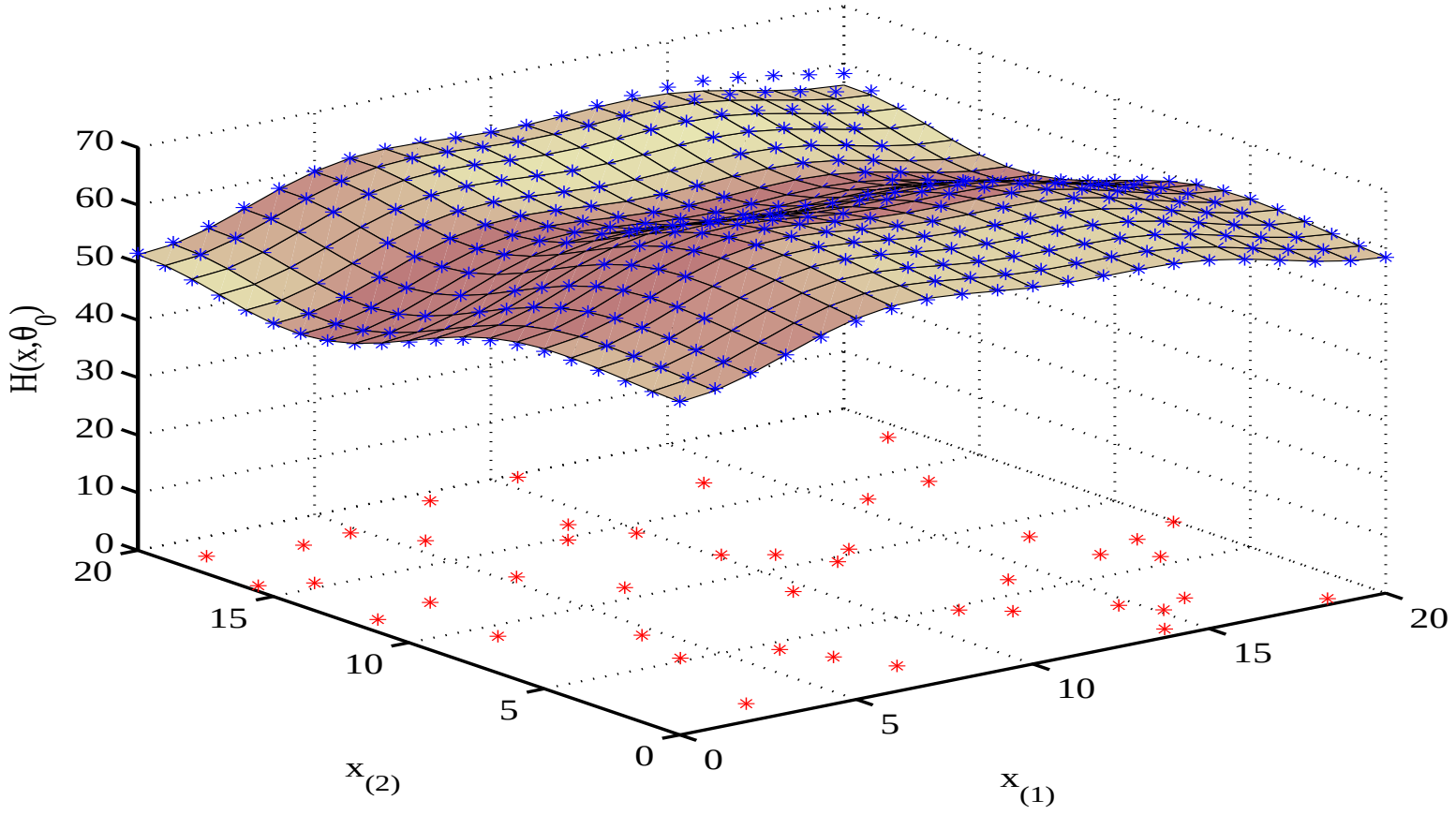
where each $\mathbf{K}_i$ is a deterministic matrix.

Simulation of random field - 1



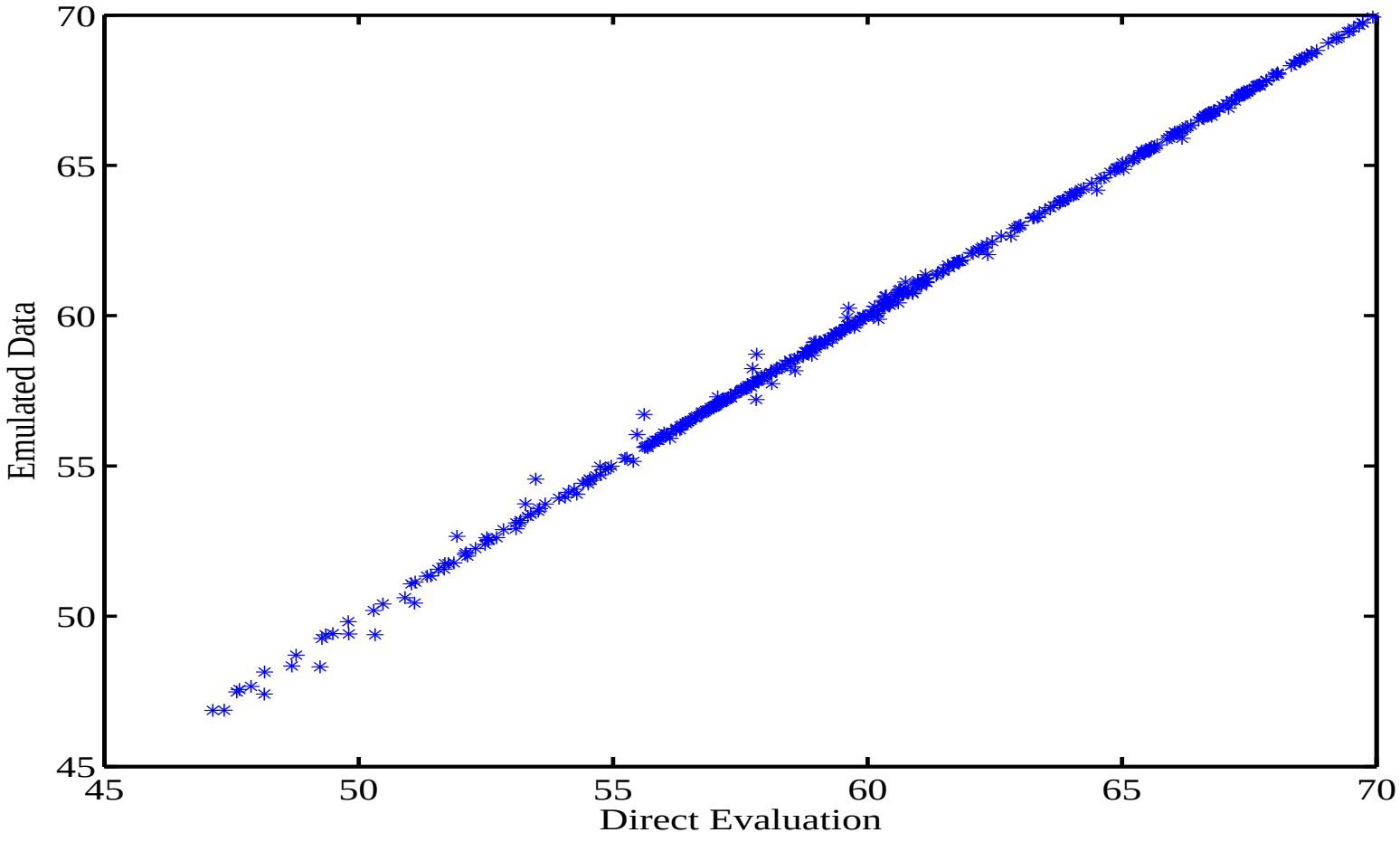
Realization of the Gaussian homogeneous random field $\mathcal{H}(\mathbf{x}, \theta)$; autocorrelation $\rho(\mathbf{x}, \mathbf{x}') = e^{-\alpha_1 |\mathbf{x}_{(1)} - \mathbf{x}'_{(1)}| - \alpha_2 |\mathbf{x}_{(2)} - \mathbf{x}'_{(2)}|}$; mean $\mu = 50$; variance $\sigma^2 = 0.09$.

Simulation of random field - 2



Emulation of the values of $\mathcal{H}(\mathbf{x}, \theta)$ at the nodal points. The initial design is shown lying on the lower plane.

Simulation of random field - 3



Correlation of the emulated values at the nodal points

Computational effort

No. Nodes	Time (secs.) Direct	Time (secs.) Emulator
121	9.56	0.07
256	19.92	0.24
441	34.43	0.75
961	76.23	6.05
1681	131.29	17.76
2601	273.18	59.66

Number of nodes vs. CPU time employed for a typical sample of the random field

Emulator: Future works

- Parametric eigenvalue problem:
  - Express the eigenvalues of interest by emulator (probabilistic response surface)
  - Exploit explicit parametric sensitivity expressions
- Representation of stochastic response field:
  - Monte Carlo simulation using emulator
  - polynomial chaos representation by emulator
- Domain decomposition and substructure approach
  - Possibility of parallel and high-performance computing

Non-parametric approaches

Random discrete dynamical systems

The equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{25}$$

- Due to the presence of (parametric/nonparametric or both) uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objectives in the 'forward problem' are:
  - to quantify uncertainties in the system matrices
  - to predict the variability in the response vector $\mathbf{q}$
- Probabilistic solution of this problem is expected to have more credibility compared to a deterministic solution

Random Matrix Method (RMM)

- The objective: To have an unified method which will work across the frequency range.
- The methodology:
  - Derive the matrix variate probability density functions of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ [a]
  - Propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)

[a] AIAA Journal, 45[7] (2007), pp. 1748-1762

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \rightarrow \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} \det\{\mathbf{\Sigma}\}^{-p/2} \det\{\mathbf{\Psi}\}^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (26)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \Gamma_n \left(\frac{1}{2} p \right) \det \{\mathbf{\Sigma}\}^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \text{etr} \left\{ -\frac{1}{2} \mathbf{\Sigma}^{-1} \mathbf{S} \right\} \tag{27}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$.

Note: If $p = n + 1$, then the matrix is non-negative definite.

Matrix variate Gamma distribution

A $n \times n$ symmetric positive definite matrix random $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a) \det\{\mathbf{\Psi}\}^{-a}\right\}^{-1} \det\{\mathbf{W}\}^{a-\frac{1}{2}(n+1)} \operatorname{etr}\{-\mathbf{\Psi}\mathbf{W}\}; \quad \Re(a) \tag{28}$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \text{ for } \Re(a) > (n-1)/2 \tag{29}$$

Wishart random matrix approach

- The probability density function of the mass ($\mathbf{M}$), damping ($\mathbf{C}$) and stiffness ($\mathbf{K}$) matrices should be such that they are symmetric and non-negative matrices.
- Wishart random matrix (a non-Gaussian matrix) is the simplest mathematical model which can satisfy these two criteria: $[\mathbf{M}, \mathbf{C}, \mathbf{K}] \equiv \mathbf{G} \sim W_n(p, \mathbf{\Sigma})$.
- Suppose we 'know' (e.g, by measurement or stochastic modeling) the mean ($\mathbf{G}_0$) and the (normalized) standard deviation (σ_G) of the system matrices:

$$\sigma_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}[\mathbf{G}]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}[\mathbf{G}]\|_{\mathrm{F}}^2}. \tag{30}$$

Wishart parameter selection - 1

The parameters p and $\boldsymbol{\Sigma}$ can be obtained based on what criteria we select. We investigate four possible choices.

1. Criteria 1 (proposed by Soize, 2000): $\mathrm{E}\left[\mathbf{G}\right] = \mathbf{G}_0$ and $\sigma_G = \widetilde{\sigma}_G$ which results

$$p = n + 1 + \theta \quad \text{and} \quad \boldsymbol{\Sigma} = \mathbf{G}_0/p \tag{31}$$

$$\theta = (1+\beta)/\widetilde{\sigma}_G^2 - (n+1),\ \beta = \left\{\mathrm{Trace}\left(\mathbf{G}_0\right)\right\}^2 / \mathrm{Trace}\left(\mathbf{G}_0{}^2\right).$$

2. Criteria 2: $\left\|\mathbf{G}_0 - \mathrm{E}\left[\mathbf{G}\right]\right\|_{\mathrm{F}}$ and $\left\|\mathbf{G}_0{}^{-1} - \mathrm{E}\left[\mathbf{G}^{-1}\right]\right\|_{\mathrm{F}}$ are minimum and $\sigma_G = \widetilde{\sigma}_G$. This results:

$$p = n + 1 + \theta \quad \text{and} \quad \boldsymbol{\Sigma} = \mathbf{G}_0/\alpha \tag{32}$$

where $\alpha = \sqrt{\theta(n+1+\theta)}$.

Wishart parameter selection - 2

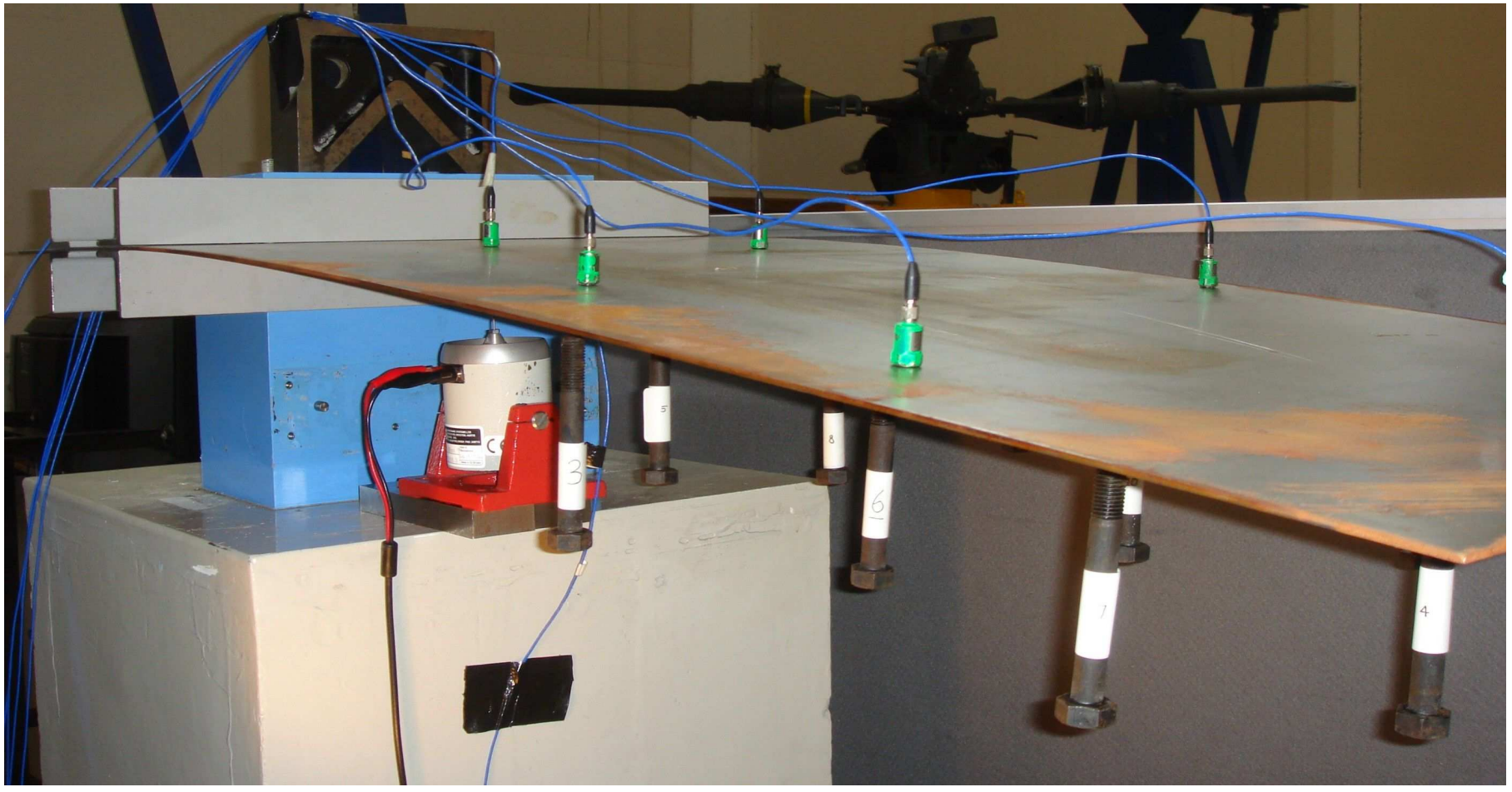
1. Criteria 3: $\mathrm{E}\left[\mathbf{G}^{-1}\right] = {\mathbf{G}_0}^{-1}$ and $\sigma_G = \widetilde{\sigma}_G$. This results:

$$p = n + 1 + \theta \quad \text{and} \quad \boldsymbol{\Sigma} = \mathbf{G}_0/\theta \tag{33}$$

2. Criteria 4: The mean of the eigenvalues of the distribution is same as the 'measured' eigenvalues of the mean matrix and the (normalized) standard deviation is same as the measured standard deviation:

$$\mathrm{E}\left[\mathbf{M}^{-1}\right] = {\mathbf{M}_0}^{-1},\ \mathrm{E}\left[\mathbf{K}\right] = \mathbf{K}_0,\ \sigma_M = \widetilde{\sigma}_M \quad \text{and} \quad \sigma_K = \widetilde{\sigma}_K. \tag{34}$$

A cantilever plate: front view



The test rig for the cantilever plate; front view (to appear in Probabilistic Engineering Mechanics).

A cantilever plate: side view

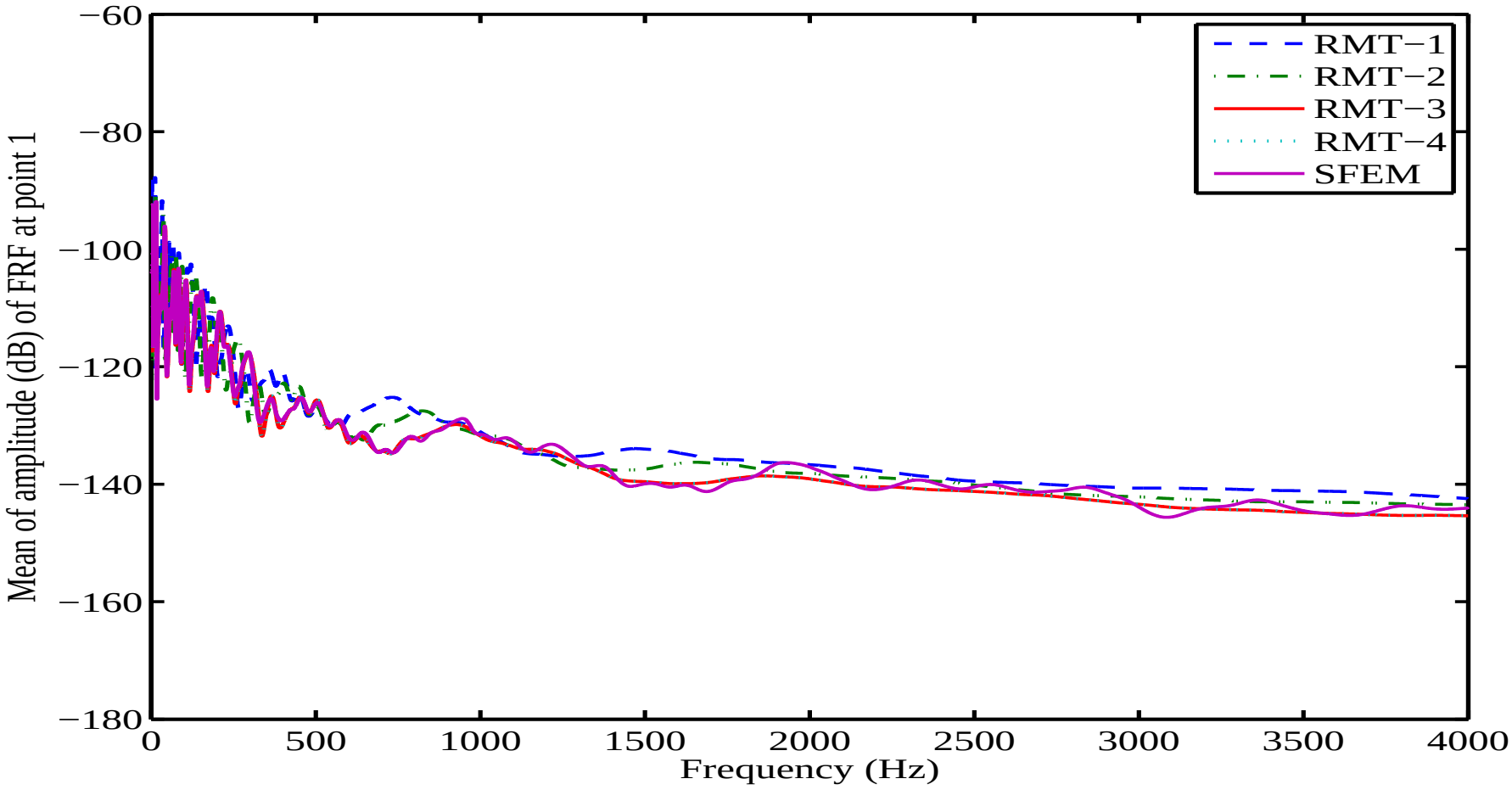
The test rig for the cantilever plate; side view.

Physical properties

Plate Properties	Numerical values
Length (L_x)	998 mm
Width (L_y)	530 mm
Thickness (t_h)	3.0 mm
Mass density (ρ)	7860 kg/m^3
Young's modulus (E)	2.0×10^5 MPa
Poisson's ratio (μ)	0.3
Total weight	12.47 kg

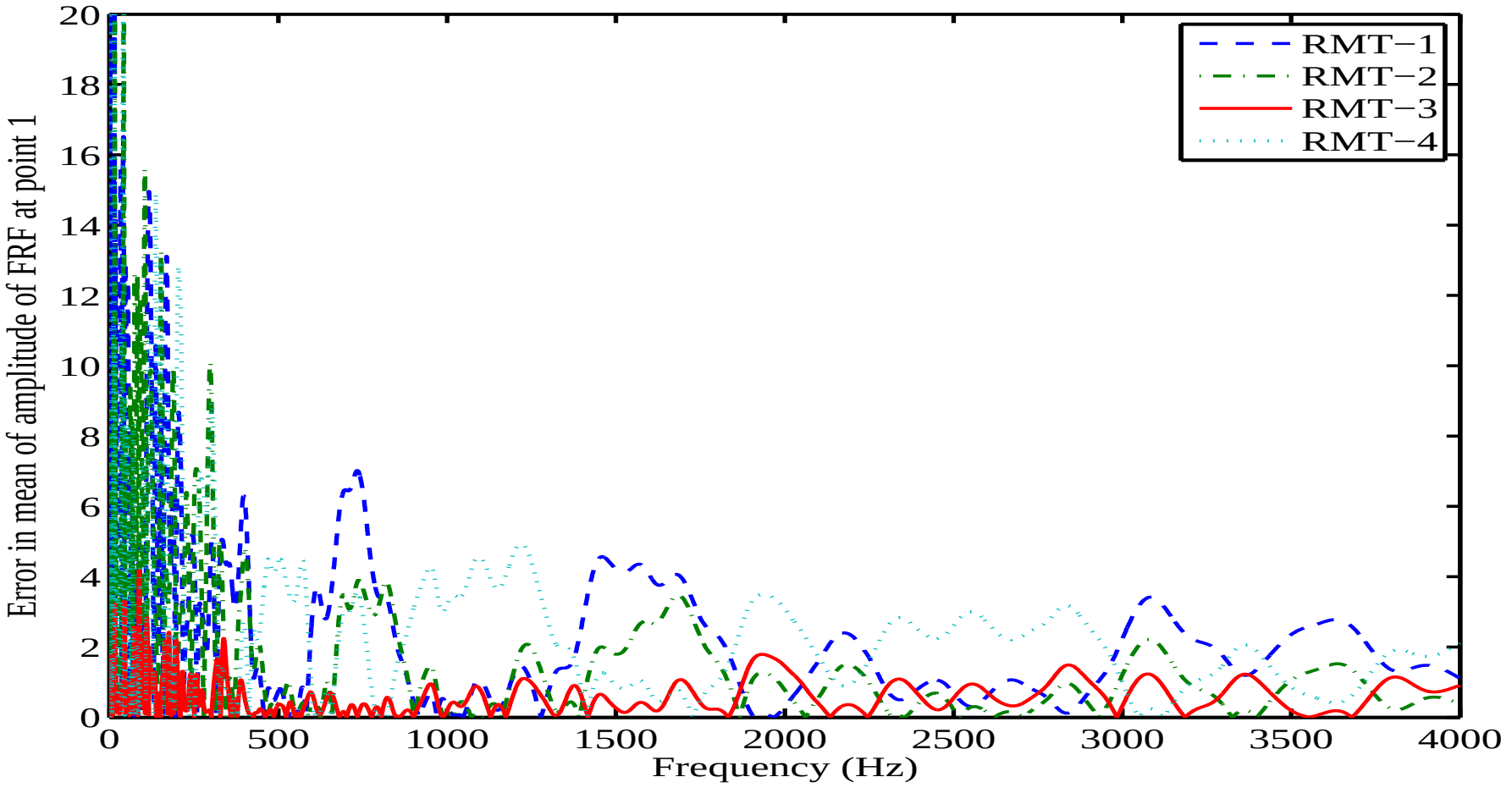
Material and geometric properties of the cantilever plate considered for the experiment. The data presented here are available from http://engweb.swan.ac.uk/∼adhikaris/uq/.

Mean of cross-FRF



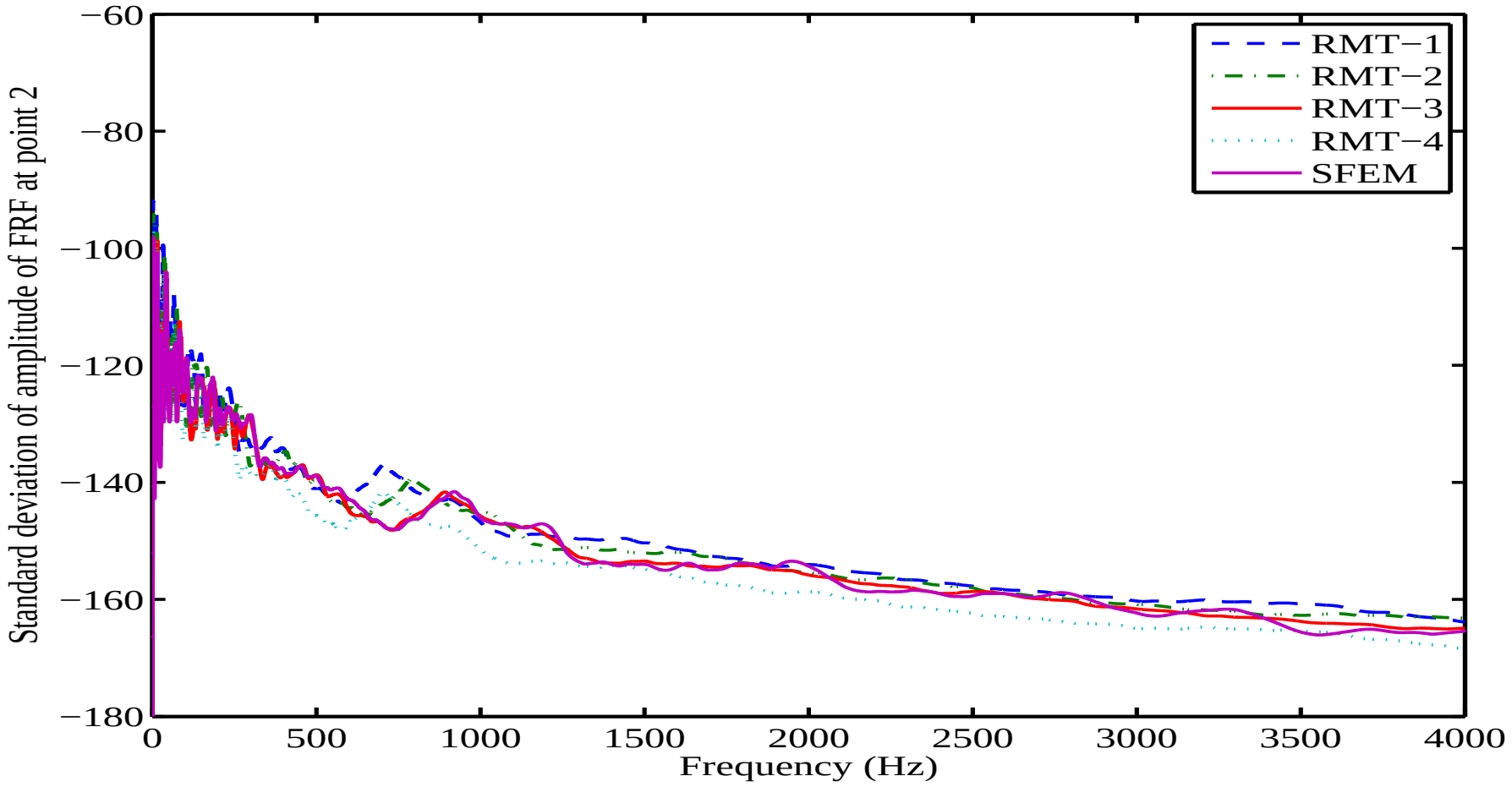
Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Error in the mean of cross-FRF



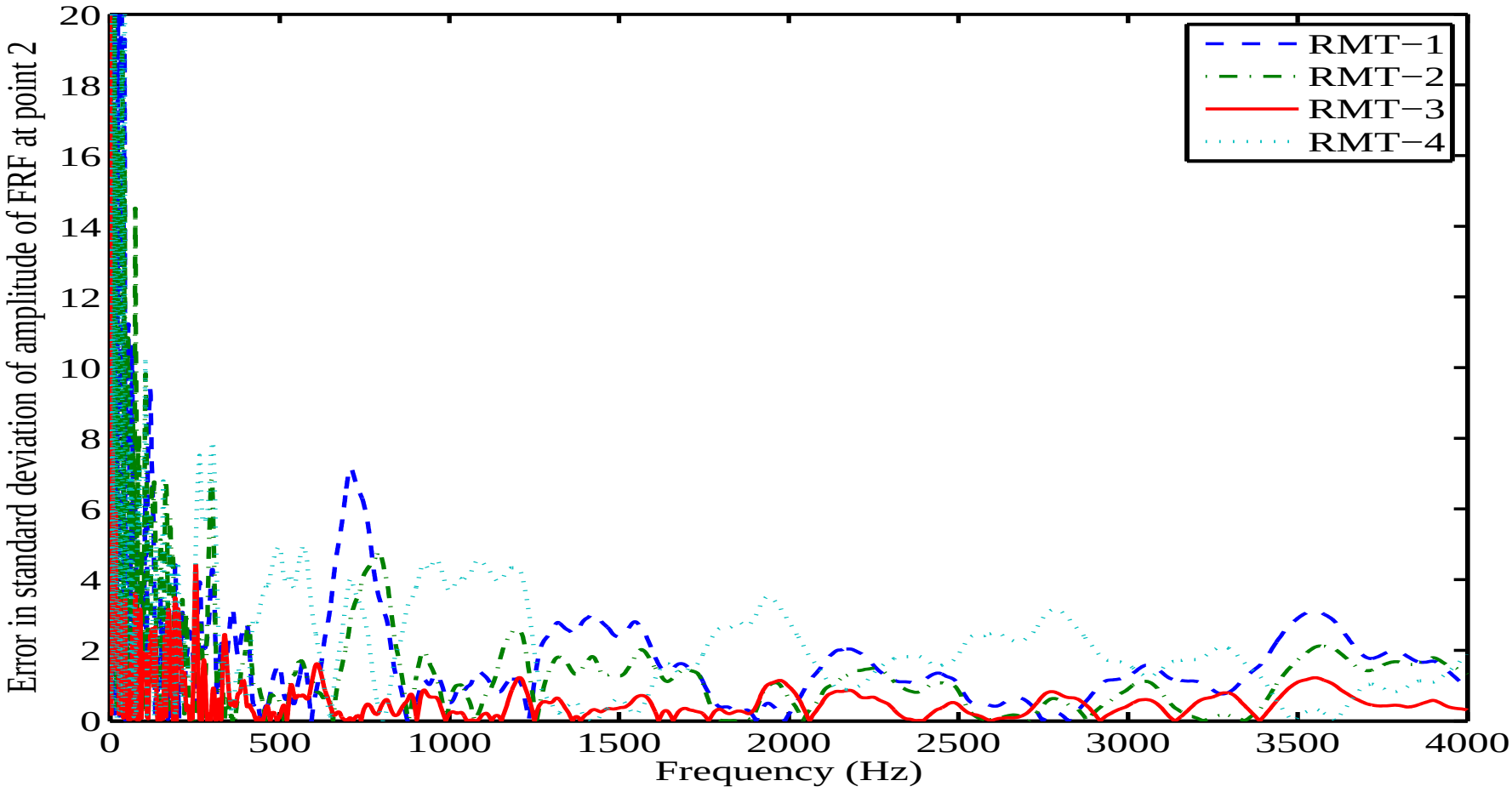
Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Standard deviation of driving-point-FRF



Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Error in the standard deviation of driving-point-FRF



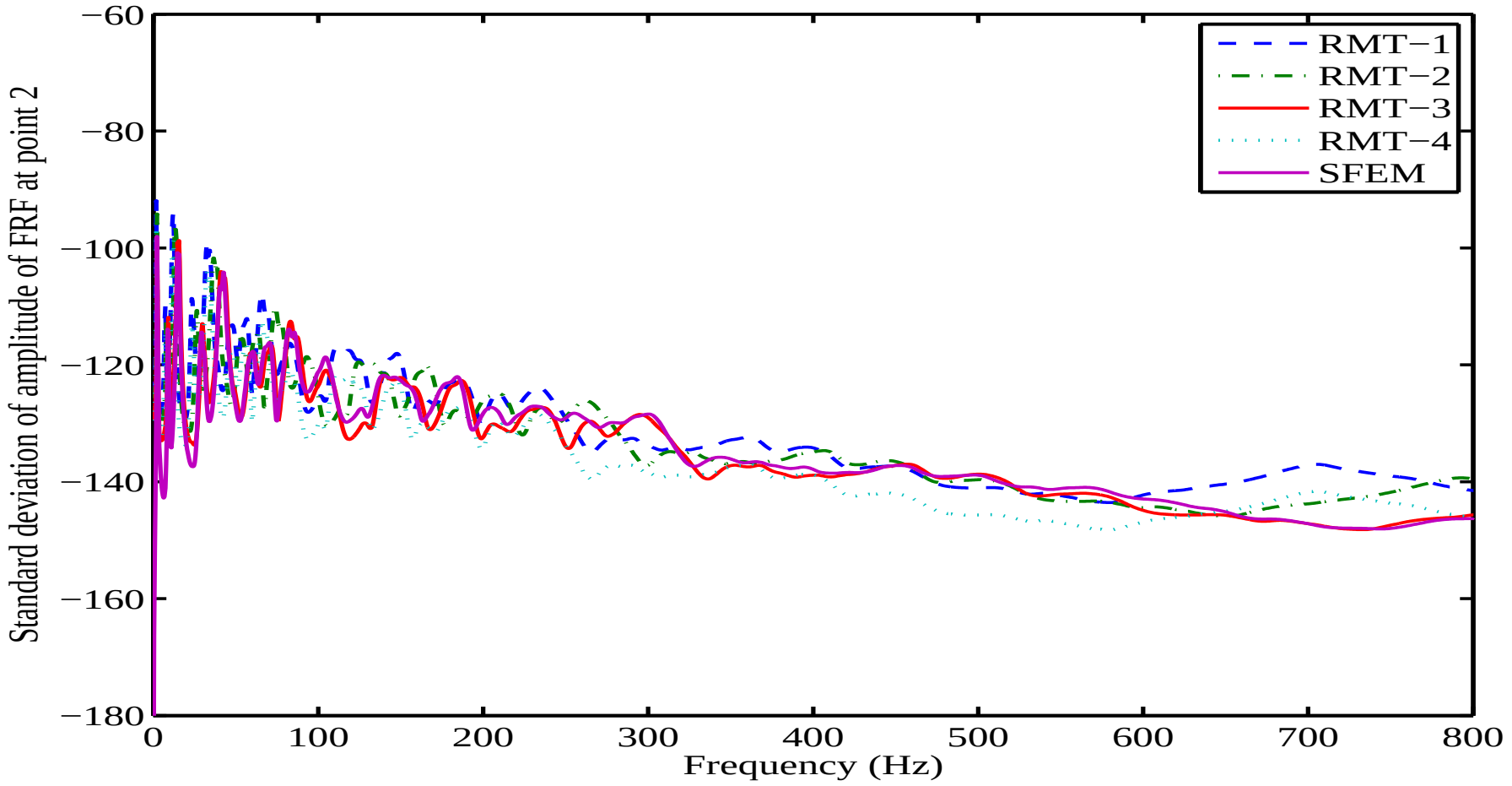
Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Main observations

- Error in the low frequency region is higher than that in the higher frequencies [a]
- In the high frequency region all methods are similar
- Overall, parameter selection 3 performs best; especially in the low frequency region.

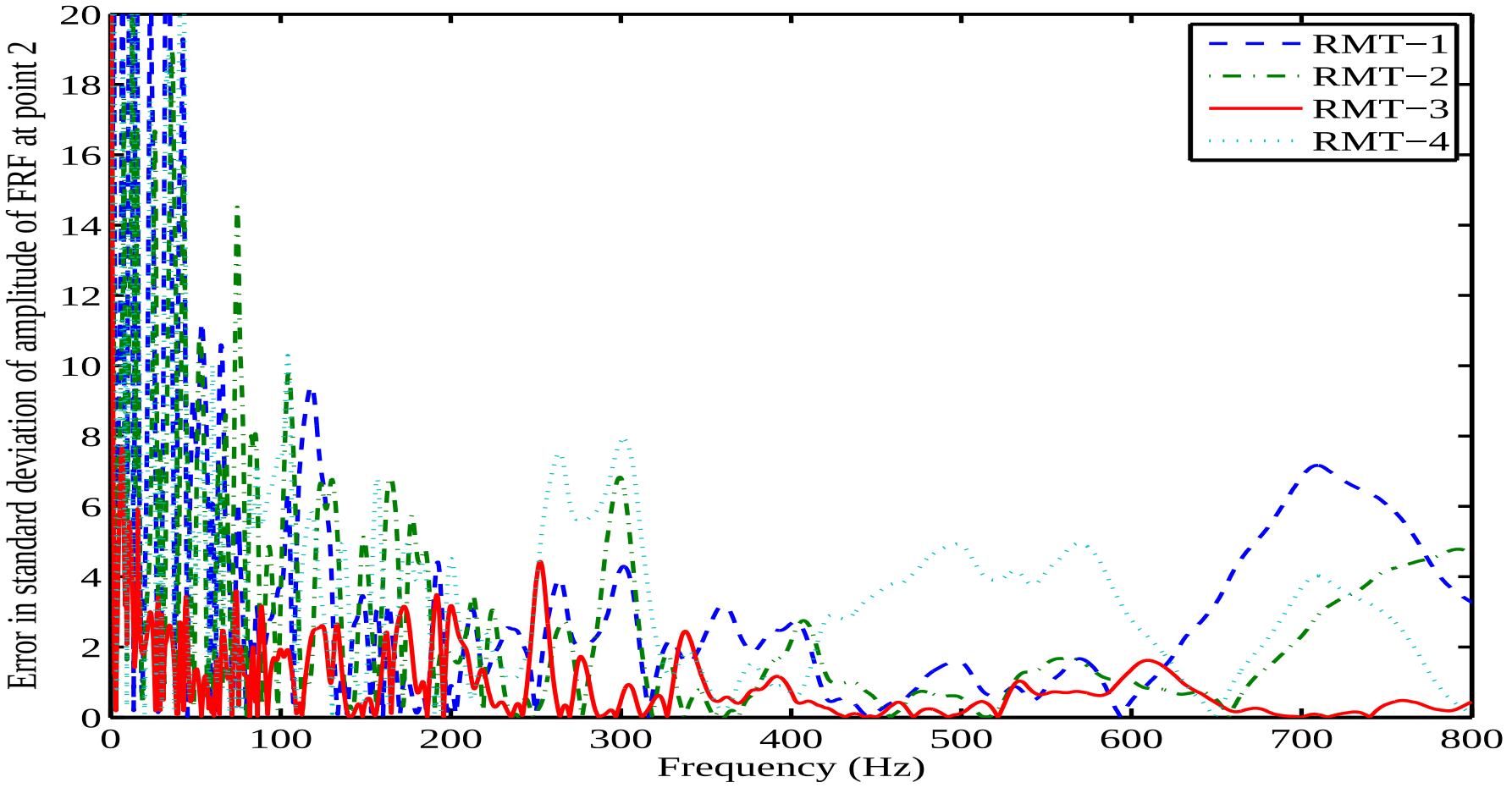
[a] to appear in ASCE J. of Engineering Mechanics

Standard deviation: low frequency



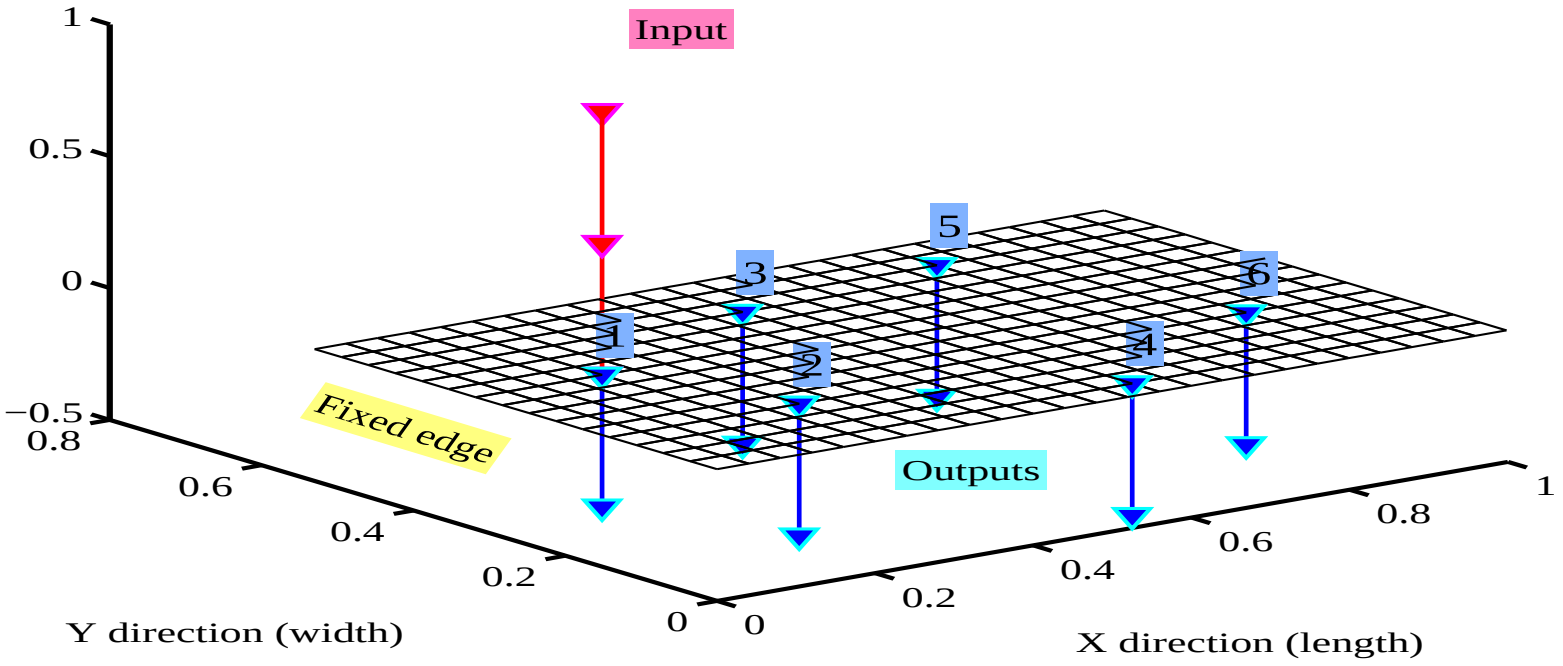
Standard deviation of the amplitude of the response of the driving-point-FRF of the plate in the low frequency region, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Error in the standard deviation: low frequency



Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate in the low frequency region, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

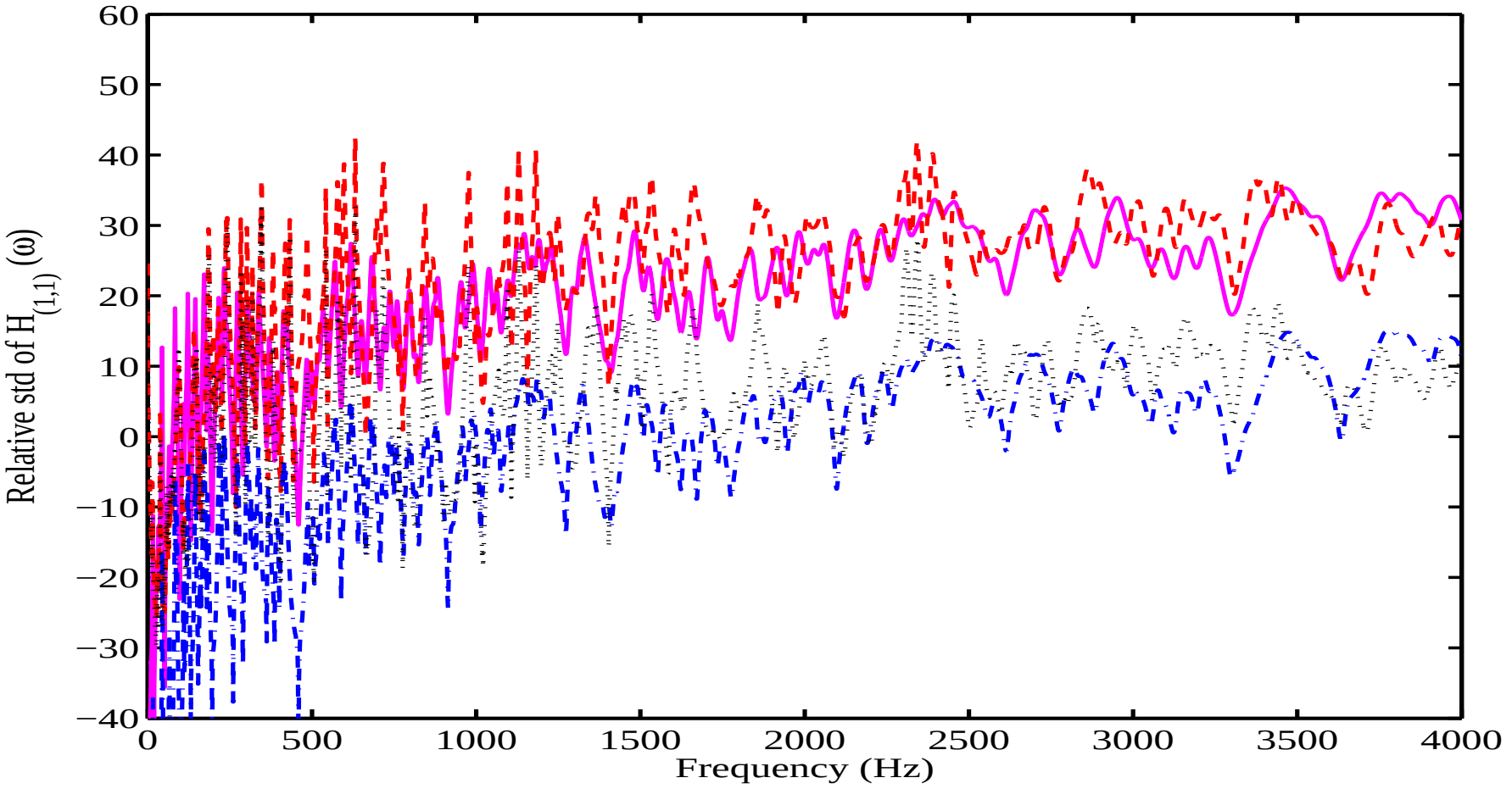
FE model of the cantilever plate



Baseline Model: 25×15 elements, 416 nodes, 1200 degrees-of-freedom. Input node number: 481, Output node numbers: 481, 877, 268, 1135, 211 and 844, 0.7% modal damping is assumed for all modes [a].

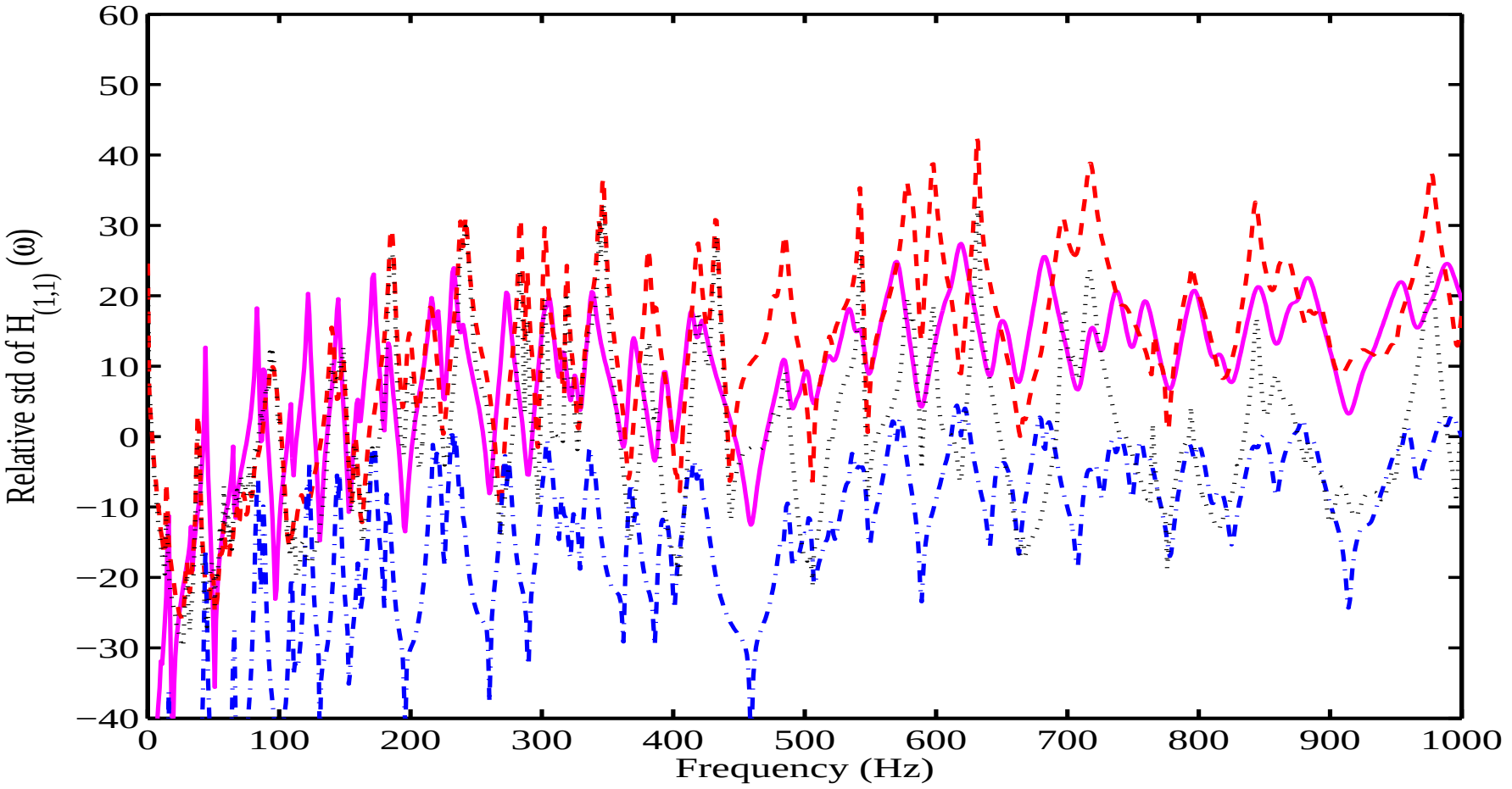
[a] to appear in Journal of Sound and Vibration

Comparison of driving-point-FRF



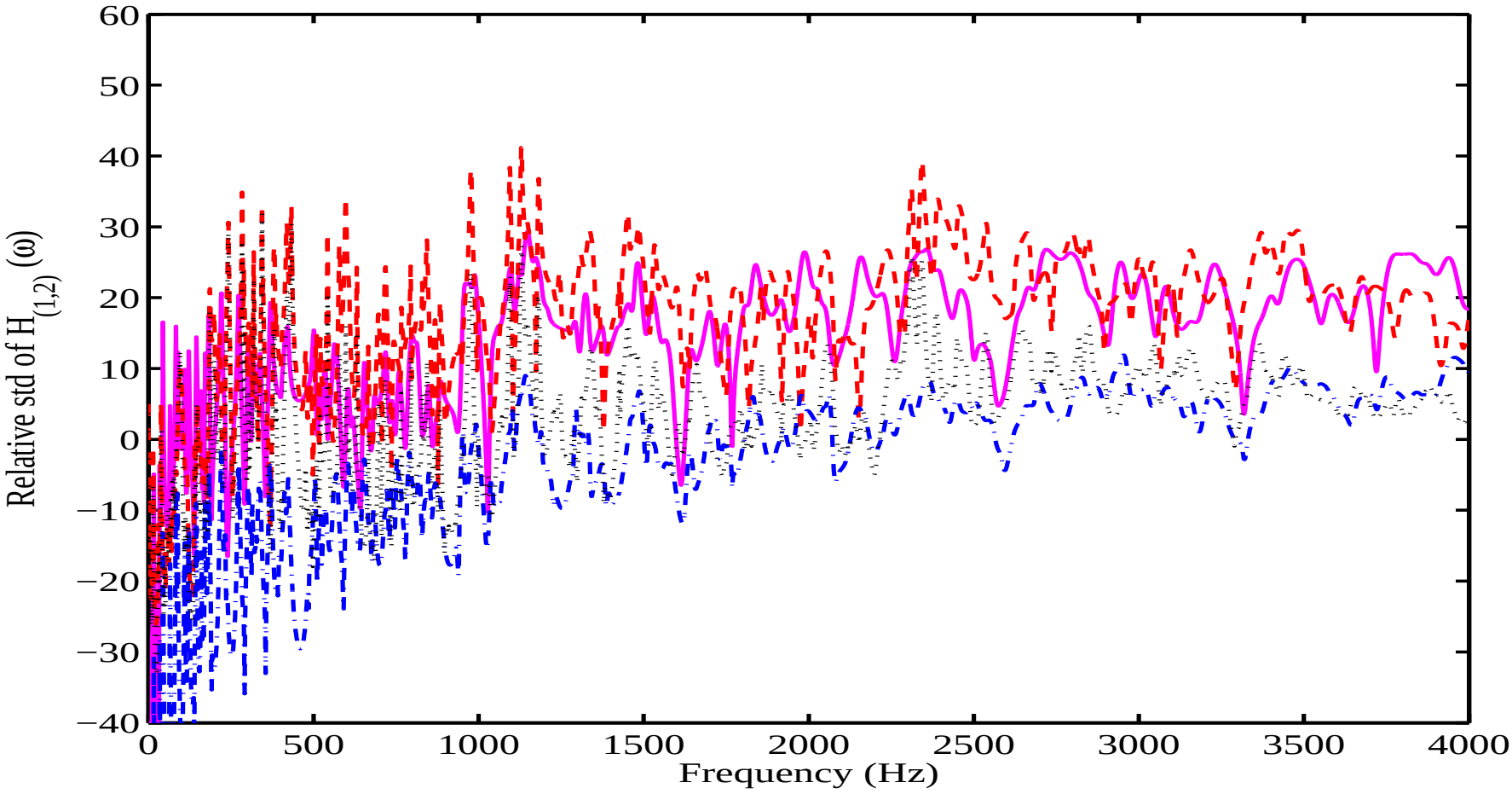
Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$. (dash and dot lines are from experiment)

Comparison of driving-point-FRF: Low Freq



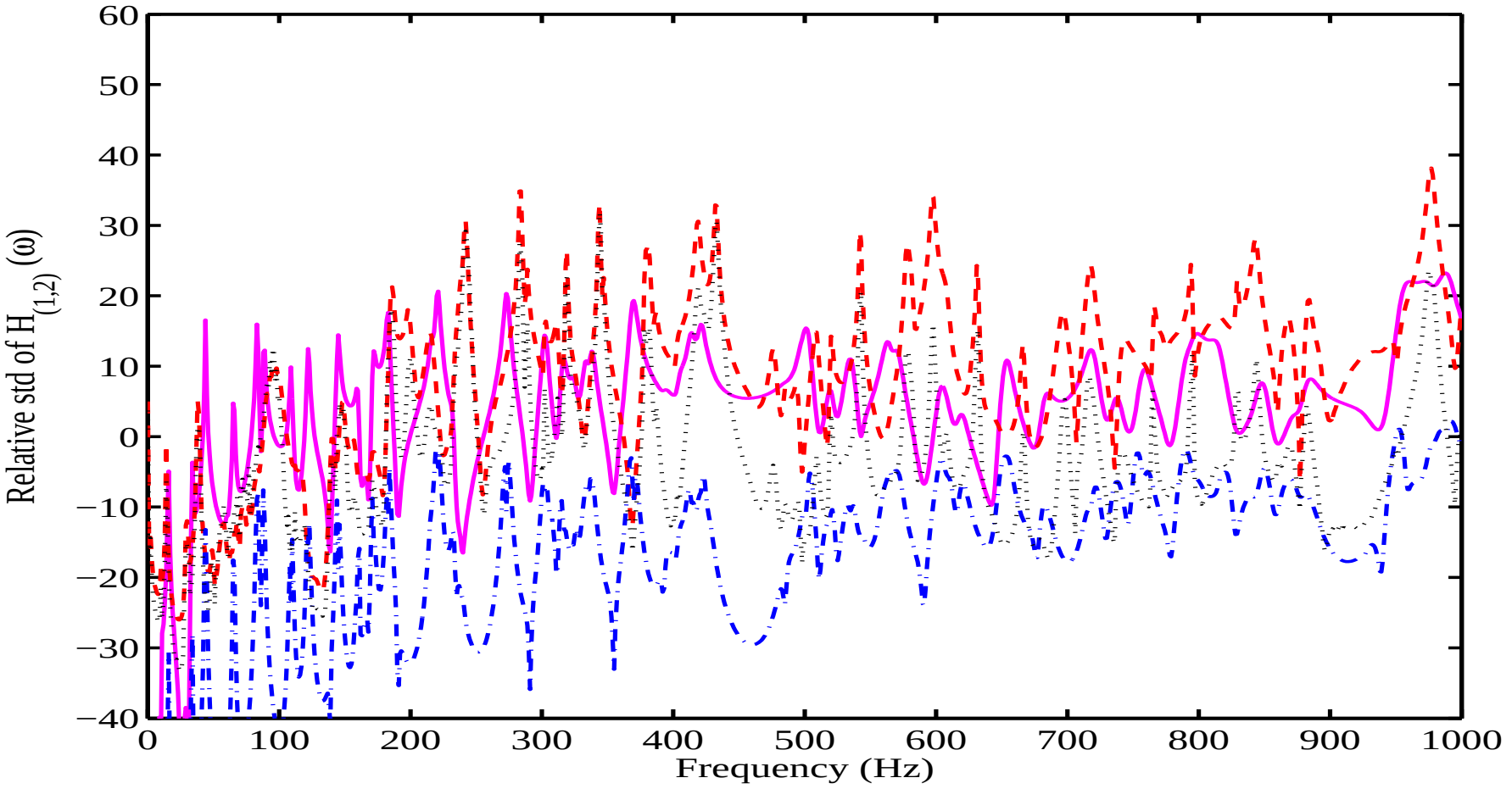
Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$. (dash and dot lines are from experiment)

Comparison of cross-FRF



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$. (dash and dot lines are from experiment)

Comparison of cross-FRF: Low Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$. (dash and dot lines are from experiment)

Random matrix approach: Future works

- Random matrix inversion based computational method:
  - utilize analytical inverted matrix variate probability density functions for response moment calculation
  - explore different random matrix parameter fitting options
- Random eigenvalue based computational method:
  - utilize eigensolution density function of Wishart matrices in response statistics calculation
  - simple analytical expressions via asymptotic approach applicable for large systems
- Non-central Wishart matrices and beta distribution:
  - better approximation of the covariance of the system matrices

Conclusions

- Uncertainties need to be taken into account for credible predictions using computational methods.
- This talk concentrated on Uncertainty Quantification (UQ) in structural dynamic problems.
- Two different approaches are discussed:
  - non-parametric uncertainty problem: $\longrightarrow$ Wishart random matrix method
  - parametric uncertainty problem: $\longrightarrow$ Gaussian emulator method, stochastic spectral finite element method
- A general UQ approach based on Wishart random matrix is discussed and the results are compared with experimental results.

Future directions-1

- Model calibration/updating (data assimilation): taking model and measurement uncertainties into account
- Model validation and predictive capability assessment: how good are our model prediction when very little or no data is available to validate them?

Future directions-2

- High performance computing for uncertain systems: how the UQ approaches can be integrated with high performance computing? This is becoming a very important issue due the availability of relatively inexpensive ‘clusters’ (open research area!).
- UQ in other areas such as fluid-structure interaction problems, electromagnetism, smart systems - how some of the proposed techniques might be integrated - multidisciplinary collaborative works.