Random Matrix Theory For Stochastic Structural Dynamics

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Outline of the presentation

- Uncertainty in structural dynamics
- Critical review of current UQ approaches
- Random matrix models
- Derivation of noncentral Wishart distribution
- Experimental validation of the proposed method
- Conclusions & discussions

Overview of predictive approaches

There are five key steps:

- Physics (mechanics) model building
- Uncertainty Quantification (UQ)
- Uncertainty Propagation (UP)
- Model Verification & Validation (V & V)
- Prediction

Tools are available for each of these steps. My focus in this talk is on UQ in linear dynamical systems.

Why uncertainty?

Different sources of uncertainties in the modeling and parameters of dynamic systems may be attributed, but not limited, to the following factors:

- Mathematical models: equations (linear, non-linear), geometry, damping model (viscous, non-viscous, fractional derivative), boundary conditions/initial conditions, input forces;
- Model parameters: Young's modulus, mass density, Poisson's ratio, damping model parameters (damping coefficient, relaxation modulus, fractional derivative order)

Why uncertainty?

- Numerical algorithms: weak formulations, discretisation of displacement fields (in finite element method), discretisation of stochastic fields (in stochastic finite element method), approximate solution algorithms, truncation and roundoff errors, tolerances in the optimization and iterative methods, artificial intelligent (AI) method (choice of neural networks)
- Measurements: noise, resolution (number of sensors and actuators), experimental hardware, excitation method (nature of shakers and hammers), excitation and measurement point, data processing (amplification, number of data points, FFT), calibration

Structural dynamics

The equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objectives in the 'forward problem' are:
  - to quantify uncertainties in the system matrices
  - to predict the variability in the response vector $\mathbf{x}$

Current UQ approaches

Two different approaches are currently available

- Parametric approaches : Such as the Stochastic Finite Element Method (SFEM):
    - aim to characterize aleatoric uncertainty
    - assumes that stochastic fields describing parametric uncertainties are known in details
    - suitable for low-frequency dynamic applications

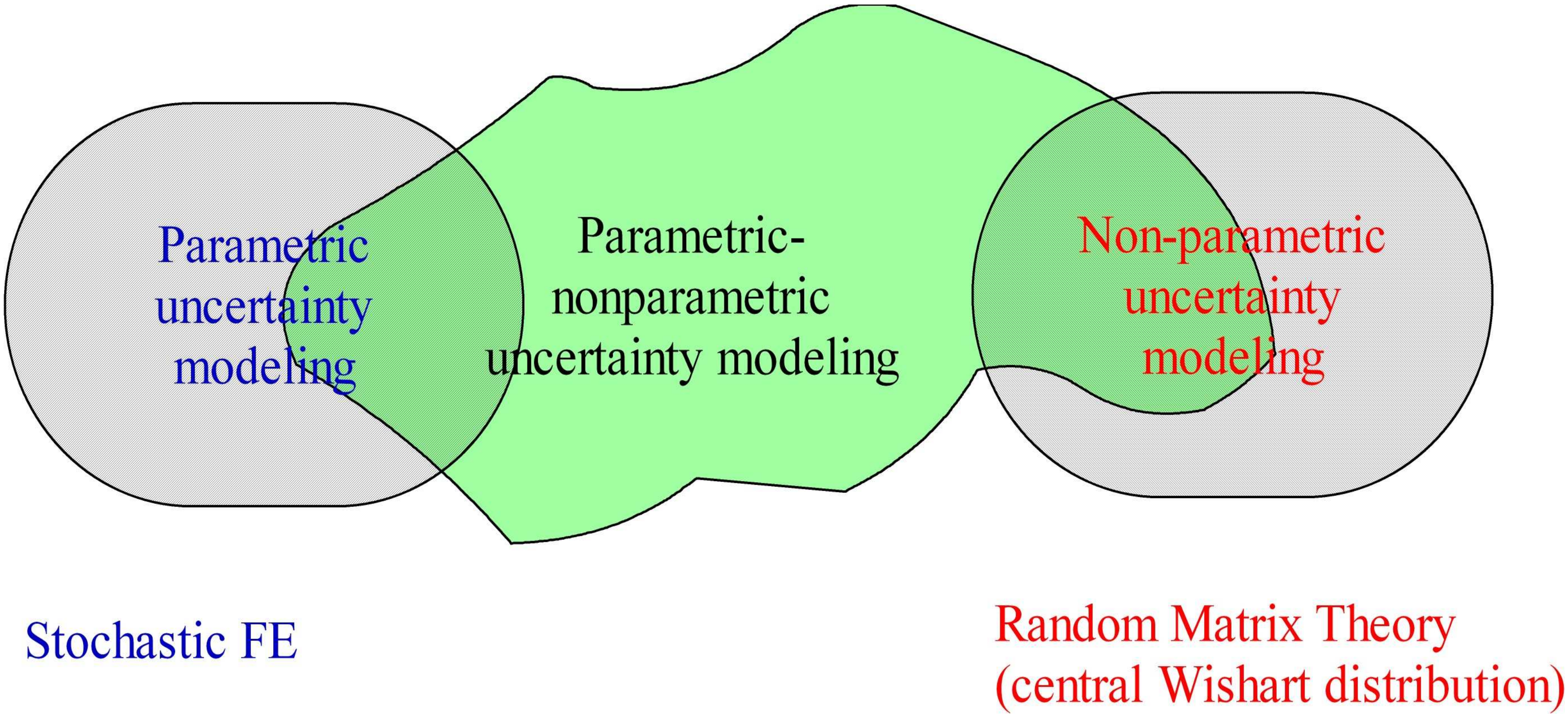
Current UQ approaches

- Nonparametric approaches : Such as the Statistical Energy Analysis (SEA) and Wishart random matrix theory:
  - aim to characterize episematic uncertainty
  - does not consider parametric uncertainties in details
  - suitable for high-frequency dynamic applications

Limitations of current UQ approaches

- Although we have mentioned and made differences between the two different types of uncertainties, in practical problems it is in general very difficult, if not impossible, to distinguish them.
- Recently conducted experimental studies by our group on one hundred nominally identical beams and plates emphasize this fact.
- For credible numerical models of complex dynamical systems, we need to quantify and model both types of uncertainties simultaneously.
- A hybrid approach is required.

Overview of proposed approach



Schematic representation of the proposed parametric-nonparametric uncertainly modeling in structural dynamics.

Proposed unified approach

- The objective : To develop a hybrid approach which takes both parametric and nonparametric uncertainties into account.
- The rationale : No matter what the nature of uncertainty is (parametric/nonparametric or both), at the end it will result in random $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ matrices.
- The methodology : Derive the matrix variate probability density functions of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ based on parametric information (e.g. mean and covariance of the elements) and overall physically realistic mathematical constraints (such as the symmetry and positive definiteness).

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} |\mathbf{\Sigma}|^{-p/2} |\mathbf{\Psi}|^{-n/2} \\ \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (2)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Central Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{ 2^{\frac{1}{2}np} \, \Gamma_n\left(\frac{1}{2}p\right) |\boldsymbol{\Sigma}|^{\frac{1}{2}p} \right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \mathrm{etr}\left\{ -\frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{S} \right\} \tag{3}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma})$.

Note: This distribution is used in current nonparametric UQ methods.

Noncentral Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a noncentral Wishart distribution with parameters $p \geq n$, $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$ and $\boldsymbol{\Theta} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}\left(\mathbf{S}\right) = \left\{ 2^{\frac{1}{2}np}\, \Gamma_n\left(\frac{1}{2}p\right) \left|\boldsymbol{\Sigma}\right|^{\frac{1}{2}p} \right\}^{-1} \operatorname{etr}\left\{-\frac{1}{2}\boldsymbol{\Theta}\right\} \operatorname{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right\} \left|\mathbf{S}\right|^{\frac{1}{2}(p-n-1)}\; {}_0F_1(p/2, \boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\mathbf{S}/4). \quad (4)$$

where ${}_0F_1$ the hypergeometric function (Bessel function) of a matrix argument. This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$. Note that if the noncentrality parameter $\boldsymbol{\Theta}$ is a null matrix, then it reduces to the central Wishart distribution.

Distribution of the system matrices

The distribution of the random system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ should be such that they are

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix $\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ should exist $\forall\, \omega$

Current nonparametric approach

- Suppose $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$
- $\mathbf{G} \sim W_n\left(p, \boldsymbol{\Sigma}\right)$ where $p = n + 1 + \theta$, $\boldsymbol{\Sigma} = \overline{\mathbf{G}} / \sqrt{\theta(n+1+\theta)}$ and $\theta = \frac{1}{\delta_G^2}\left\{1 + \{\mathrm{Trace}\left(\overline{\mathbf{G}}\right)\}^2 / \mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)\right\} - (n+1)$
- $\delta_G^2 = \frac{\mathrm{E}\left[\| \mathbf{G} - \mathrm{E}[\mathbf{G}] \|_{\mathrm{F}}^2\right]}{\| \mathrm{E}[\mathbf{G}] \|_{\mathrm{F}}^2} = \frac{\mathrm{Trace}\left(\mathrm{cov}\left(\mathrm{vec}\left(\mathbf{G}\right)\right)\right)}{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)}$ (normalized std) .
- The main limitation: $\mathrm{cov}\left(G_{ij}, G_{kl}\right) = \frac{1}{\theta}\left(\overline{G}_{ik}\overline{G}_{jl} + \overline{G}_{il}\overline{G}_{jk}\right)$
- Only one parameter controls the uncertainty

Current nonparametric approach

- The covariance matrix of $\mathbf{G}$ can have $n(n+1) \times (n(n+1)+2)/8$ number of independent parameters.
- Current nonparametric approach, only offers a single parameter to quantify uncertainty which can potentially be expressed by $n(n+1)(n(n+1)+2)/8$ number of independent parameters - a gross oversimplification.
- To account for parametric uncertainties, we need a matrix variate distribution which not only satisfy the mathematical constrains, but also must offer more parameters to fit the 'known' covariance tensor of $\mathbf{G}$.

Matrix factorization approach

- Because $\mathbf{G}$ is a symmetric and positive-definite random matrix, it can be always factorized as

$$\mathbf{G} = \mathbf{X}\mathbf{X}^T \tag{5}$$

where $\mathbf{X} \in \mathbb{R}^{n \times p}, p \geq n$ is in general a rectangular matrix.

- Extending the standard maximum entropy argument to the matrix case we can say that the pdf of $\mathbf{X}$ is given by the matrix variate Gaussian distribution, that is, $\mathbf{X} \sim N_{n,p}\left(\boldsymbol{\mathcal{M}}, \boldsymbol{\Sigma} \otimes \mathbf{I}_p\right)$.
- This shows that $\mathbf{G}$ has non central Wishart distribution.

The main result

Theorem 1. *The unified parametric-nonparametric probability density function a random system matrix* $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ *follows the noncentral Wishart distribution, that is* $\mathbf{G} \sim W_n(p, \mathbf{\Sigma}, \mathbf{\Theta})$ *where* $p > n$ *is a real scalar,* $\mathbf{\Sigma}$ *and* $\mathbf{\Theta}$ *are symmetric positive-definite* $n \times n$ *real matrices.*

Noncentral distribution

- If the noncentrality parameter $\mathbf{\Theta}$ is a null matrix, the unified distribution reduces to the nonparametric distribution (central Wishart distribution).
- The unified distribution derived here is therefore further generalization of the nonparametric distribution.
- The additional $n(n+1)/2$ parameters provided by the matrix $\mathbf{\Theta} \in \mathbb{R}_n^+$ allow to model parametric uncertainty which is not available within the scope of the nonparametric distribution.

Parameter estimation

- We match the mean and covariance of the distribution of $\mathbf{G}$ with 'measured/known' quantities.

$$\mathrm{E}\left[\mathbf{G}\right] = p\boldsymbol{\Sigma} + \boldsymbol{\Omega},$$

$$\mathrm{cov}\left(\mathrm{vec}\left(\mathbf{G}\right)\right) = \left(\mathbf{I}_{n^2} + \mathbf{K}_{nn}\right)\left(p\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma} + \boldsymbol{\Omega}\otimes\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\otimes\boldsymbol{\Omega}\right).$$

- Mean is satisfied exactly while the covariance is satisfied in least-square sense.
- Suppose $\overline{\mathbf{G}} \in \mathbb{R}_n^+$, the mean matrix and $\mathbf{C}_G = \mathrm{cov}\left(\mathrm{vec}\left(\mathbf{G}\right)\right) \in \mathbb{R}_{n^2}^+$, the covariance matrix, are known.

Parameter estimation

- Obtain the normalized standard deviation δ_G of $\mathbf{G}$:
  $$\delta_G^2 = \frac{\mathrm{E}\left[\left\| \mathbf{G} - \mathrm{E}[\mathbf{G}] \right\|_{\mathrm{F}}^2\right]}{\left\| \mathrm{E}[\mathbf{G}] \right\|_{\mathrm{F}}^2} = \frac{\mathrm{Trace}\left(\mathbf{C}_G\right)}{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)}$$
- $$p = \frac{1}{\delta_G^2} \frac{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right) + \left\{\mathrm{Trace}\left(\overline{\mathbf{G}}\right)\right\}^2}{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)}$$
- Form the matrix $\boldsymbol{\mathcal{A}} = \overline{\mathbf{G}} \otimes \overline{\mathbf{G}} - p\mathbf{C}_G/2 \in \mathbb{R}^{n^2 \times n^2}$ and obtain $\boldsymbol{\Omega} \in \mathbb{R}^{n \times n}$ by least-square minimization of the Frobenius norm $\|\boldsymbol{\mathcal{A}} - \boldsymbol{\Omega} \otimes \boldsymbol{\Omega}\|_{\mathrm{F}}$.
- Calculate $\boldsymbol{\Sigma} = \left(\overline{\mathbf{G}} - \boldsymbol{\Omega}\right)/p$ and $\boldsymbol{\Theta} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega}$.

Numerical recipe

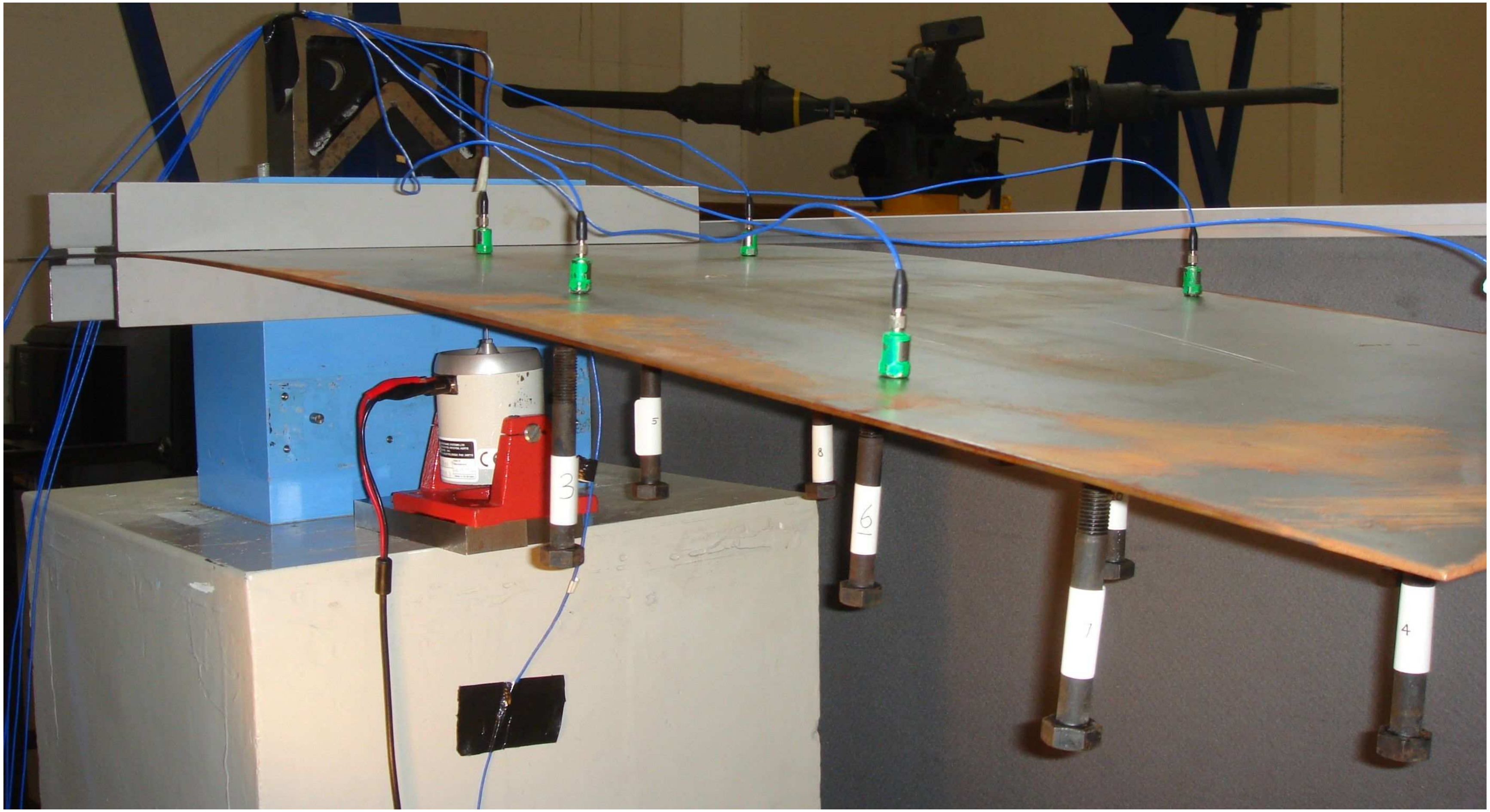
- Obtain the distribution parameters $p \in \mathbb{R}$, $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Omega} \in \mathbb{R}_n^+$ from $\overline{\mathbf{G}}$ and $\mathbf{C}_G$
- Perform the Cholesky factorizations of the positive definite matrices $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Omega} \in \mathbb{R}_n^+$ as $\mathbf{\Sigma} = \mathbf{D}\mathbf{D}^T$, $\mathbf{D} \in \mathbb{R}^{n \times n}$ and $\mathbf{\Omega} = \widehat{\boldsymbol{\mathcal{M}}}\widehat{\boldsymbol{\mathcal{M}}}^T$, $\widehat{\boldsymbol{\mathcal{M}}} \in \mathbb{R}^{n \times n}$.
- Calculate the $n \times n$ square matrix $\widetilde{\boldsymbol{\mathcal{M}}} = \mathbf{D}^{-1}\widehat{\boldsymbol{\mathcal{M}}}$
- Construct the $n \times p$ rectangular mean matrix $\boldsymbol{\mathcal{M}} = [\widetilde{\boldsymbol{\mathcal{M}}}, \mathbf{O}_{n,n-p}] \in \mathbb{R}^{n \times p}$.

Numerical recipe

- Obtain the matrix $\mathbf{Y} \in \mathbb{R}^{n \times p}$ containing uncorrelated Gaussian random numbers with mean $\boldsymbol{\mathcal{M}}$ and unit standard deviation.
- Generate the samples of a system matrix as $\mathbf{G} = \mathbf{D}\mathbf{Y}\mathbf{Y}^T\mathbf{D}^T \in \mathbb{R}_n^+$.
- In MATLAB®, the following four lines of code will generate the samples of the system matrices:

```
D=[chol(Sigma)]'; Mhat=[chol(Omega)]';
Mtilde=D\Mhat;
Y=[Mtilde zeros(n,p-n)] + randn(n,p);
G=D*Y*Y'*D';
```

A cantilever plate: front view



The test rig for the cantilever plate; front view.

A cantilever plate: side view

The test rig for the cantilever plate; side view.

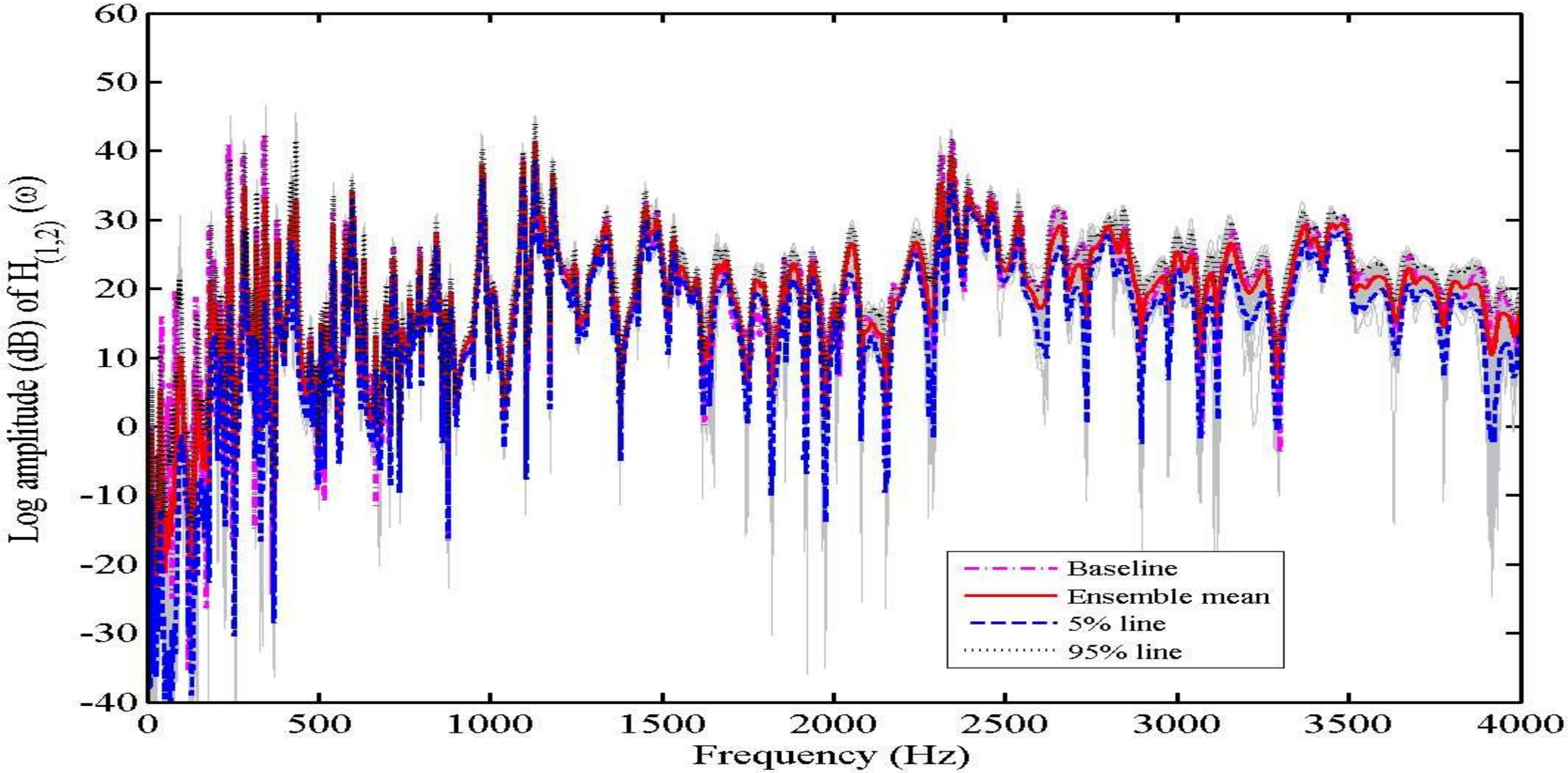
Physical Properties

Plate Properties	Numerical values
Length (L_x)	998 mm
Width (L_y)	530 mm
Thickness (t_h)	3.0 mm
Mass density (ρ)	7860 kg/m^3
Young's modulus (E)	2.0×10^5 MPa
Poisson's ratio (μ)	0.3
Total weight	12.47 kg

Material and geometric properties of the cantilever plate considered for the experiment.

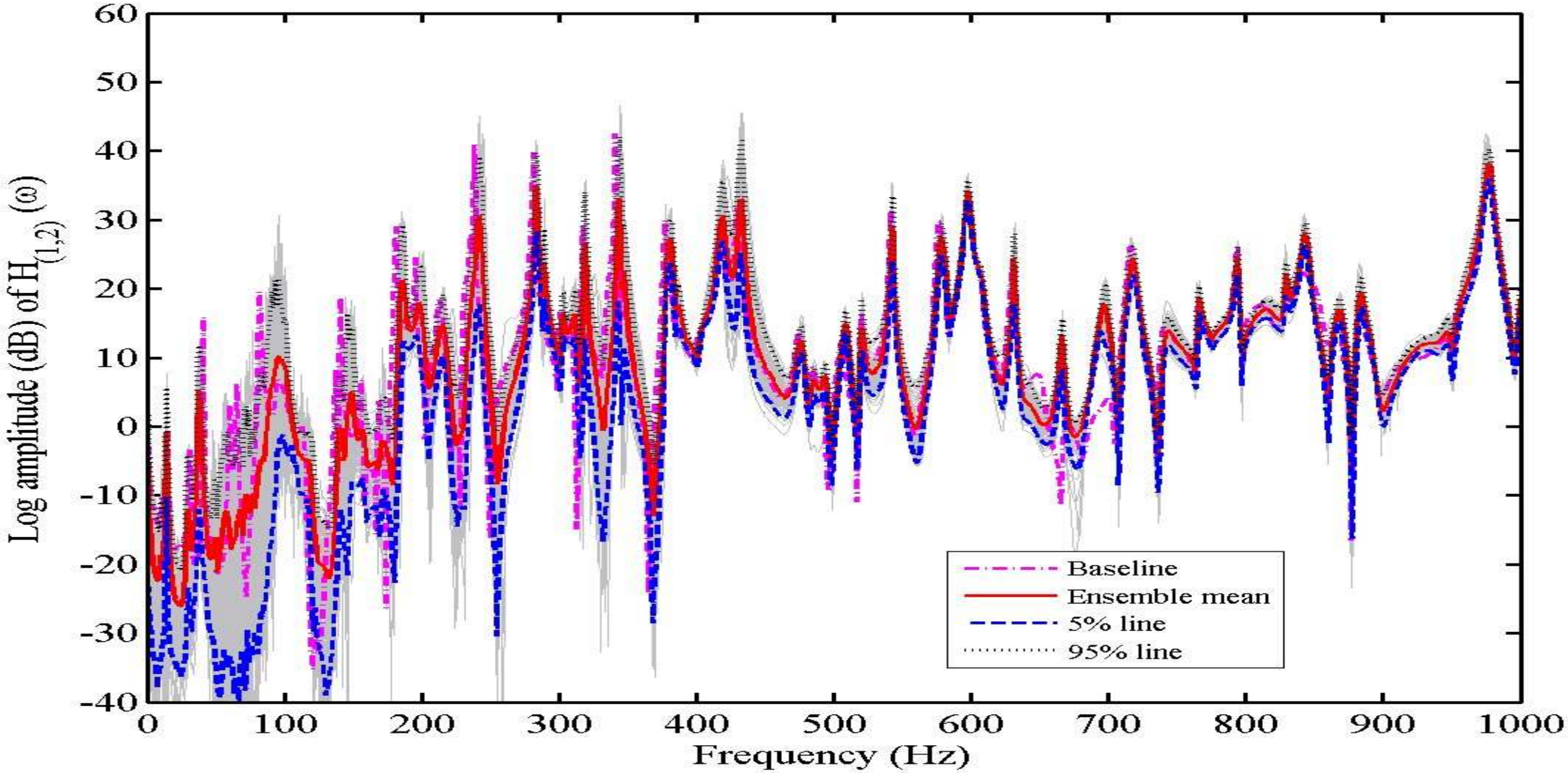
The data presented here are available from http://engweb.swan.ac.uk/$\sim$adhikaris/uq/.

FRF amplitude: complete spectrum



Experimentally measured amplitude of the cross-FRF of the plate at point 2 (nodal coordinate: (6,11)) with 10 randomly placed oscillators. 100 FRFs, together with the ensemble mean, 5% and 95% probability points are shown.

FRF amplitude: Low Freq



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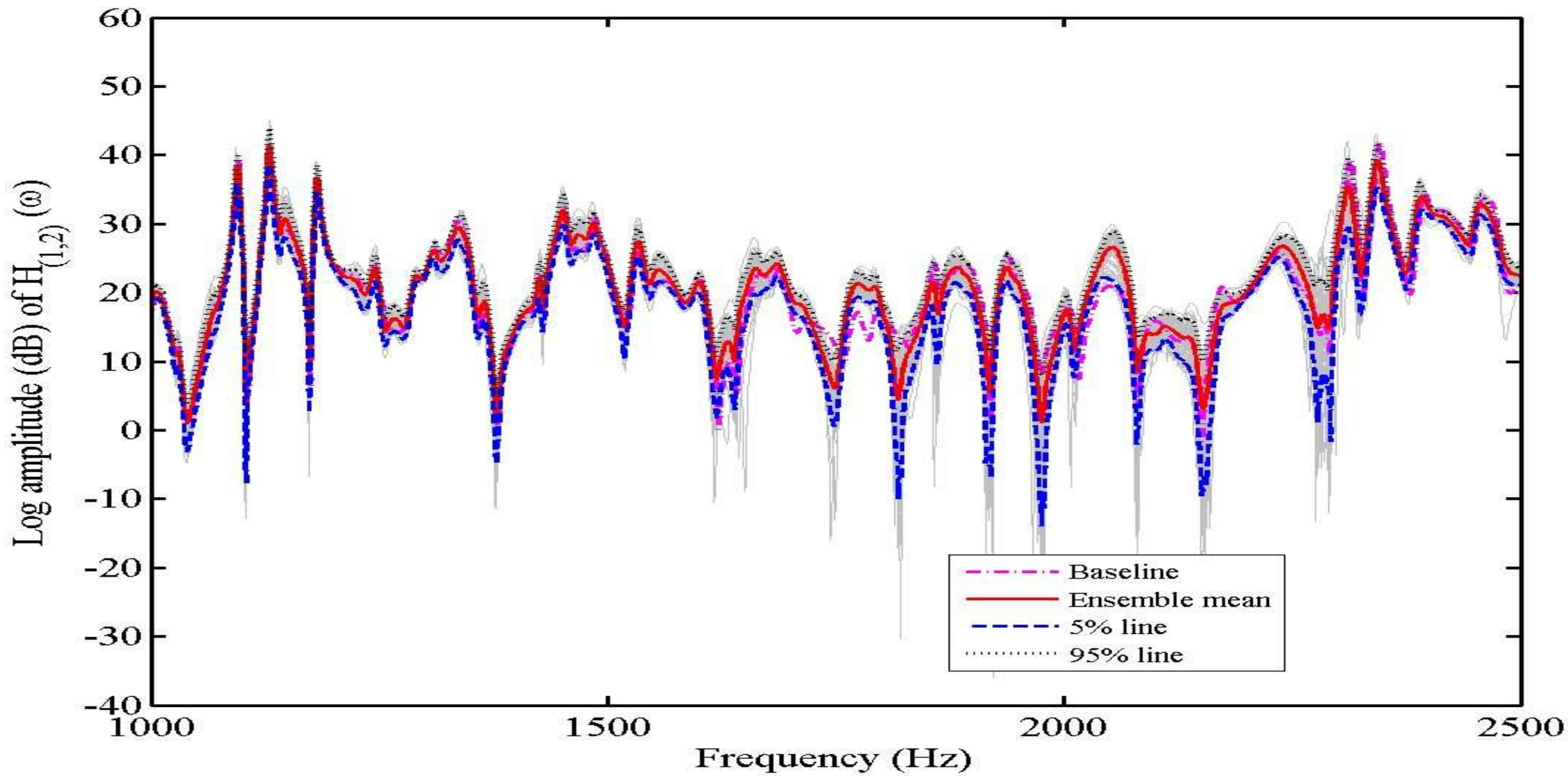
FRF amplitude: Mid Freq



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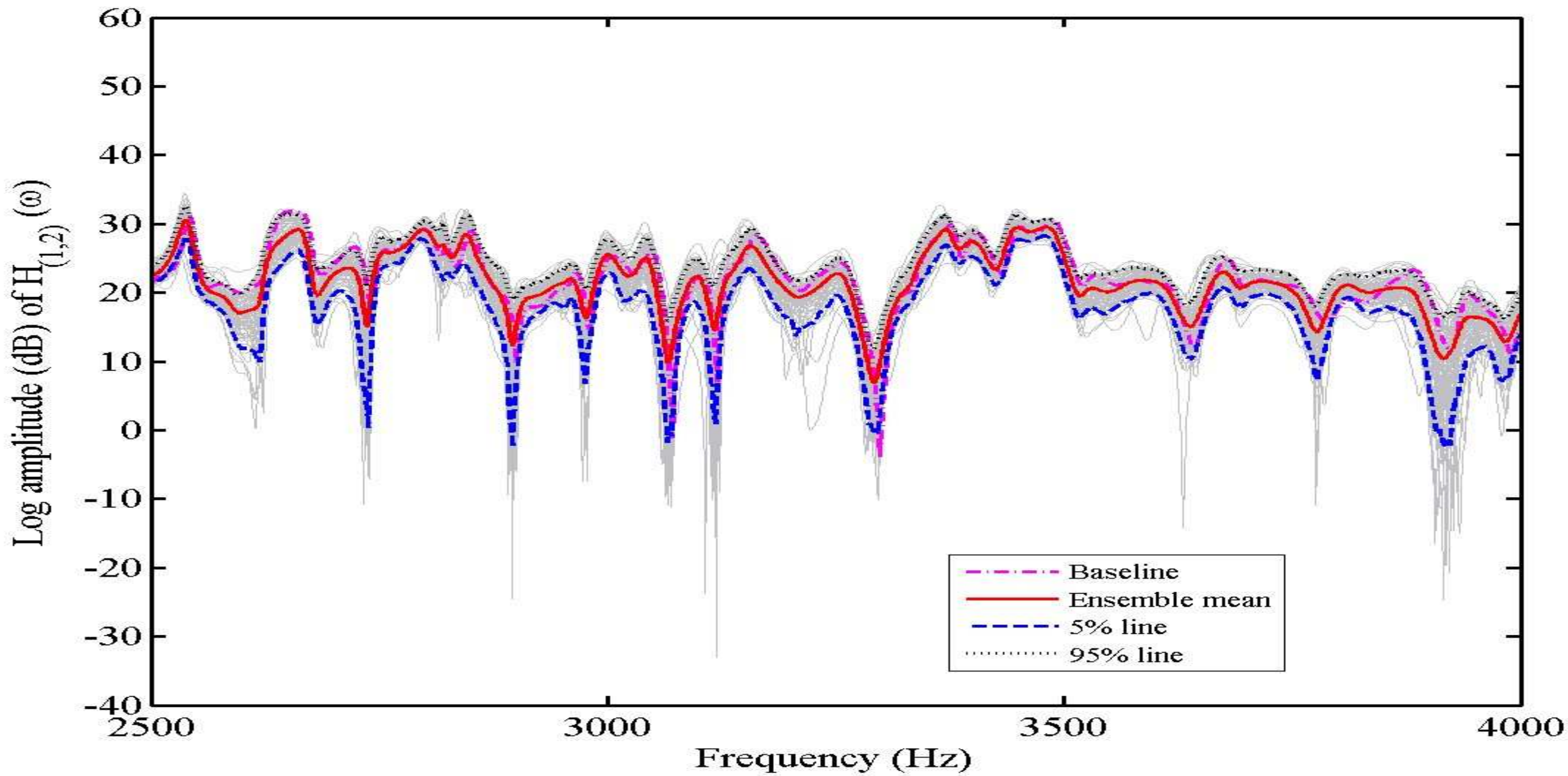
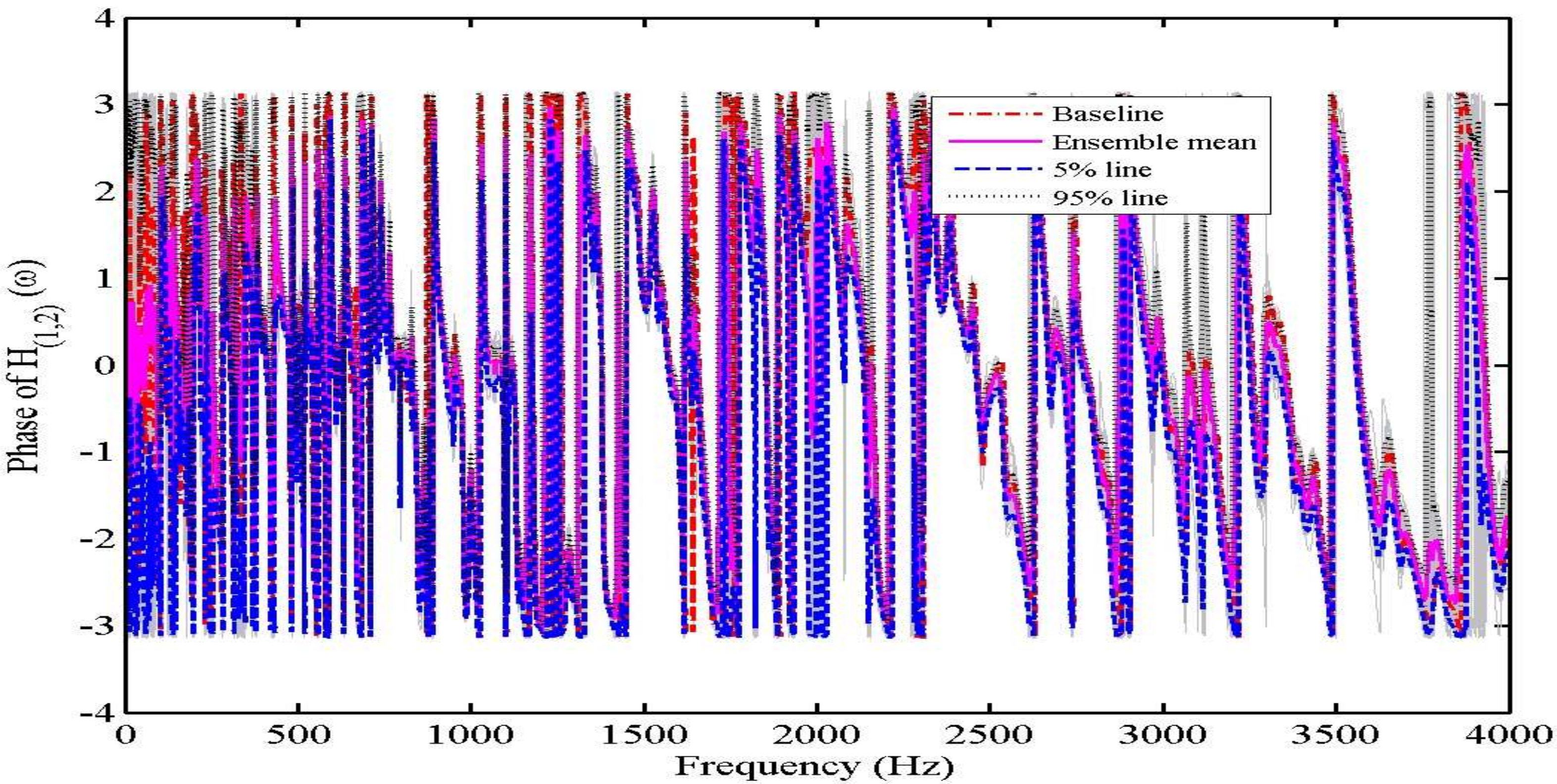
FRF amplitude: High Freq



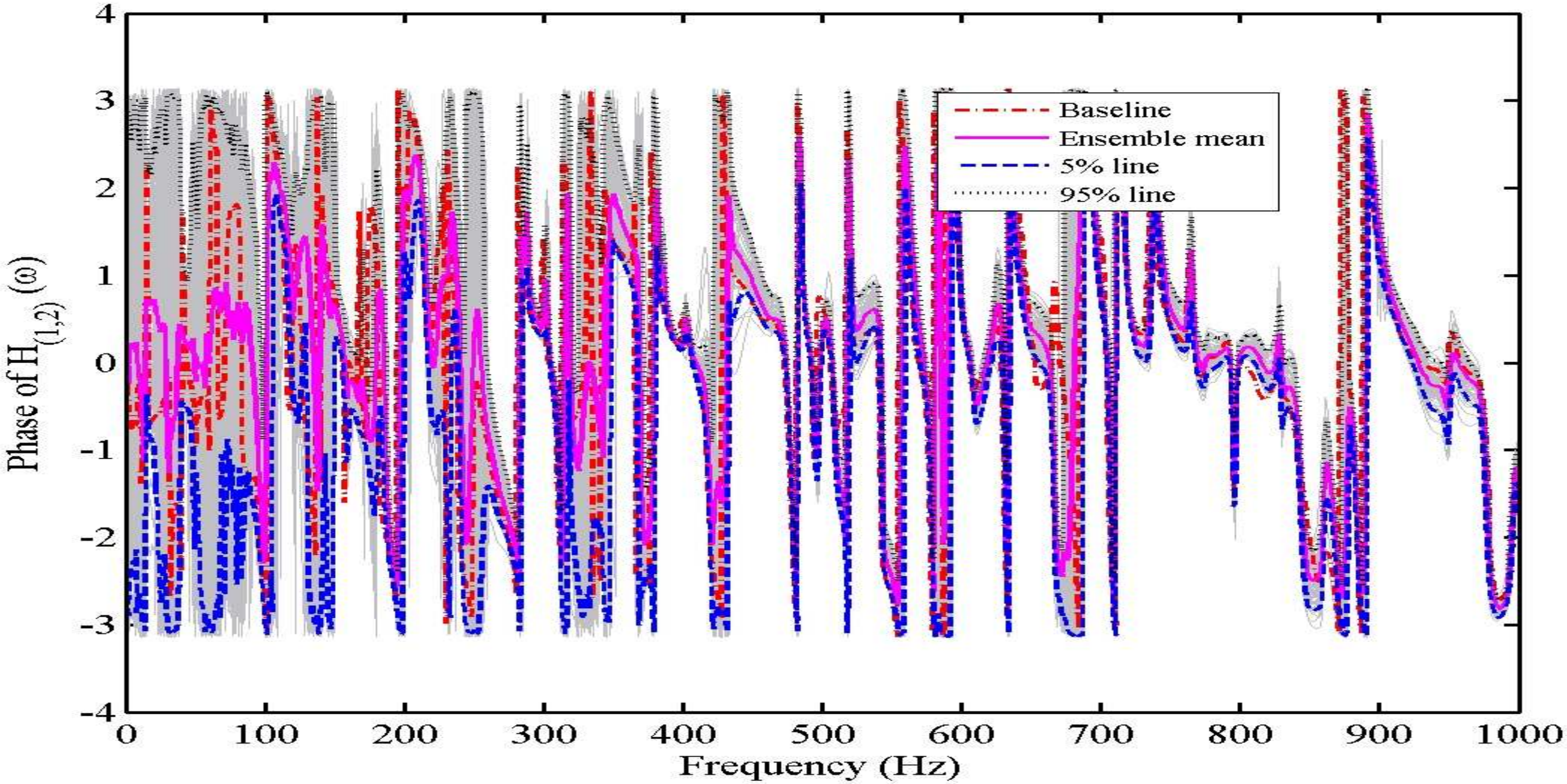
Experimentally measured amplitude of the cross-FRF of the plate at point 2 (nodal coordinate: (6,11)) with 10 randomly placed oscillators. 100 FRFs, together with the ensemble mean, 5% and 95% probability points are shown.

FRF phase: complete spectrum



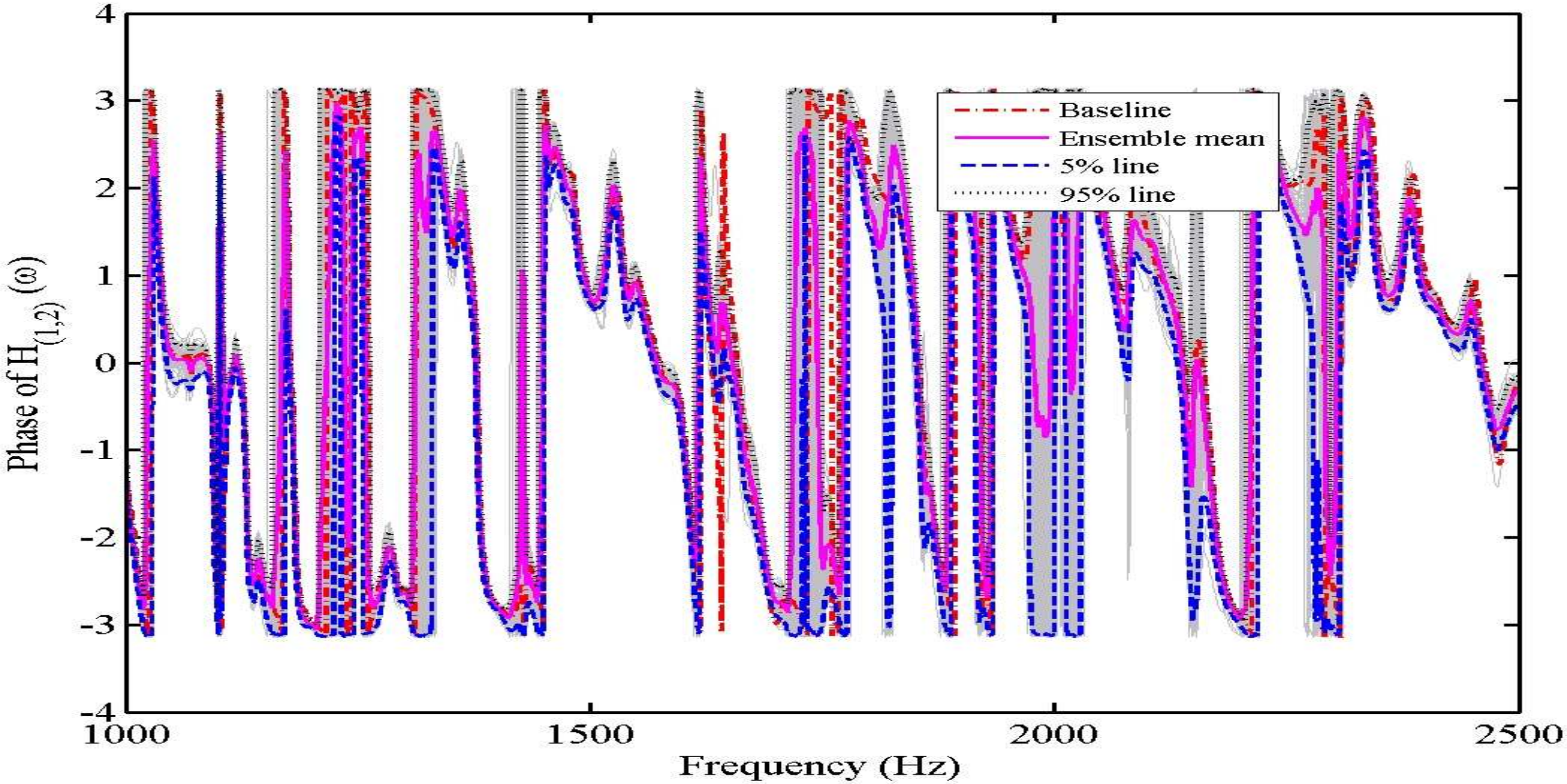
Experimentally measured phase of the cross-FRF of the plate at point 2 (nodal coordinate: (6,11)) with 10 randomly placed oscillators. 100 FRFs, together with the ensemble mean, 5% and 95% probability points are shown.

FRF phase: Low Freq



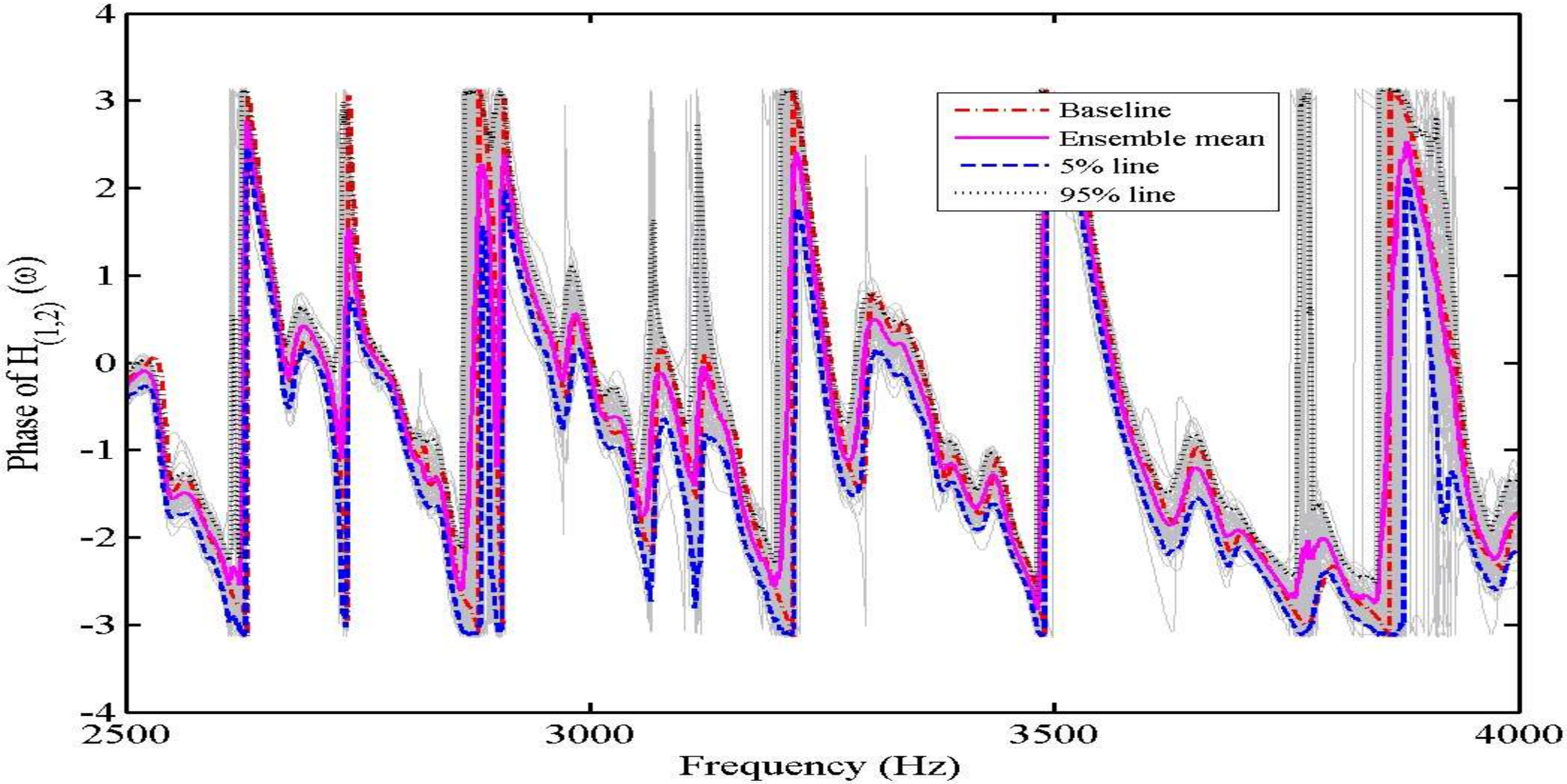
Experimentally measured phase of the cross-FRF of the plate at point 2 (nodal coordinate: (6,11)) with 10 randomly placed oscillators. 100 FRFs, together with the ensemble mean, 5% and 95% probability points are shown.

FRF phase: Mid Freq



Experimentally measured phase of the cross-FRF of the plate at point 2 (nodal coordinate: (6,11)) with 10 randomly placed oscillators. 100 FRFs, together with the ensemble mean, 5% and 95% probability points are shown.

FRF phase: High Freq



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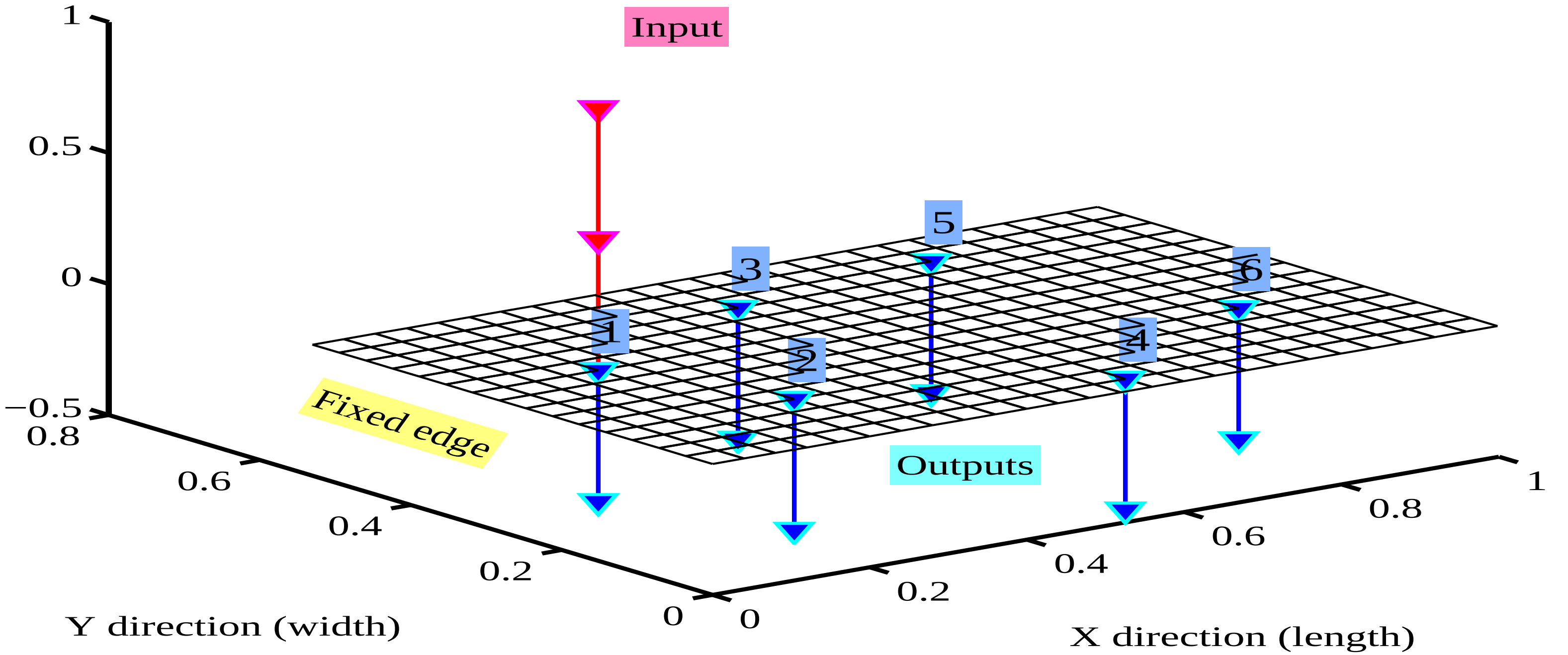
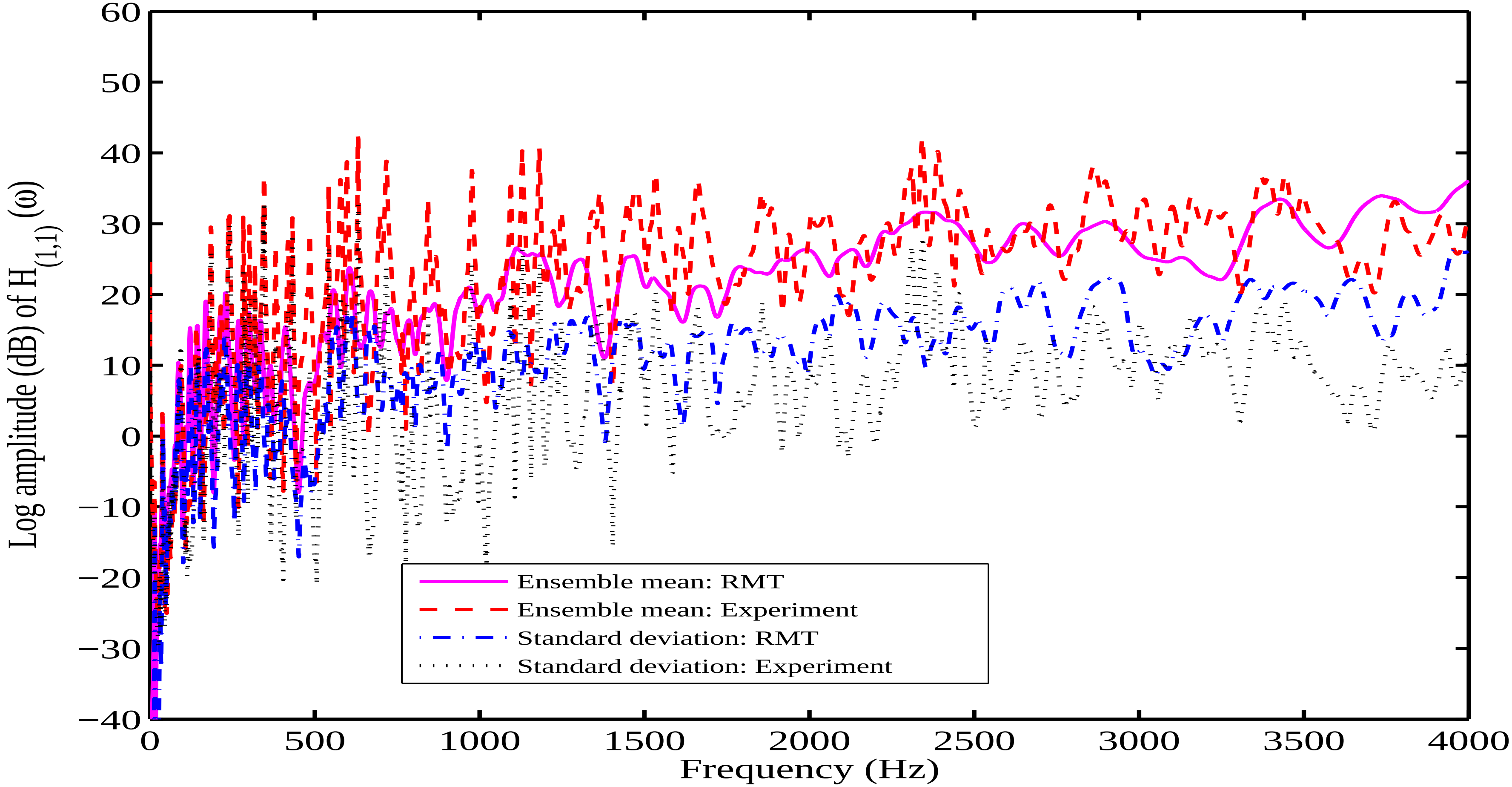
Finite Element Model



25×15 elements, 416 nodes, 1200 degrees-of-freedom. Input node number: 481, Output node numbers: 481, 877, 268, 1135, 211 and 844, 0.7% modal damping is assumed for all modes..

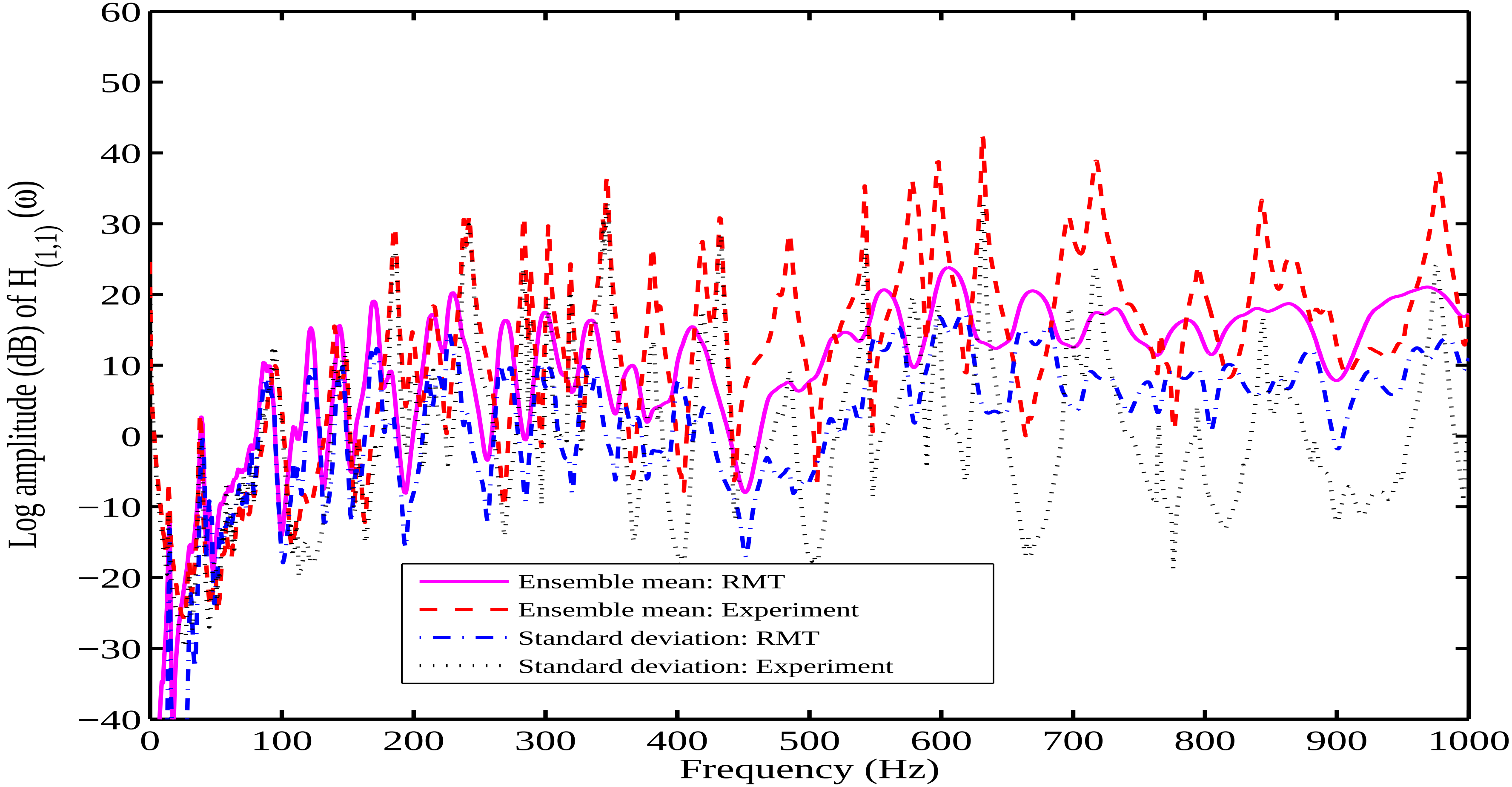
Comparison of driving-point-FRF



Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF,

$n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

Comparison of driving-point-FRF: Low Freq



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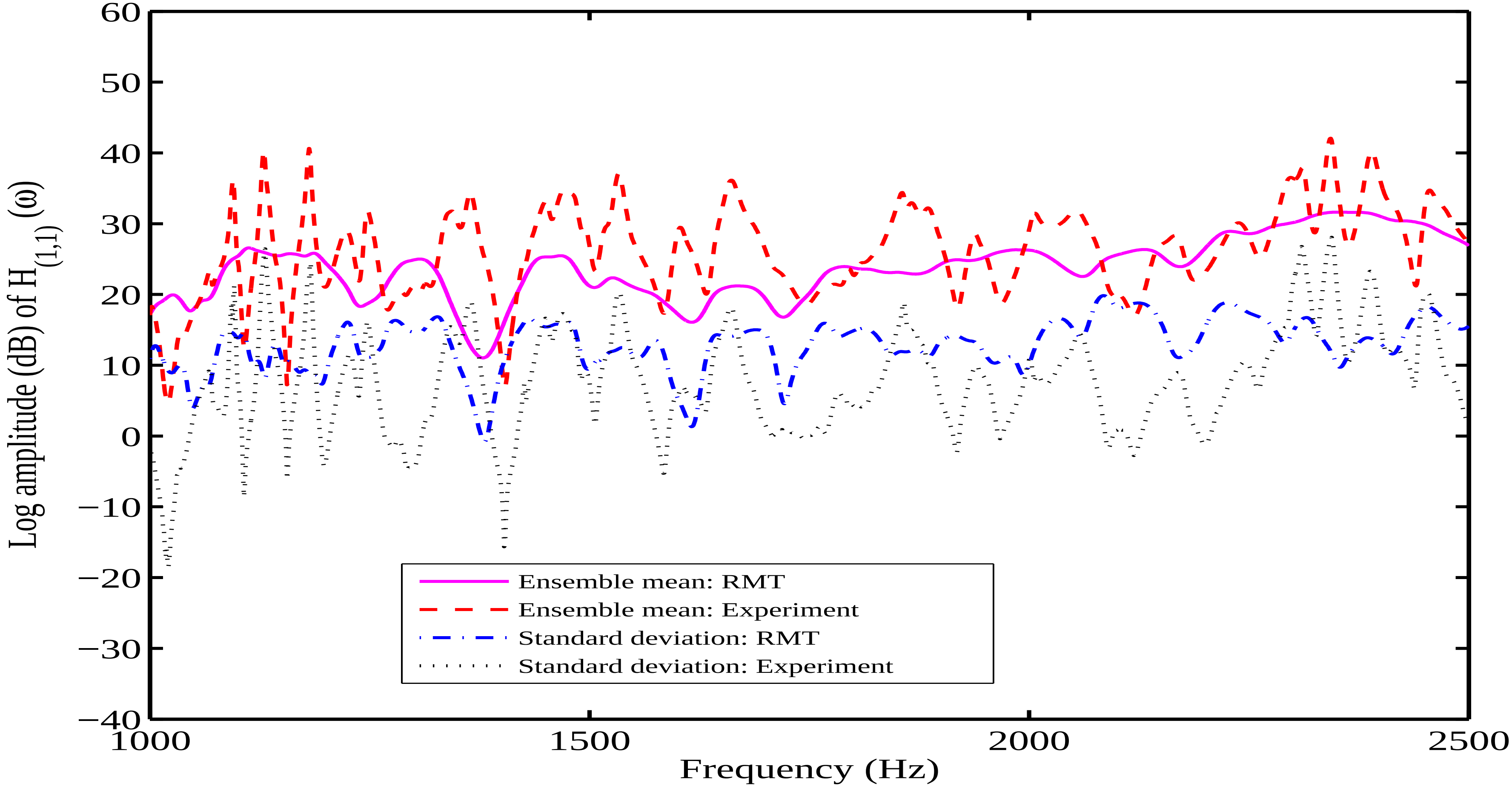

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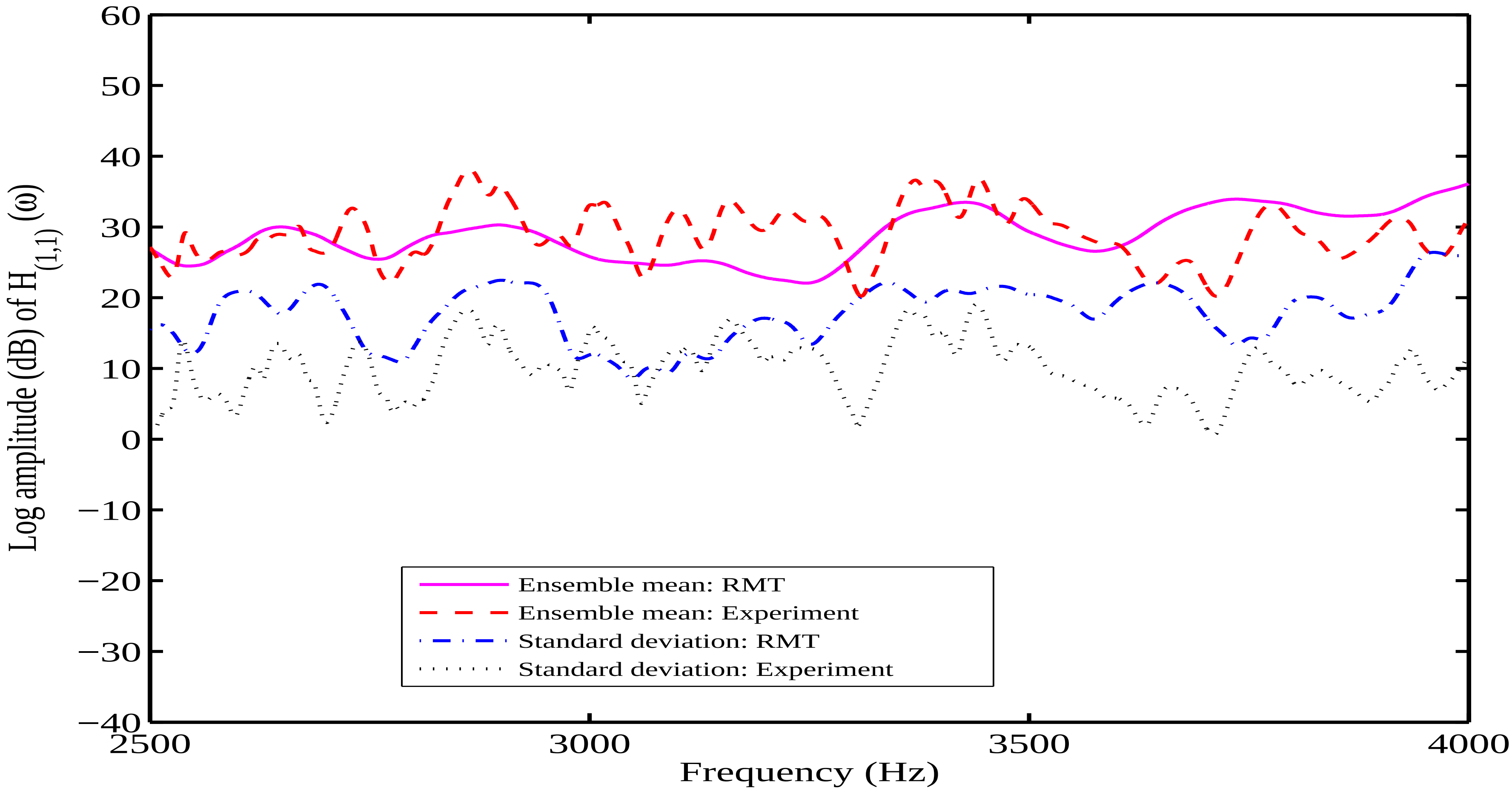
Comparison of driving-point-FRF: High Freq



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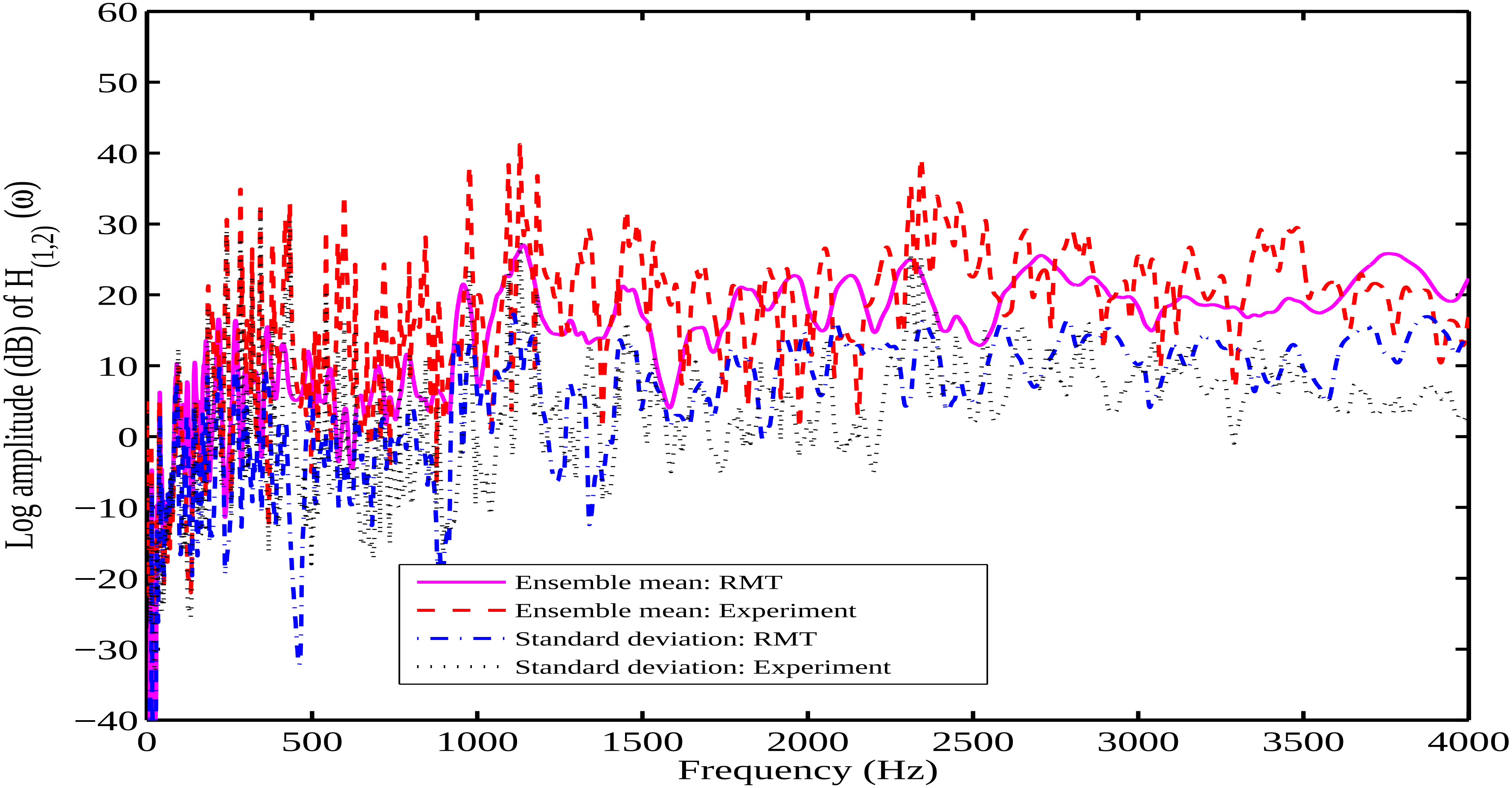
Comparison of cross-FRF



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$,

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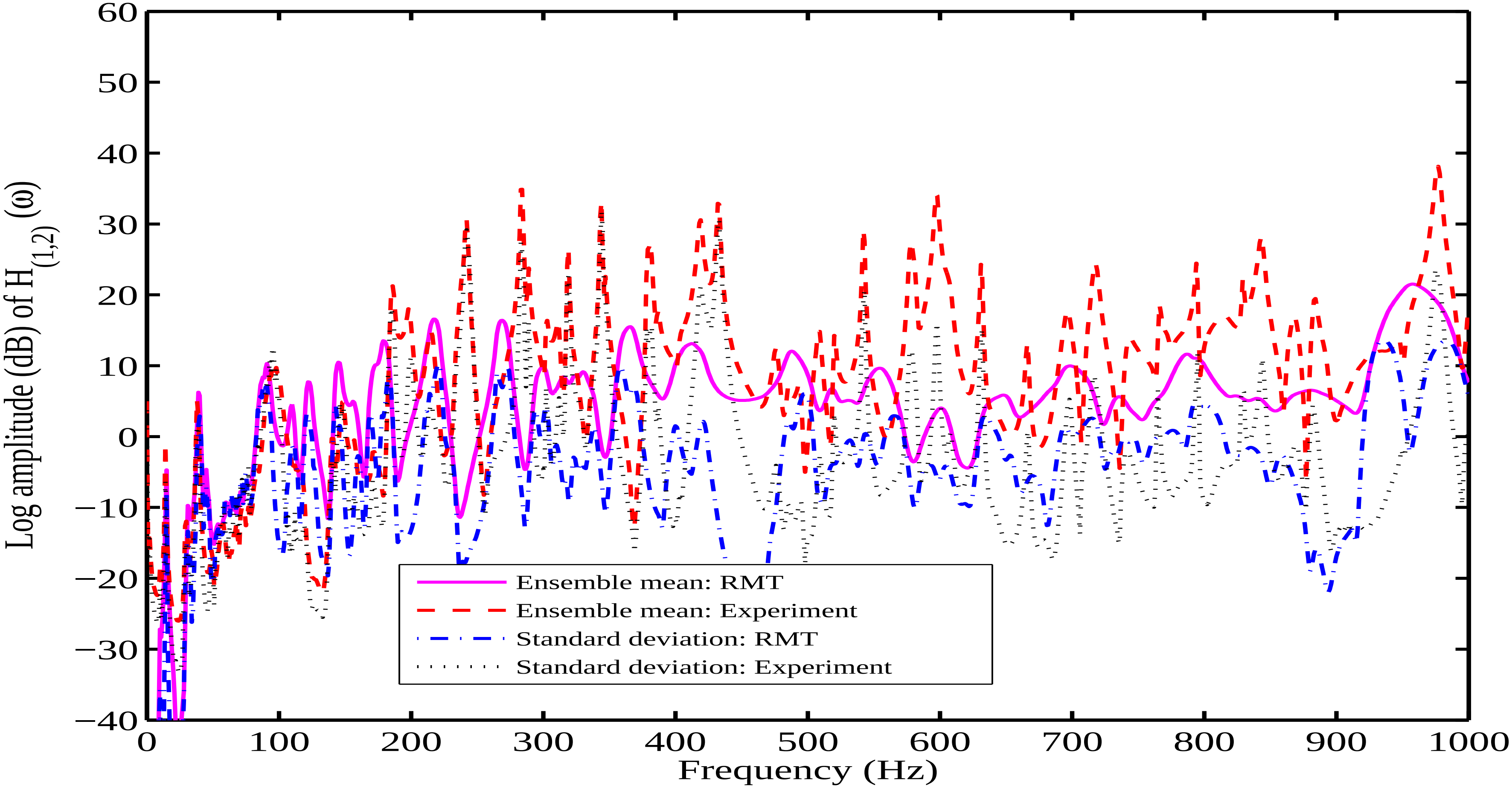
Comparison of cross-FRF: Low Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$,

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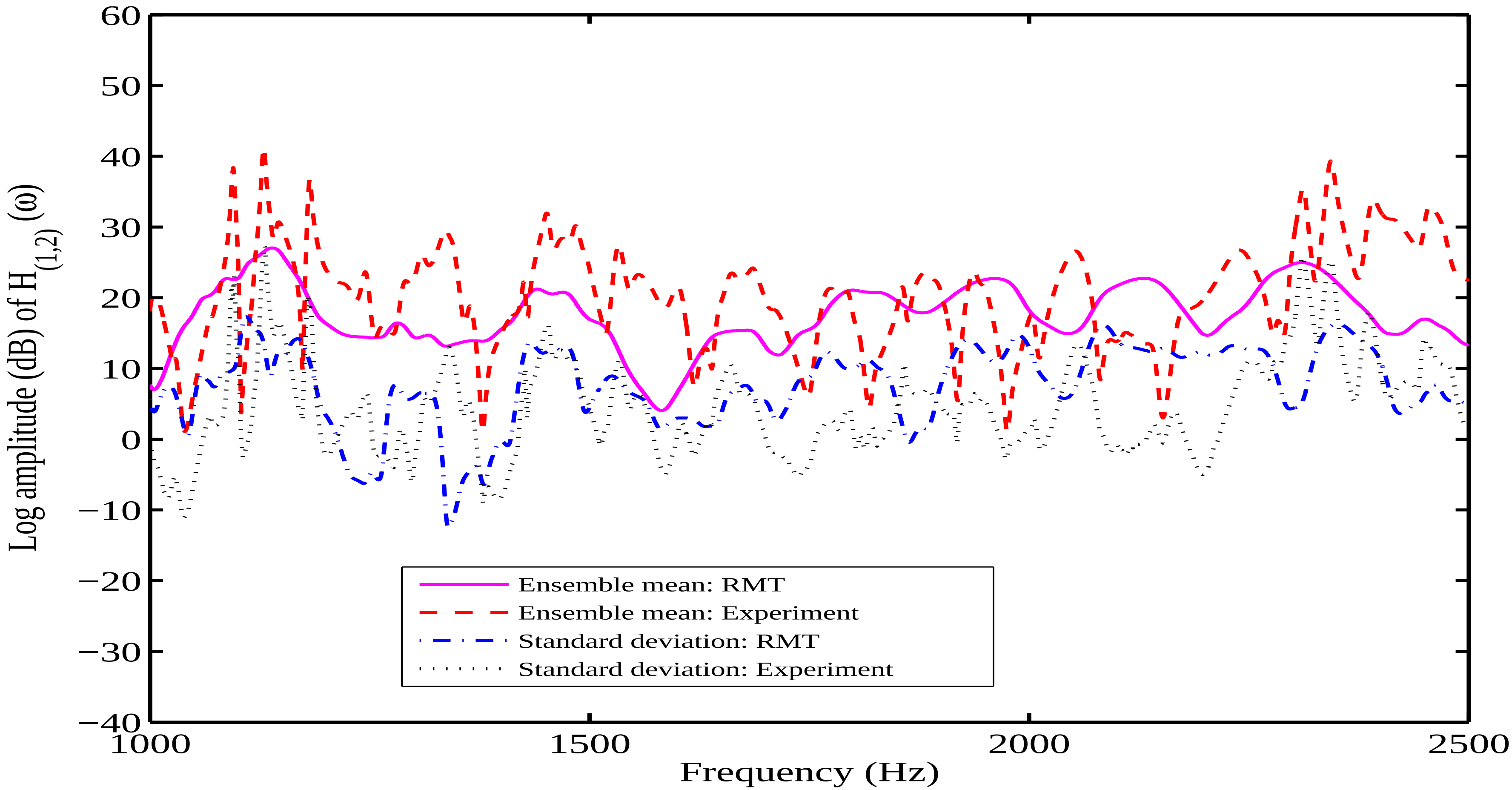
Comparison of cross-FRF: Mid Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$,

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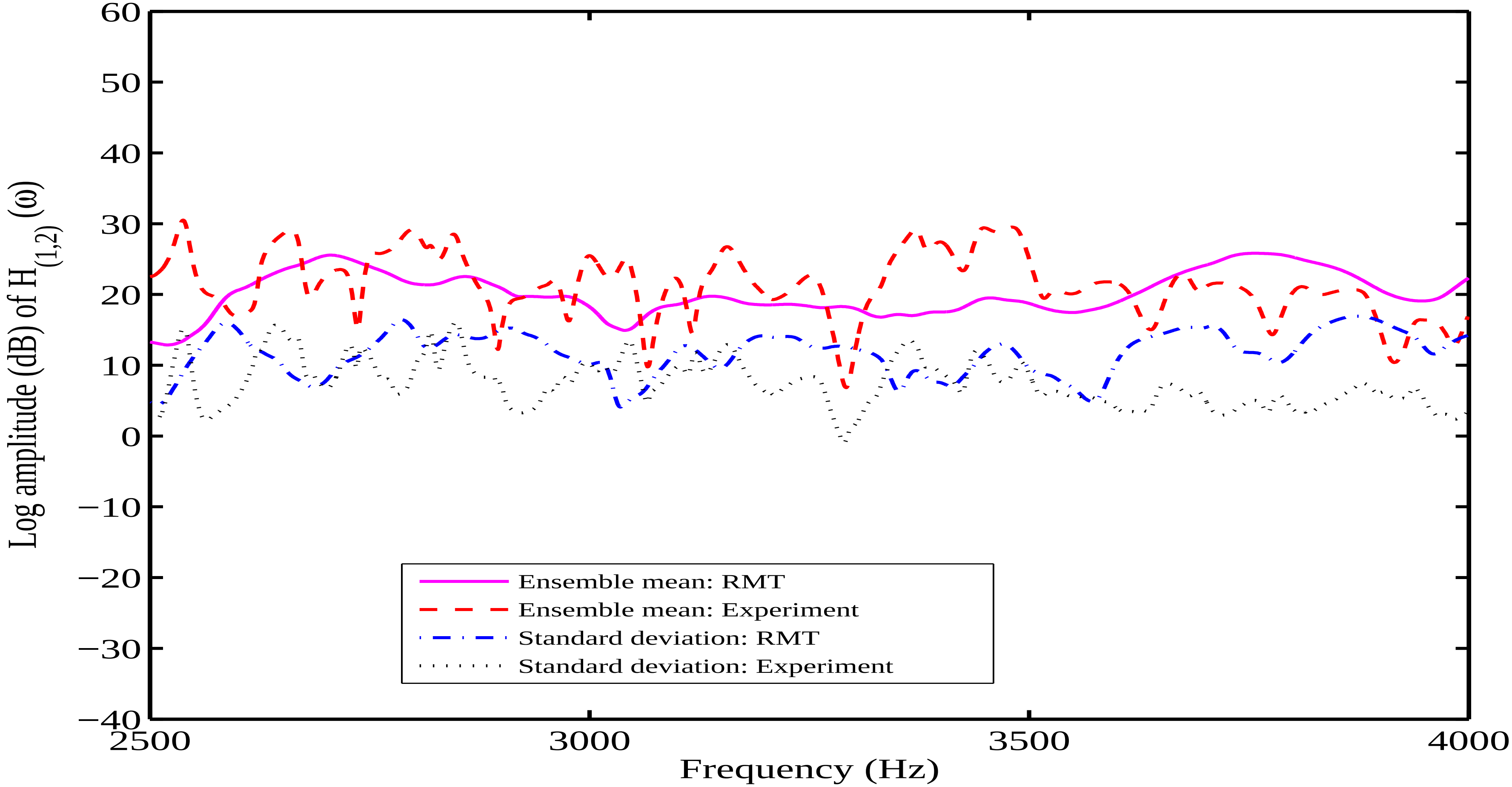
Comparison of cross-FRF: High Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$,

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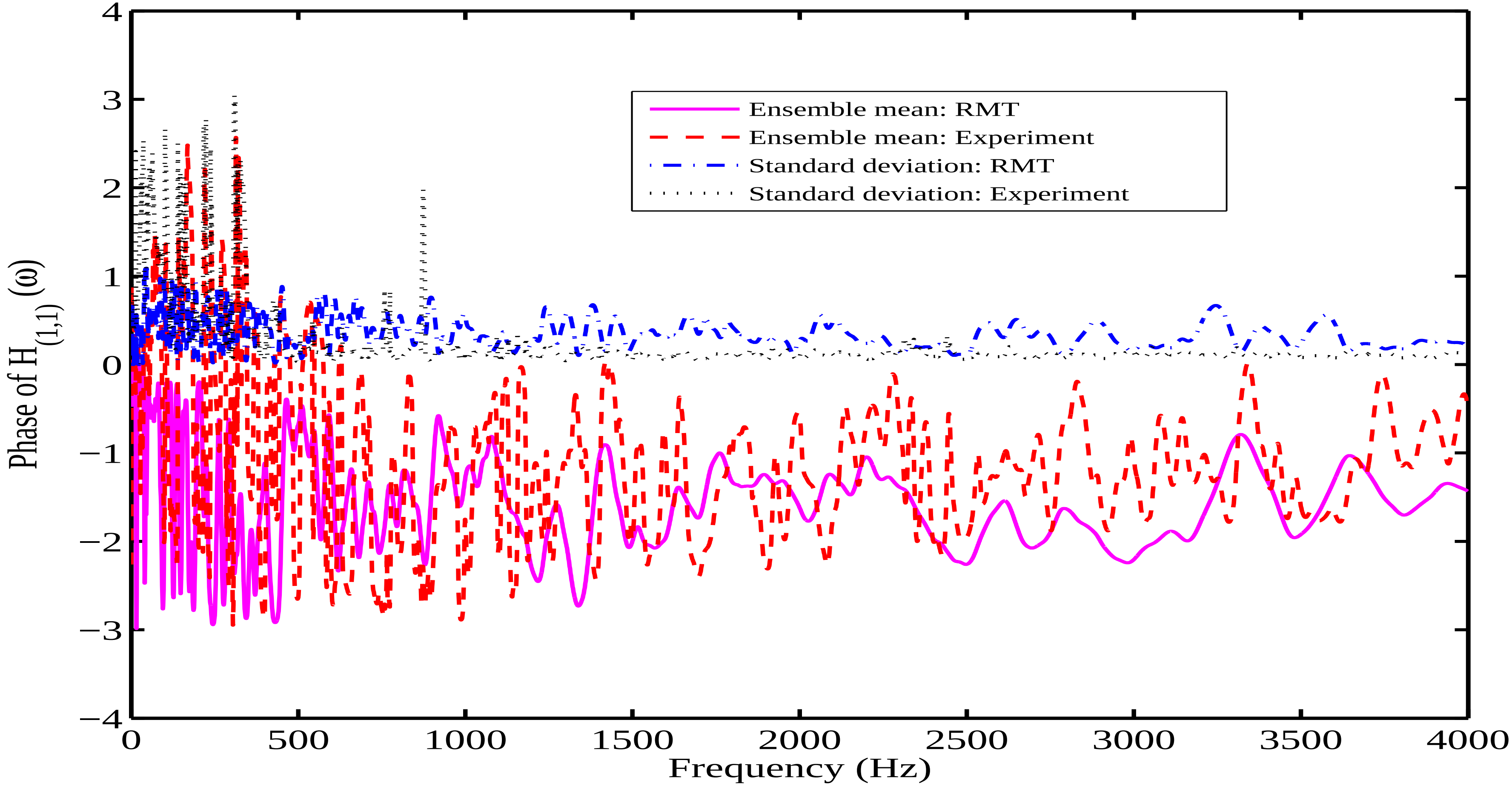
Comparison of driving-point-FRF



Comparison of the mean and standard deviation of the phase of the driving-point-FRF,

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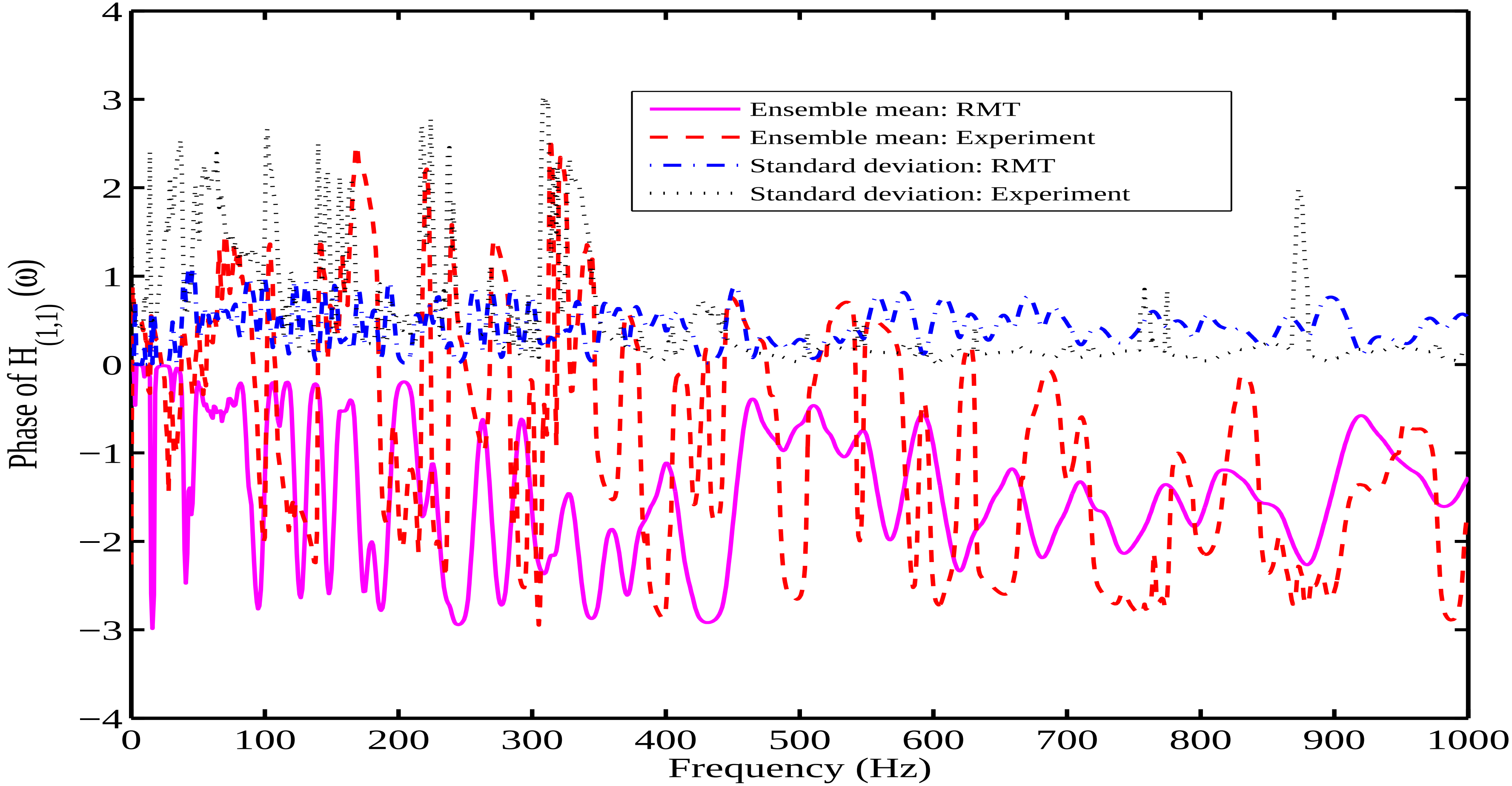
Comparison of driving-point-FRF: Low Freq



Comparison of the mean and standard deviation of the phase of the driving-point-FRF,

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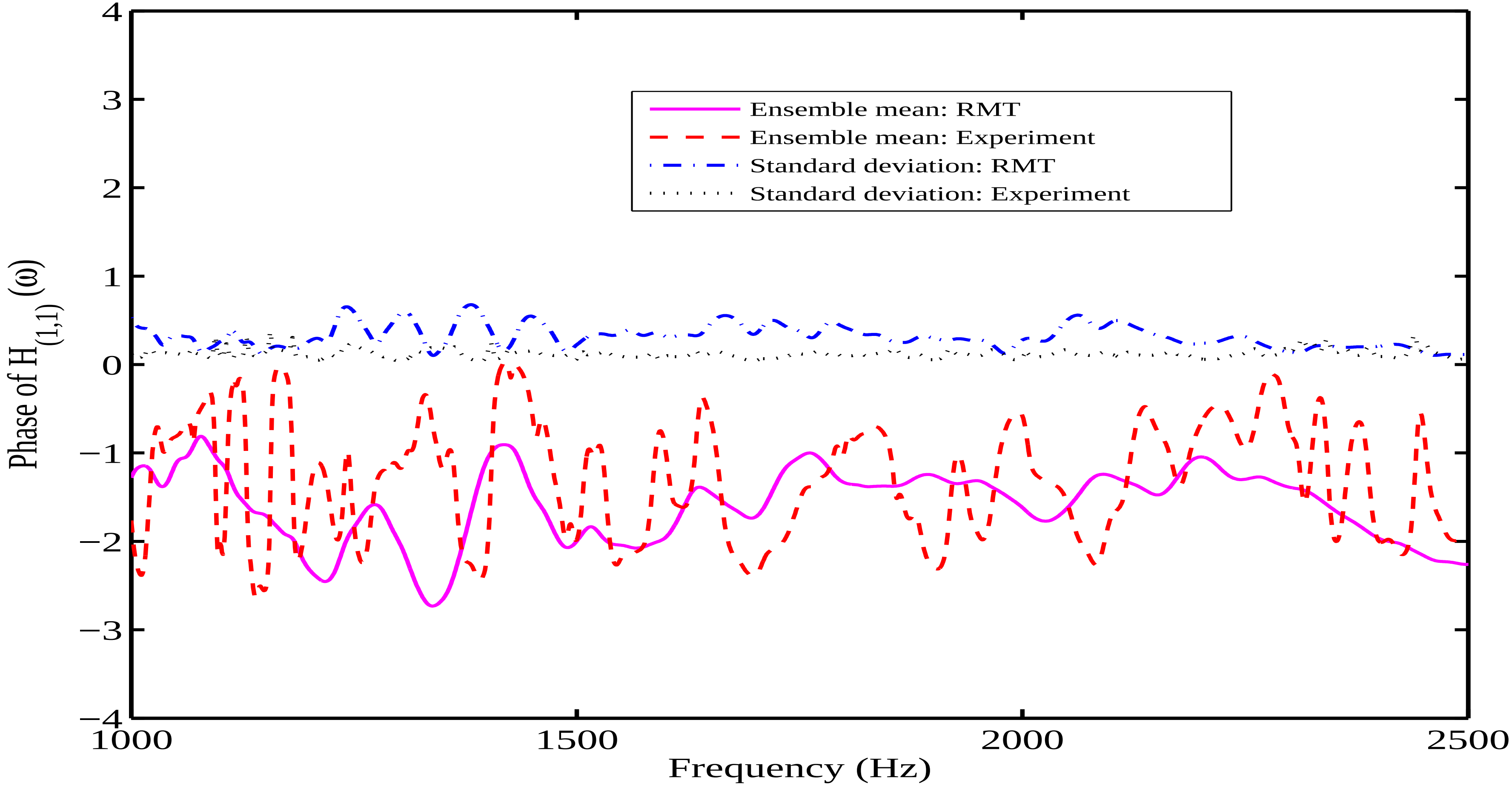
Comparison of driving-point-FRF: Mid Freq



Comparison of the mean and standard deviation of the phase of the driving-point-FRF,

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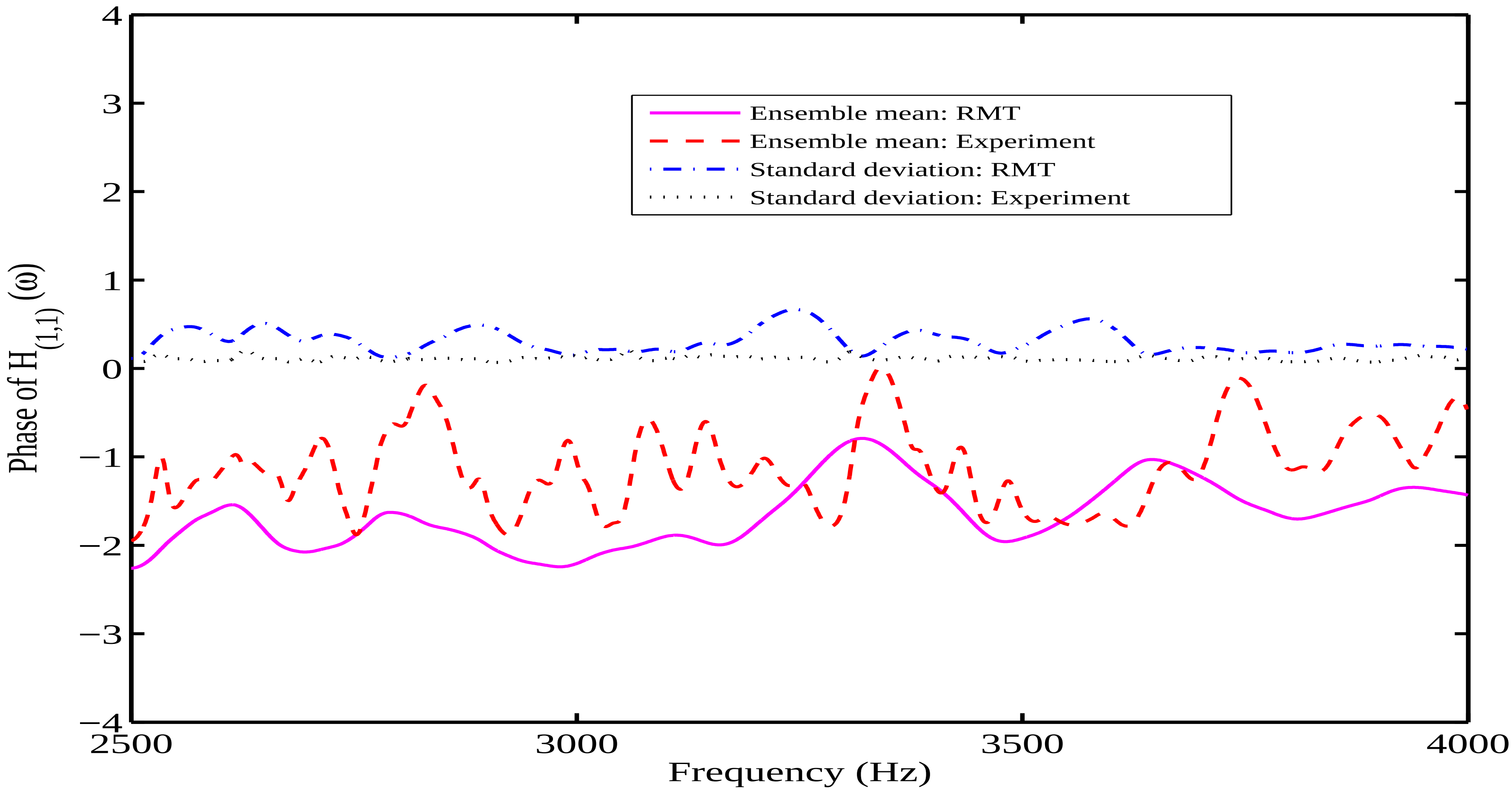
Comparison of driving-point-FRF: High Freq



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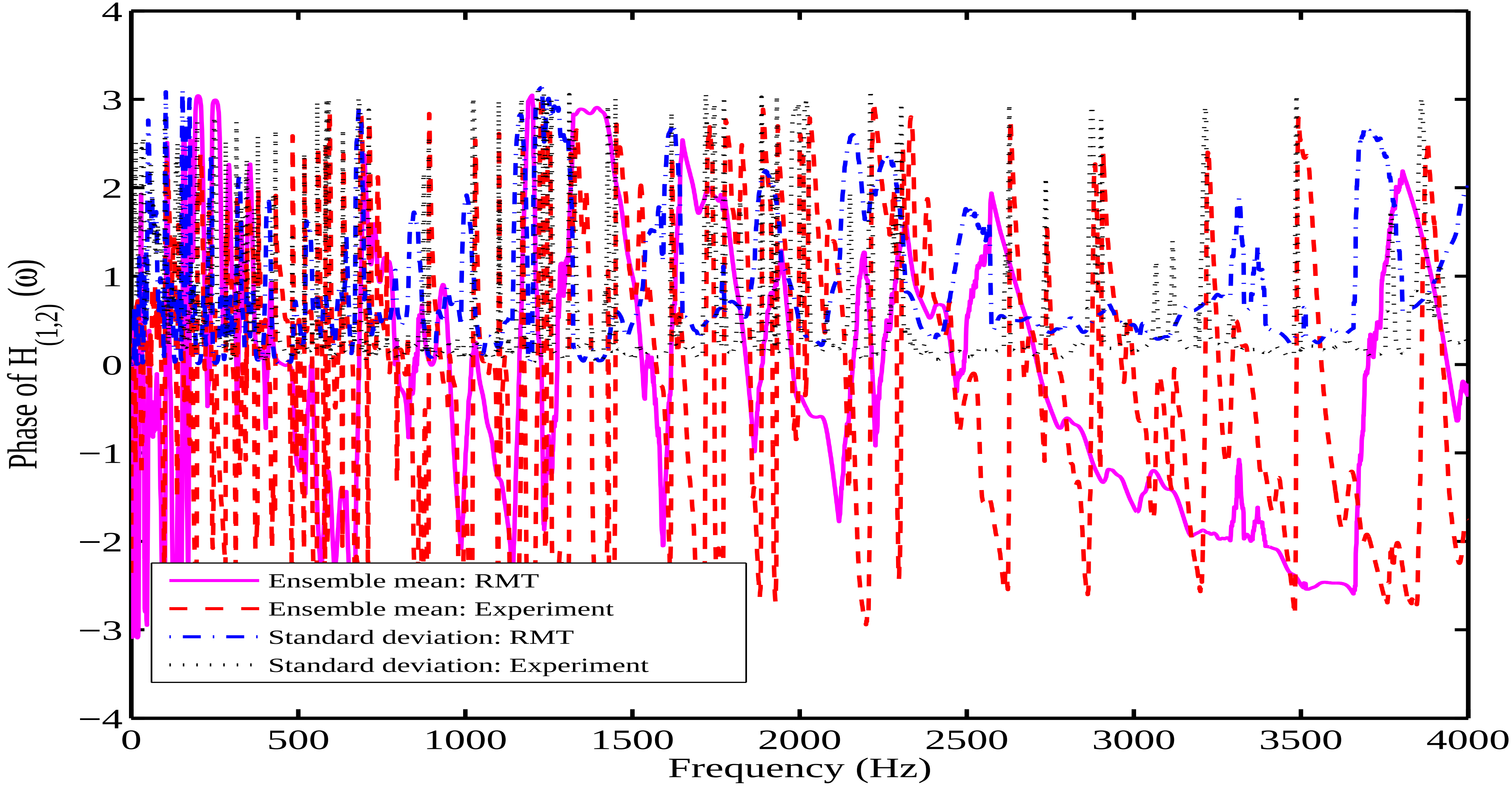
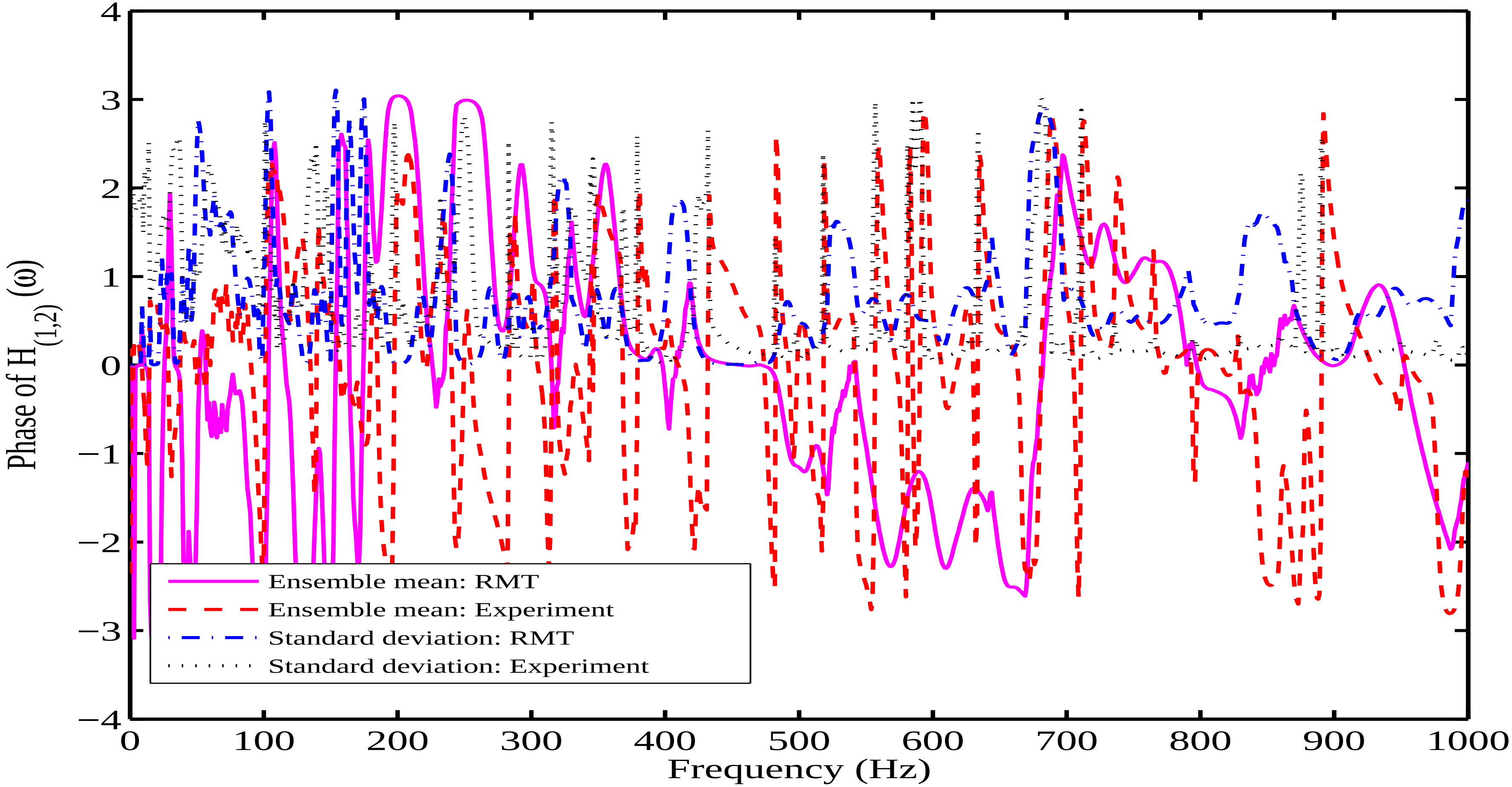
Comparison of cross-FRF



Comparison of the mean and standard deviation of the phase of the cross-FRF, $n = 1200$,

$\delta_M = 0.1166$ and $\delta_K = 0.2711$.

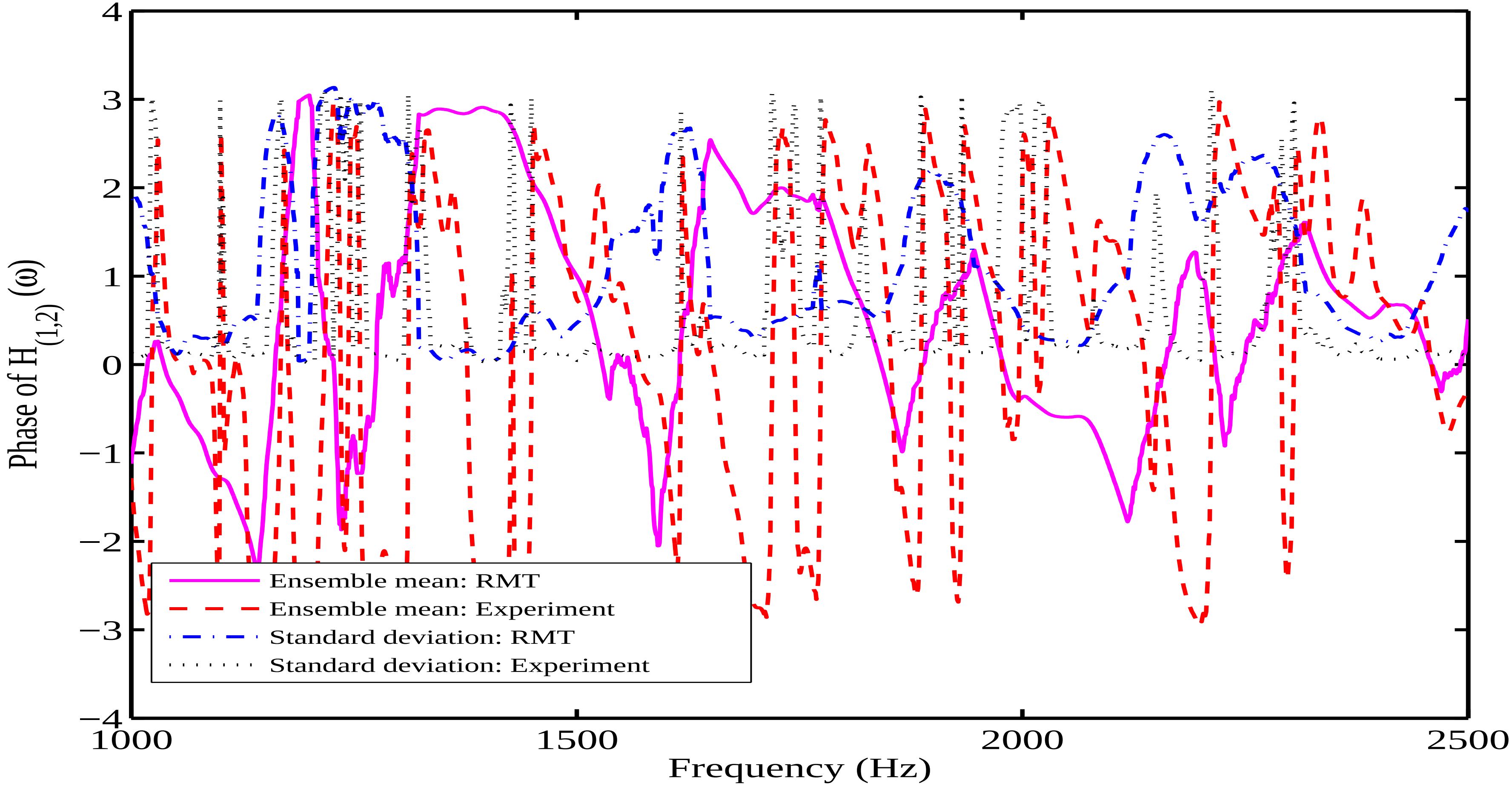
Comparison of cross-FRF: Low Freq



Comparison of the mean and standard deviation of the phase of the cross-FRF, $n = 1200$,

$\delta_M = 0.1166$ and $\delta_K = 0.2711$.

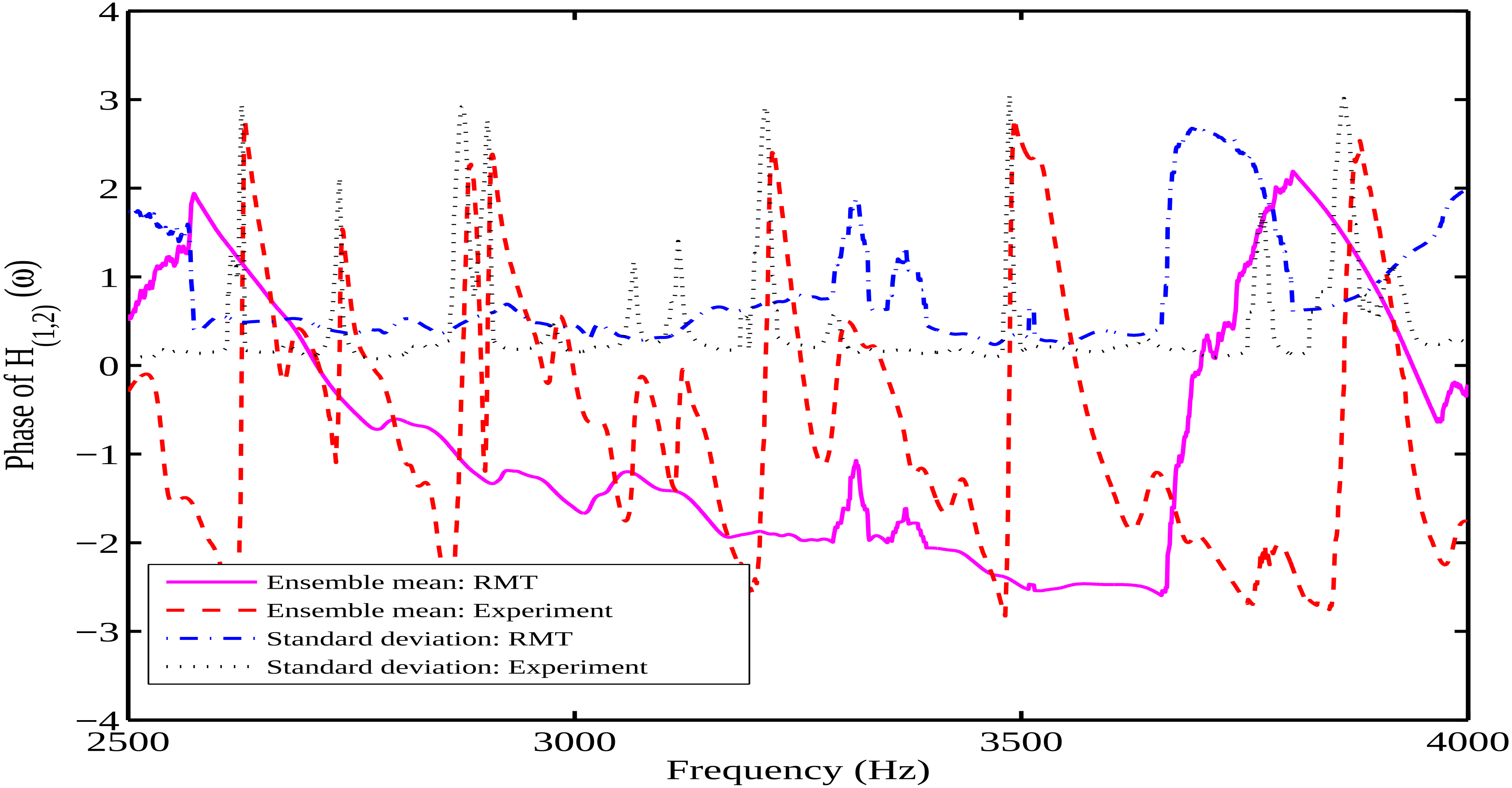
Comparison of cross-FRF: Mid Freq



Comparison of the mean and standard deviation of the phase of the cross-FRF, $n = 1200$,

$\delta_M = 0.1166$ and $\delta_K = 0.2711$.

Comparison of cross-FRF: High Freq



Comparison of the mean and standard deviation of the phase of the cross-FRF, $n = 1200$,

$\delta_M = 0.1166$ and $\delta_K = 0.2711$.

Conclusions

- When uncertainties in the system parameters (parametric uncertainty) and modelling (nonparametric) are considered, the discretized equation of motion of linear dynamical systems is characterized by random mass, stiffness and damping matrices.
- A new unified parametric-nonparametric UQ method for linear dynamical systems has been proposed and experimentally validated.
- The matrix variate probability density function of the random system matrices can be represented by noncentral Wishart distribution. Existing nonparametric distribution is a special case of the proposed distribution.