Response Variability of Viscoelastically Damped Systems

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Outline of the presentation

- Overview of viscoelastically damped systems
- Eigensolutions
  - State-space approach
  - Approximate methods in N-space
- Dynamic response calculation
- Parametric sensitivity of eigensolutions
- Parametric sensitivity of dynamic response
- Numerical results
- Conclusions

Damping models

- Viscous damping is the most widely used damping model for complex aerospace dynamic systems.
- In general a physically realistic model of damping may not be a viscous damping model.
- Damping models in which the dissipative forces depend on any quantity other than the instantaneous generalized velocities are non-viscous (e.g., viscoelastic) damping models.
- Possibly the most general way to model damping within the linear range is to use non-viscous damping models which depend on the past history of motion via convolution integrals over kernel functions.

Equation of motion

The equations of motion of a N-DOF linear system:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\,\dot{\mathbf{u}}(\tau)\,\mathrm{d}\tau + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) \tag{1}$$

together with the initial conditions

$$\mathbf{u}(t=0) = \mathbf{u_0} \in \mathbb{R}^N \quad \text{and} \quad \dot{\mathbf{u}}(t=0) = \dot{\mathbf{u}}_\mathbf{0} \in \mathbb{R}^N. \tag{2}$$

$\mathbf{u}(t)$: displacement vector, $\mathbf{f}(t)$: forcing vector, $\mathbf{M}, \mathbf{K}$: mass and stiffness matrices.

In the limit when $\boldsymbol{\mathcal{G}}(t-\tau) = \mathbf{C}\,\delta(t-\tau)$, where $\delta(t)$ is the Dirac-delta function, this reduces to viscous damping.

Damping functions - 1

Model Number	Damping function	Author and year of publication
1	$G(s) = \sum_{k=1}^{n} \dfrac{a_k s}{s + b_k}$	Biot[1] - 1955
2	$G(s) = \dfrac{E_1 s^\alpha - E_0 b s^\beta}{1 + b s^\beta} \quad (0 < \alpha, \beta < 1)$	Bagley and Torvik[2] - 1983
3	$sG(s) = G^\infty \left[1 + \sum_k \alpha_k \dfrac{s^2 + 2\xi_k \omega_k s}{s^2 + 2\xi_k \omega_k s + \omega_k^2}\right]$	Golla and Hughes[3] - 1985
		and McTavish and Hughes[4] - 1993
4	$G(s) = 1 + \sum_{k=1}^{n} \dfrac{\Delta_k s}{s + \beta_k}$	Lesieutre and Mingori[5] - 1990
5	$G(s) = c \dfrac{1 - e^{-st_0}}{st_0}$	Adhikari[6] - 1998
6	$G(s) = \dfrac{c}{st_0} \dfrac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}$	Adhikari[6] - 1998
7	$G(s) = c\, e^{s^2/4\mu} \left[1 - \operatorname{erf}\left(\dfrac{s}{2\sqrt{\mu}}\right)\right]$	Adhikari and Woodhouse[7] - 2001

Some damping functions in the Laplace domain.

Damping functions - 2

- We use a damping model for which the kernel function matrix:

$$\boldsymbol{\mathcal{G}}(t) = \sum_{k=1}^{n} \mu_k e^{-\mu_k t} \mathbf{C}_k \tag{3}$$

- The constants $\mu_k \in \mathbb{R}^+$ are known as the relaxation parameters and n denotes the number relaxation parameters.
- When $\mu_k \to \infty, \forall\, k$ this reduces to the viscous damping model:

$$\mathbf{C} = \sum_{k=1}^{n} \mathbf{C}_k. \tag{4}$$

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Non-linear Eigenvalue Problem

- The eigenvalue problem associated with a linear system with exponential damping model:

$$\left[s_j^2 \mathbf{M} + s_j \sum_{k=1}^{n} \frac{\mu_k}{s_j + \mu_k} \mathbf{C}_k + \mathbf{K} \right] \mathbf{z}_j = \mathbf{0}, \quad \text{for } j = 1, \cdots, m. \tag{5}$$

- Two types of eigensolutions:
  - $2N$ complex conjugate solutions - underdamped/vibrating modes
  - p real solutions [$p = \sum_{k=1}^{n} \operatorname{rank}(\mathbf{C}_k)$] - overdamped modes

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State-space Approach - 1

The equation of motion can be transformed to ($m = 2N + nN$) dimensional system

$$\mathbf{B}\,\dot{\mathbf{z}}(t) = \mathbf{A}\,\mathbf{z}(t) + \mathbf{r}(t) \tag{6}$$

$$\mathbf{B} = \begin{bmatrix} \sum_{k=1}^{n} \mathbf{C}_k & \mathbf{M} & -\mathbf{C}_1/\mu_1 & \cdots & -\mathbf{C}_n/\mu_n \\ \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{C}_1/\mu_1 & \mathbf{O} & \mathbf{C}_1/\mu_1^2 & \mathbf{O} & \mathbf{O} \\ \vdots & \mathbf{O} & \mathbf{O} & \ddots & \mathbf{O} \\ -\mathbf{C}_n/\mu_n & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{C}_n/\mu_n^2 \end{bmatrix}, \quad \mathbf{r}(t) = \begin{Bmatrix} \mathbf{f}(t) \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{Bmatrix} \tag{7}$$

$$\mathbf{A} = \begin{bmatrix} -\mathbf{K} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{C}_1/\mu_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mathbf{C}_n/\mu_n \end{bmatrix}, \quad \mathbf{z}(t) = \begin{Bmatrix} \mathbf{u}(t) \\ \mathbf{v}(t) \\ \mathbf{y}_1(t) \\ \vdots \\ \mathbf{y}_n(t) \end{Bmatrix} \tag{8}$$

State-space Approach - 2

The eigenvalue problem in the sate-space is given by

$$\mathbf{A}\,\mathbf{z}_j = \lambda_j \mathbf{B}\,\mathbf{z}_j \tag{9}$$

The ‘size’ of the eigenvalue problem is ($2N + nN$)-dimensional.

- although exact in nature, the state-space approach is computationally very intensive for real-life systems;
- the physical insights offered by methods in the original space (eg, the modal analysis) is lost in a state-space based approach

Approximate eigensolutions

If ω_j and $\mathbf{x}_j$ are the undamped natural frequency and mode shape of the system satisfying $\mathbf{K}\mathbf{x}_j = \omega_j^2 \mathbf{M}\mathbf{x}_j$, the eigenvalues of the viscoelastically damped system obtained using the first-order perturbation method:

$$s_j \approx \mathrm{i}\omega_j - G'_{jj}(\mathrm{i}\omega_j)/2, \quad -\mathrm{i}\omega_j - G'_{jj}(-\mathrm{i}\omega_j)/2. \tag{10}$$

Similarly, the eigenvectors are given by

$$\mathbf{z}_j \approx \mathbf{x}_j - \sum_{\substack{k=1\\k\neq j}}^{N} \frac{s_j G'_{kj}(s_j)\mathbf{x}_k}{\omega_k^2 + s_j^2 + s_j G'_{kk}(s_j)}. \tag{11}$$

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Dynamic Response - 1

Taking the Laplace transform of the equation of motion and considering the initial conditions we have

$$s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q_0} - \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\,\mathbf{G}(s)\bar{\mathbf{q}} - \mathbf{G}(s)\mathbf{q_0} + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s)$$

or $\quad \mathbf{D}(s)\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + [s\mathbf{M} + \mathbf{G}(s)]\,\mathbf{q_0}.$

The *dynamic stiffness matrix* is defined as

$$\mathbf{D}(s) = s^2\mathbf{M} + s\,\mathbf{G}(s) + \mathbf{K} \;\in \mathbb{C}^{N\times N}. \tag{12}$$

The inverse of the dynamics stiffness matrix, known as the transfer function matrix, is given by

$$\mathbf{H}(s) = \mathbf{D}^{-1}(s) \in \mathbb{C}^{N\times N}. \tag{13}$$

Dynamic Response - 2

Using the residue-calculus, the transfer function matrix can be expressed like a viscously damped system as

$$\mathbf{H}(s) = \sum_{j=1}^{m} \frac{\mathbf{R}_j}{s - s_j}; \quad \mathbf{R}_j = \operatorname*{res}_{s=s_j} [\mathbf{H}(s)] = \frac{\mathbf{z}_j \mathbf{z}_j^T}{\mathbf{z}_j^T \frac{\partial \mathbf{D}(s_j)}{\partial s_j} \mathbf{z}_j} \tag{14}$$

where m is the number of non-zero eigenvalues (order) of the system, s_j and $\mathbf{z}_j$ are respectively the eigenvalues and eigenvectors of the system, which are solutions of the non-linear eigenvalue problem

$$[s_j^2 \mathbf{M} + s_j \mathbf{G}(s_j) + \mathbf{K}] \mathbf{z}_j = \mathbf{0}, \quad \text{for } j = 1, \cdots, m \tag{15}$$

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Dynamic Response - 3

The expression of $\mathbf{H}(s)$ allows the response to be expressed as modal summation as

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{m} \gamma_j \frac{\mathbf{z}_j^T \bar{\mathbf{f}}(s) + \mathbf{z}_j^T \mathbf{M}\dot{\mathbf{q}}_{\mathbf{0}} + s\mathbf{z}_j^T \mathbf{M}\mathbf{q}_{\mathbf{0}} + \mathbf{z}_j^T \mathbf{G}(s)\mathbf{q}_{\mathbf{0}}(s)}{s - s_j} \mathbf{z}_j \quad (16)$$

where the normalization constant

$$\gamma_j = \frac{1}{\mathbf{z}_j^T \frac{\partial \mathbf{D}(s_j)}{\partial s_j} \mathbf{z}_j}. \quad (17)$$

We use the approximate eigensolutions in the 'N'-space.

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Dynamic Response - 4

The response in the time domain can be obtained by taking the inverse transform:

$$\mathbf{q}(t) = \mathcal{L}^{-1}[\bar{\mathbf{q}}(s)] = \sum_{j=}^{m} \gamma_j a_j(t) \mathbf{z}_j \tag{18}$$

where the time-dependent scalar coefficients (for $t > 0$)

$$a_j(t) = \int_0^t e^{s_j(t-\tau)} \left\{ \mathbf{z}_j^T \mathbf{f}(\tau) + \mathbf{z}_j^T \boldsymbol{\mathcal{G}}(\tau) \mathbf{q_0} \right\} \mathrm{d}\tau + e^{s_j t} \left\{ \mathbf{z}_j^T \mathbf{M} \dot{\mathbf{q}}_\mathbf{0} + s_j \mathbf{z}_j^T \mathbf{M} \mathbf{q_0} \right\} \tag{19}$$

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Response variability: Direct approach

The dynamic response in the Laplace domain:

$$\bar{\mathbf{q}}(s) = \mathbf{D}^{-1}(s)\bar{\mathbf{p}}(s) \tag{20}$$

where

$$\mathbf{D}(s) = s^2\mathbf{M} + s\sum_{k=1}^{n}\frac{\mu_k}{s+\mu_k}\mathbf{C}_k + \mathbf{K} \tag{21}$$

$$\bar{\mathbf{p}}(s) = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_{\mathbf{0}} + \left[s\mathbf{M} + \mathbf{G}(s)\right]\mathbf{q}_{\mathbf{0}}. \tag{22}$$

Suppose the system matrices are functions of some design parameter p. We want to obtain $\dfrac{\partial \bar{\mathbf{q}}(s)}{\partial p}$.

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Response variability: Direct approach

Differentiating the equation of motion in the Laplace domain

$$\frac{\partial \bar{\mathbf{q}}(s)}{\partial p} = \frac{\partial \mathbf{D}^{-1}(s)}{\partial p}\bar{\mathbf{p}}(s) + \mathbf{D}^{-1}(s)\frac{\partial \bar{\mathbf{p}}(s)}{\partial p} \tag{23}$$

Using the direct approach,

$$\frac{\partial \mathbf{D}^{-1}(s)}{\partial p} = \mathbf{D}^{-1}(s)\frac{\partial \mathbf{D}(s)}{\partial p}\mathbf{D}^{-1}(s) \tag{24}$$

where

$$\frac{\partial \mathbf{D}(s)}{\partial p} = s^2\frac{\partial \mathbf{M}}{\partial p} + s\frac{\partial}{\partial p}\left\{\sum_{k=1}^{n}\frac{\mu_k}{s+\mu_k}\mathbf{C}_k\right\} + \frac{\partial \mathbf{K}}{\partial p} \tag{25}$$

Response variability: Modal approach

$$\mathbf{D}^{-1}(s) = \sum_{j=1}^{m} \frac{\mathbf{R}_j}{s - s_j}; \quad \mathbf{R}_j = \frac{\mathbf{z}_j \mathbf{z}_j^T}{\theta_j} \tag{26}$$

Using the modal approach,

$$\frac{\partial \mathbf{D}^{-1}(s)}{\partial p} = \sum_{j=1}^{m} \frac{\frac{\partial \mathbf{R}_j}{\partial p}}{s - s_j} - \frac{\mathbf{R}_j}{(s - s_j)^2} \frac{\partial s_j}{\partial p} \tag{27}$$

$$\frac{\partial \mathbf{R}_j}{\partial p} = \left(\frac{\partial \mathbf{z}_j}{\partial p} \mathbf{z}_j^T + \mathbf{z}_j \frac{\partial \mathbf{z}_j}{\partial p}^T \right) / \theta_j \tag{28}$$

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Eigensolution derivative

It can be shown that (Adhikari: AIAA Journal, 40[10] (2002), pp. 2061-2069)

$$\frac{\partial s_j}{\partial p} = -\frac{1}{\theta_j}\left(\mathbf{z}_j^T \frac{\partial \mathbf{D}(s)}{\partial p}\Big|_{s=s_j}\mathbf{z}_j\right). \tag{29}$$

$$\frac{\partial \mathbf{z}_j}{\partial p} = a_{jj}\mathbf{z}_j - \sum_{\substack{k=1\\k\neq j}}^{m} \frac{\mathbf{u}_k^T \frac{\partial \mathbf{D}(s)}{\partial p}\Big|_{s=s_j}\mathbf{z}_j}{\theta_k(s_j-\lambda_k)}\mathbf{u}_k \tag{30}$$

where

$$a_{jj} = -\frac{\mathbf{z}_j^T \frac{\partial^2[\mathbf{D}(s)]}{\partial s\,\partial p}\Big|_{s=s_j}\mathbf{z}_j}{2\left(\mathbf{z}_j^T \frac{\partial \mathbf{D}(s)}{\partial s}\Big|_{s=s_j}\mathbf{z}_j\right)}. \tag{31}$$

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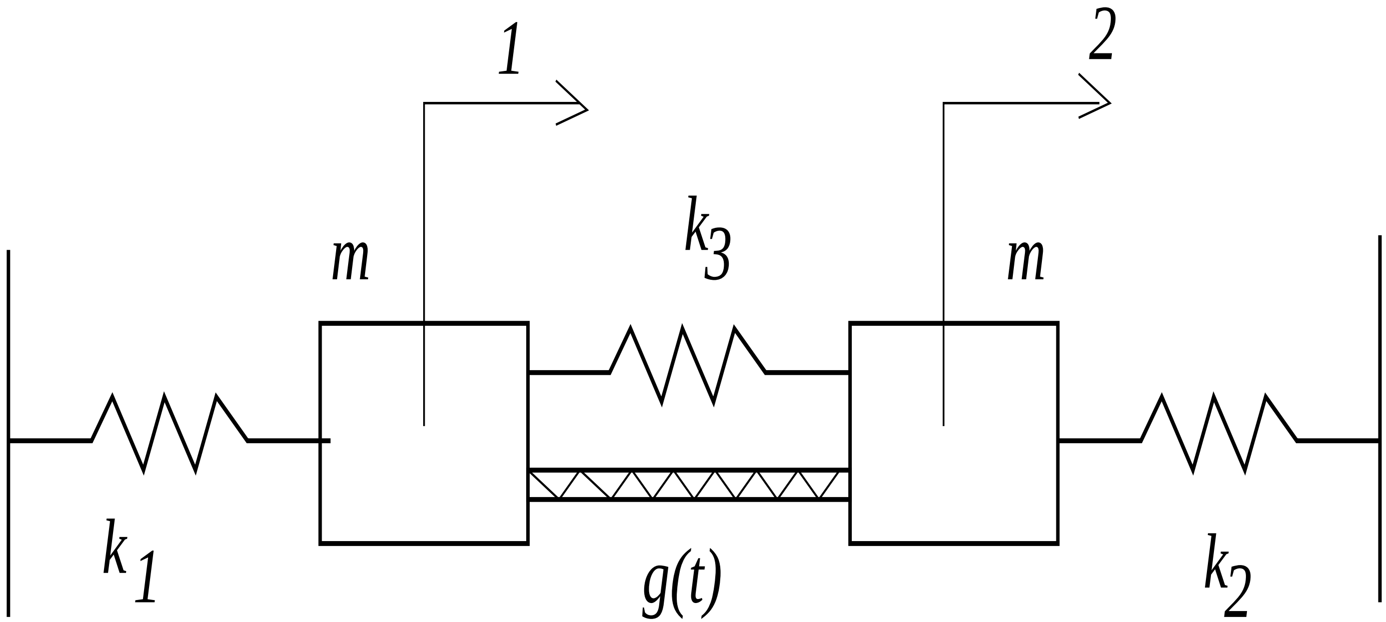
Example of a 2 DOF system



The two degrees-of-freedom spring-mass system with non-viscous damping, $m = 1$ Kg, $k_1 = 1000$ N/m, $k_3 = 100$ N/m, $g(t) = c\left(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}\right)$, $c = 4.0$ Ns/m, $\mu_1 = 10.0$ s^{-1}, $\mu_2 = 2.0$ s^{-1}

Example: system matrices

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \tag{32}$$

and

$$\boldsymbol{\mathcal{G}}(t) = g(t)\hat{\mathbf{I}}, \quad \text{where} \quad \hat{\mathbf{I}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{33}$$

The damping function $g(t)$ is assumed to be the GHM model[3, 4] so that

$$g(t) = c\left(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}\right); \quad c, \mu_1, \mu_2 \geq 0, \tag{34}$$

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System matrix derivative

We consider the derivative of eigenvalues with respect to the relaxation parameter μ_1. The derivative of the system matrices:

$$\frac{\partial \mathbf{M}}{\partial \mu_1} = \mathbf{O}, \quad \frac{\partial \mathbf{G}(s)}{\partial \mu_1} = \hat{\mathbf{I}} \frac{c\, s}{(s+\mu_1)^2} \quad \text{and} \quad \frac{\partial \mathbf{K}}{\partial \mu_1} = \mathbf{O}. \tag{35}$$

Thus we have

$$\begin{aligned} \frac{\partial \mathbf{G}(s)}{\partial s} &= -\hat{\mathbf{I}} c \left\{ \frac{\mu_1}{(s+\mu_1)^2} + \frac{\mu_2}{(s+\mu_2)^2} \right\} \\ \frac{\partial^2 [\mathbf{G}(s)]}{\partial s\, \partial \mu_1} &= -\hat{\mathbf{I}} c \frac{s-\mu_1}{(s+\mu_1)^3}. \end{aligned} \tag{36}$$

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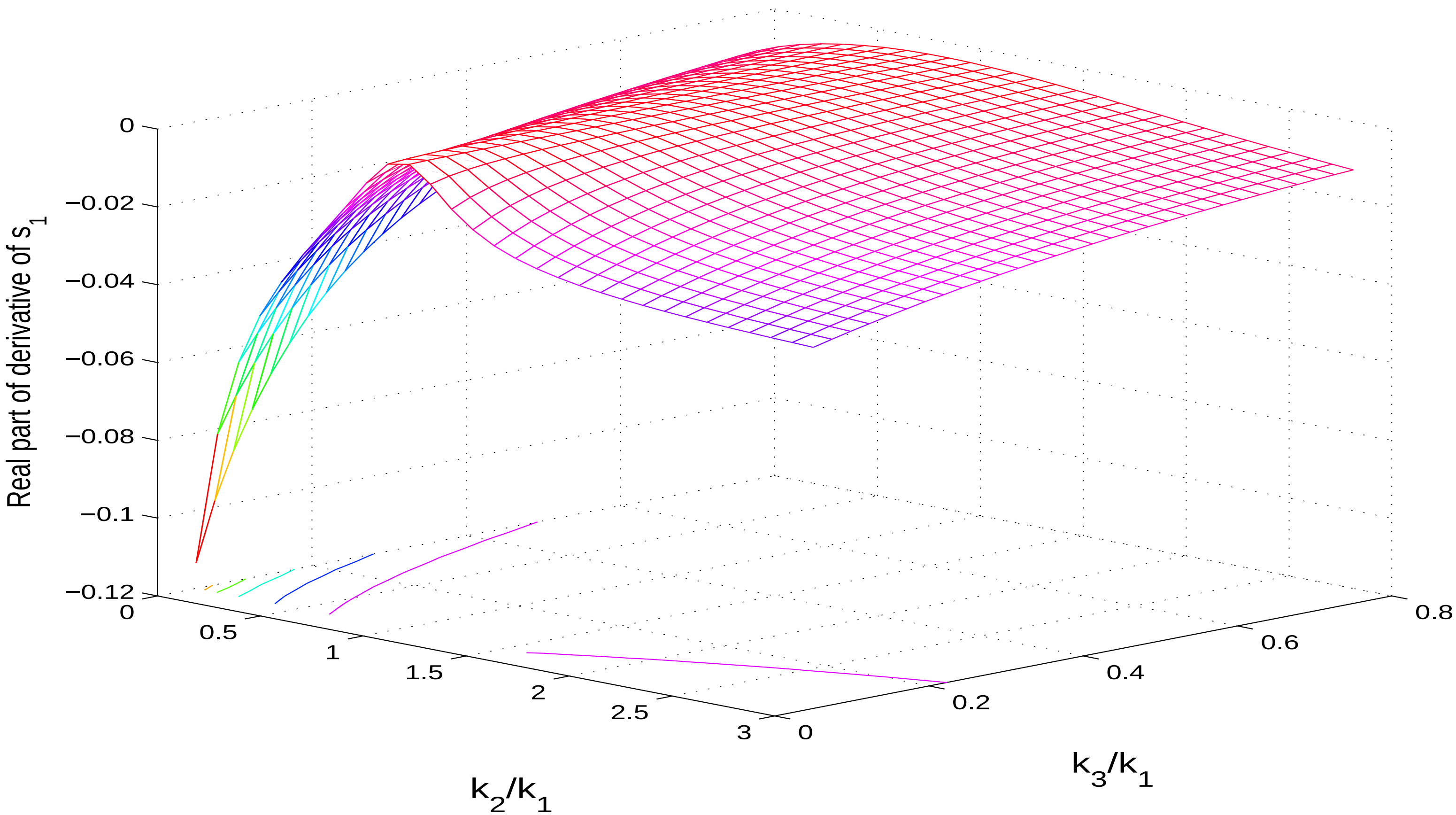
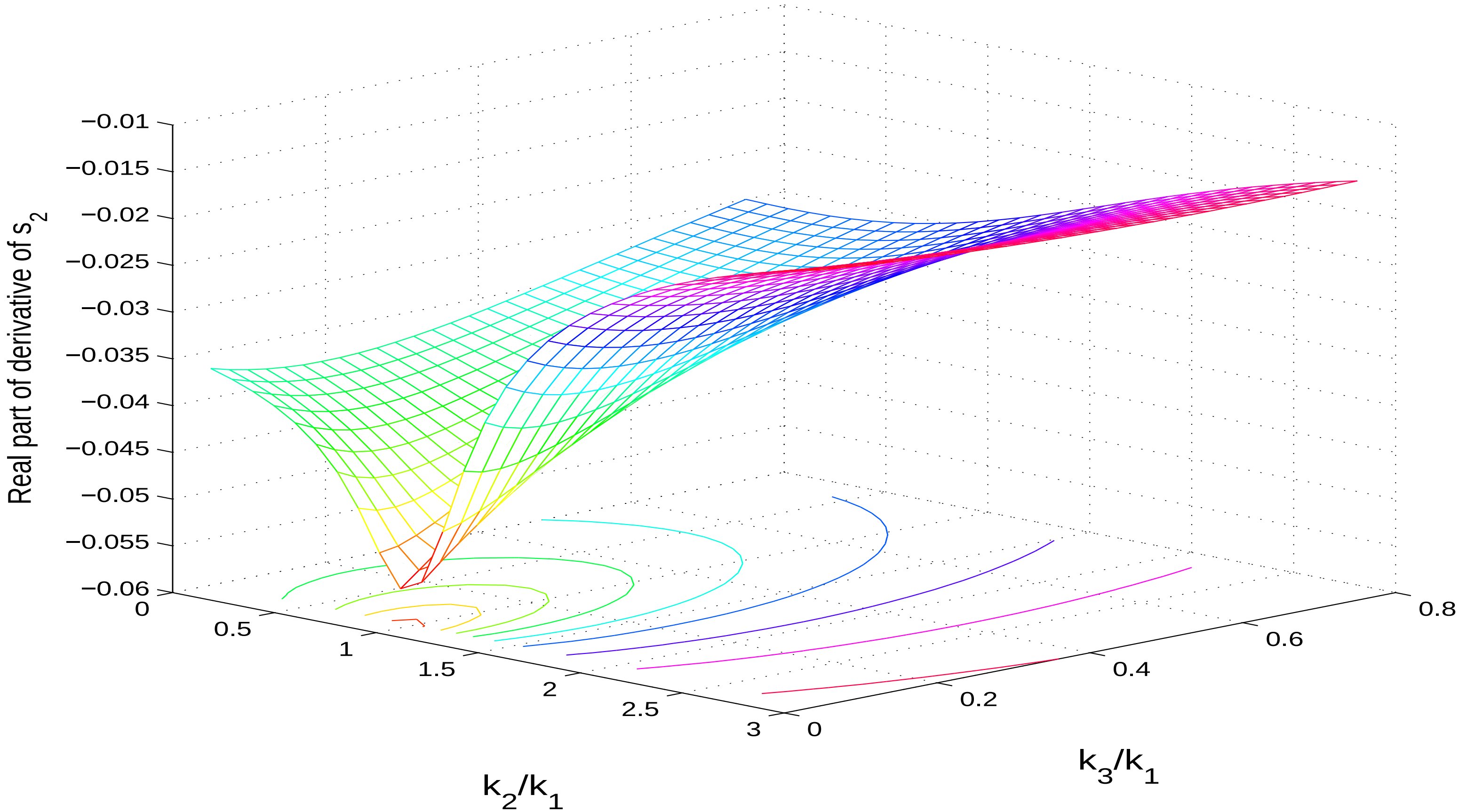
Numerical Results



Real part of the derivative of the first eigenvalue with respect to the relaxation parameter μ_1.

Numerical Results



Real part of the derivative of the second eigenvalue with respect to the relaxation parameter μ_1.

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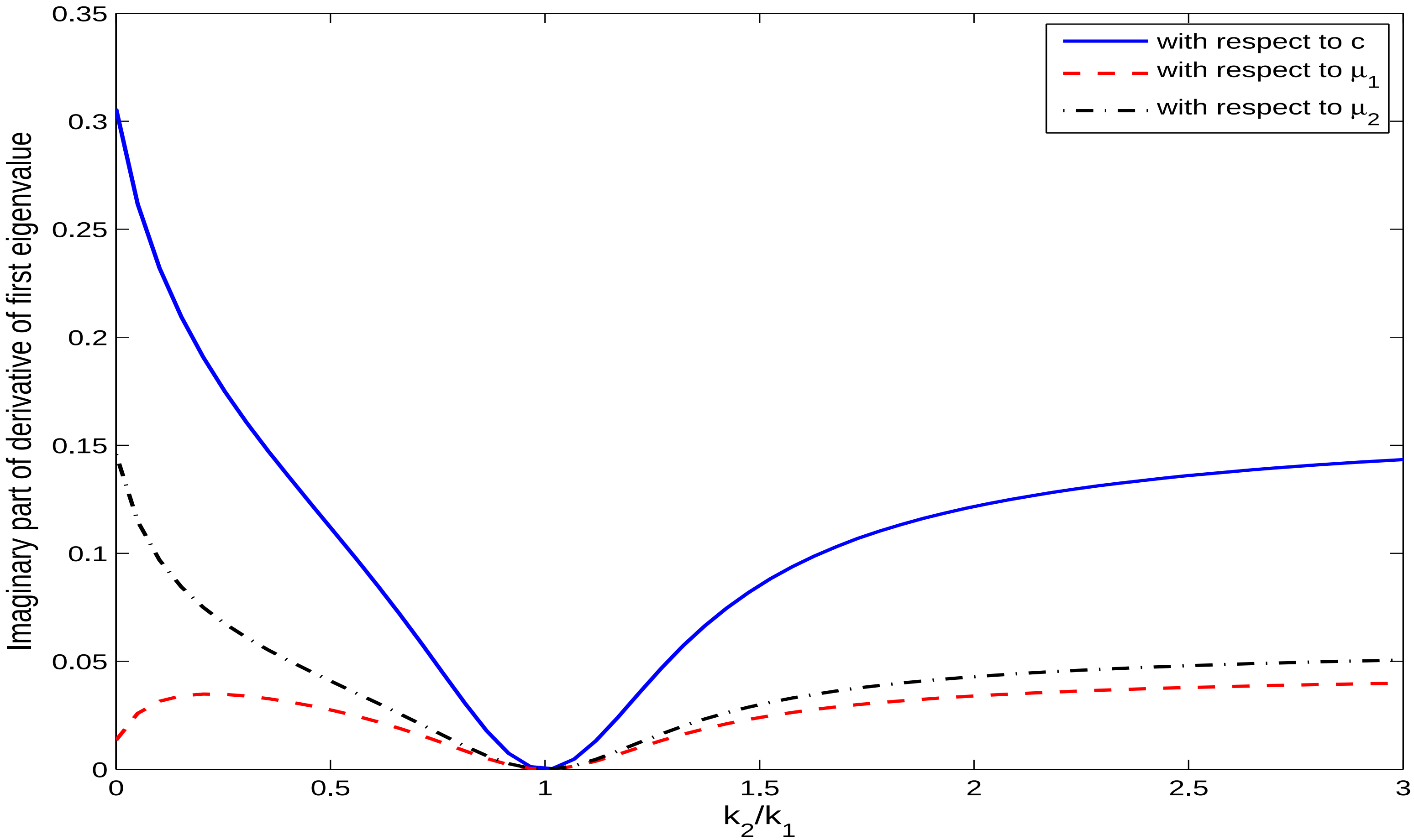
Numerical Results



Imaginary part of the derivative of the first eigenvalue with respect to the damping parameters c, μ_1 and μ_2.

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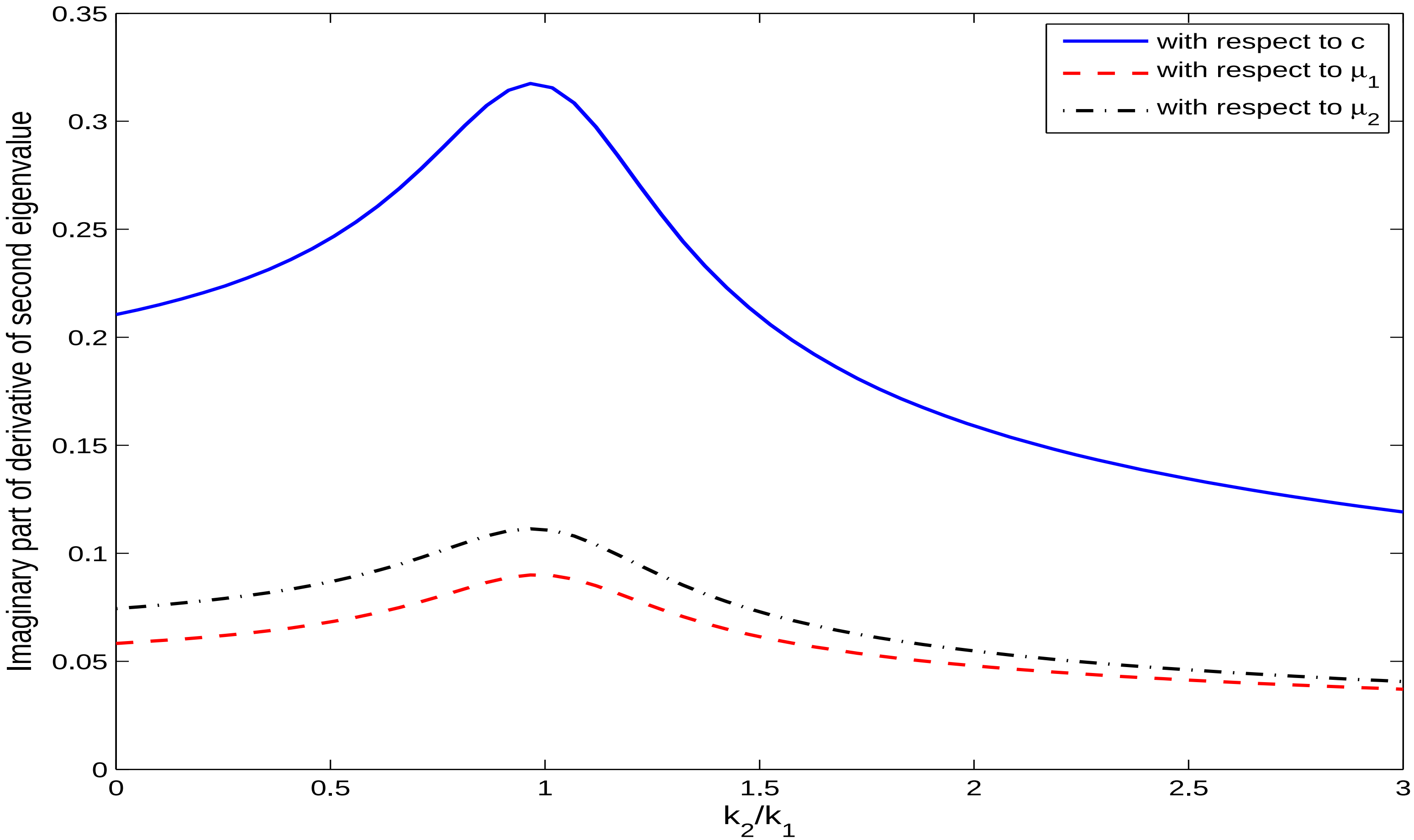
Numerical Results



Imaginary part of the derivative of the second eigenvalue with respect to the damping parameters c, μ_1 and μ_2.

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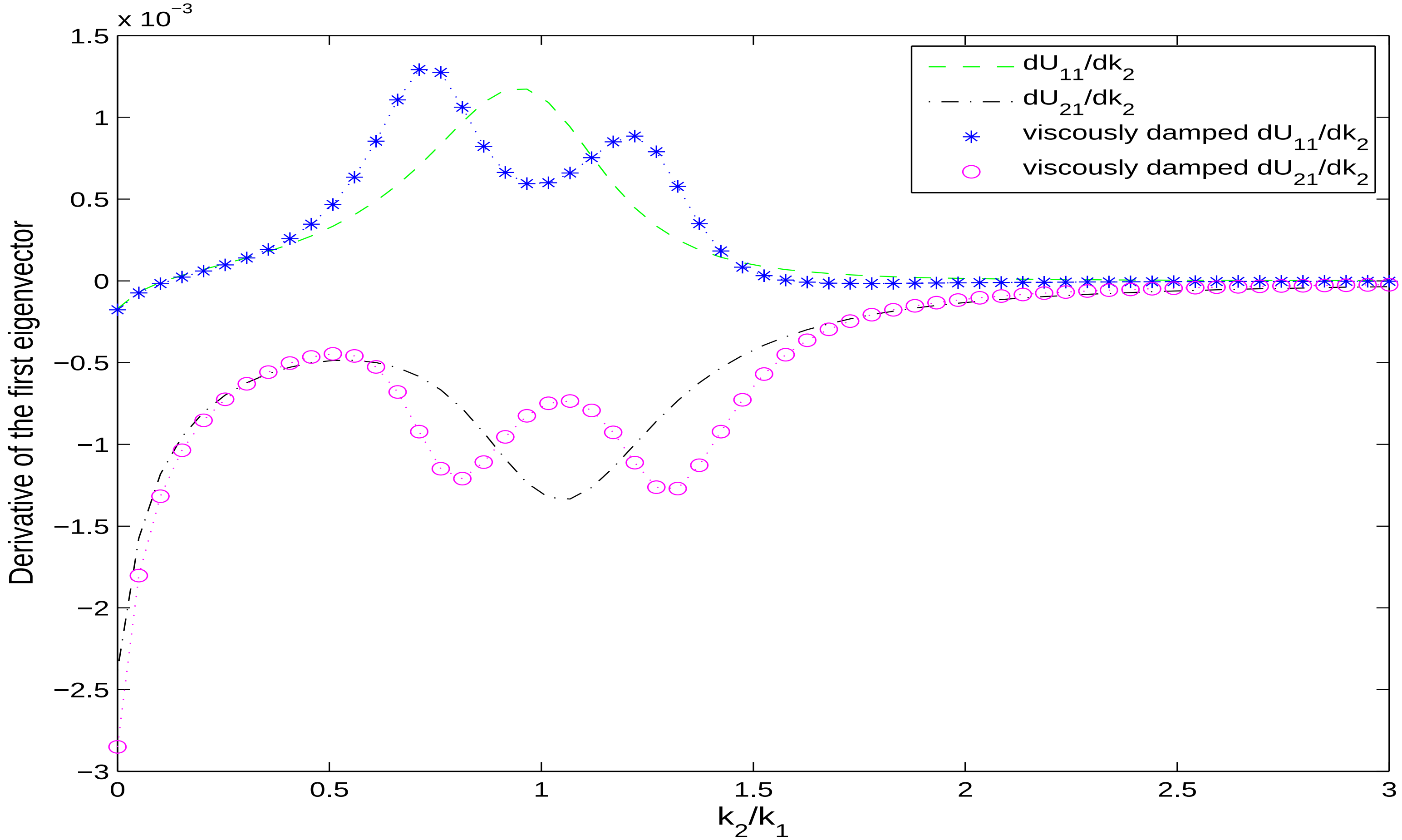
Numerical Results



Real part of the derivative of the first eigenvector with respect to k_2.

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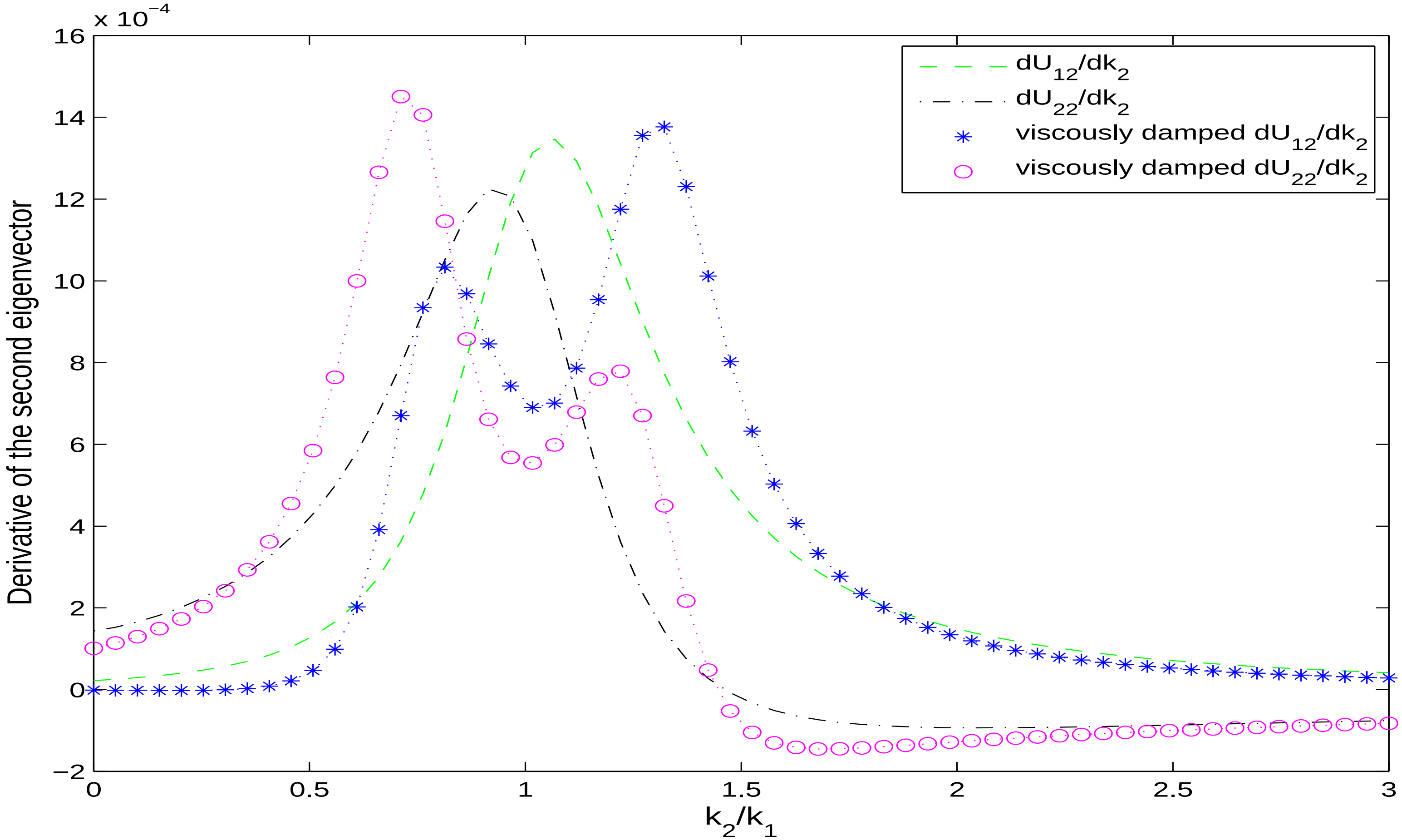
Numerical Results



Real part of the derivative of the second eigenvalue with respect to k_2.

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Conclusions - 1

- Multiple degree-of-freedom linear systems with viscoelastic damping is considered.
- The transfer function matrix of the system was derived in terms of the eigenvalues and eigenvectors of the second-order system.
- The eigensolutions are obtained using an approximate perturbation method (although an exact but computationally more expensive state-space method can be used).

Conclusions - 2

- Parametric sensitivity of the dynamic response was derived using two approaches - namely the direct approach and modal approach.
- The direct approach is easy to implement but computationally expensive as one has to differentiate the dynamic stiffness matrix at every frequency point.
- The modal approach utilizes derivatives of the complex eigensolutions and generally computationally more efficient.

Future Directions

- The results derived here extend the equivalent results available for viscously damped systems. The expressions can be useful to any problems which require parametric sensitivity information. Such problems include (a) probabilistic analysis, (b) optimal design, (c) model updating and system identification
- Future work will look into (a) sensitivity of transient dynamic response of viscoelastically damped systems in the time domain (this problem has relevance to vehicle noise reduction), and (b) joint sensitivity analysis of multiple parameters.

References

[1] Biot, M. A., "Variational principles in irreversible thermodynamics with application to viscoelasticity," *Physical Review*, Vol. 97, No. 6, 1955, pp. 1463–1469.

[2] Bagley, R. L. and Torvik, P. J., "Fractional calculus–a different approach to the analysis of viscoelastically damped structures," *AIAA Journal*, Vol. 21, No. 5, May 1983, pp. 741–748.

[3] Golla, D. F. and Hughes, P. C., "Dynamics of viscoelastic structures - a time domain finite element formulation," *Transactions of ASME, Journal of Applied Mechanics*, Vol. 52, December 1985, pp. 897–906.

[4] McTavish, D. J. and Hughes, P. C., "Modeling of linear viscoelastic space structures," *Transactions of ASME, Journal of Vibration and Acoustics*, Vol. 115, January 1993, pp. 103–110.

[5] Lesieutre, G. A. and Mingori, D. L., "Finite element modeling of frequency-dependent material properties using augmented thermodynamic fields," *AIAA Journal of Guidance, Control and Dynamics*, Vol. 13, 1990, pp. 1040–1050.

[6] Adhikari, S., *Energy Dissipation in Vibrating Structures*, Master's thesis, Cambridge University Engineering Department, Cambridge, UK, May 1998, First Year Report.

[7] Adhikari, S. and Woodhouse, J., "Identification of damping: part 1, viscous damping," *Journal of Sound and Vibration*, Vol. 243, No. 1, May 2001, pp. 43–61.