Wishart Random Matrices for Uncertainty Quantification of Complex Dynamical Systems

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Outline of the presentation

- Uncertainty Quantification (UQ) in structural dynamics
- Review of current approaches
- Wishart random matrices
  - Parameter selection
  - Computational method
  - Analytical method
- Experimental results
- Conclusions & future directions

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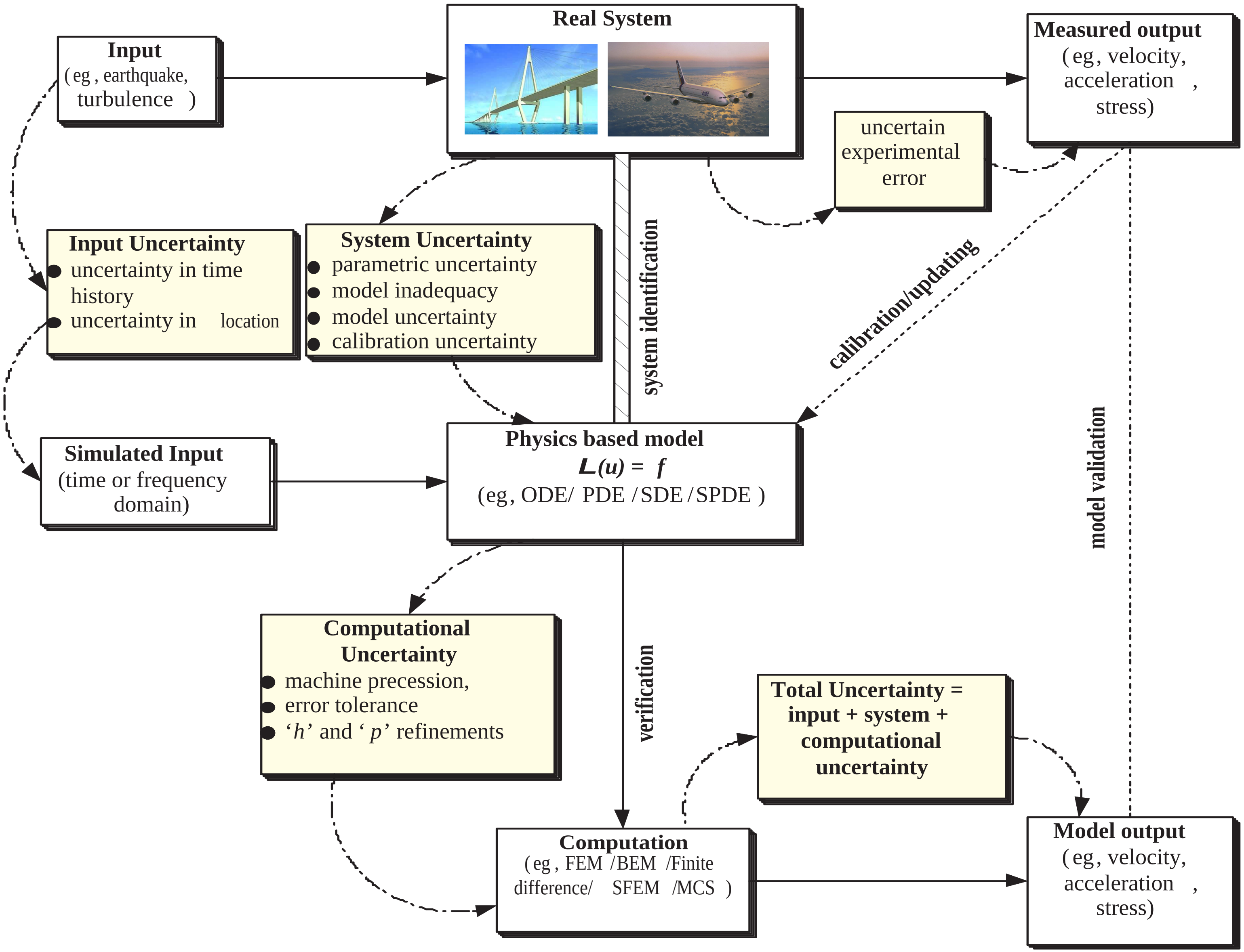
Complex aerospace system

Complex aerospace system can have millions of degrees of freedom and significant uncertainty in its numerical (Finite Element) model

The role of uncertainty in computational science



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Sources of uncertainty

(a) parametric uncertainty - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) model inadequacy - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(d) computational uncertainty - e.g, machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis, and
(e) model uncertainty - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.

Problem-types in computational sciences

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (deterministic)	Unknown	*Analysis (forward problem)*	FEM/BEM/Finite difference
Known (deterministic)	Incorrect (deterministic)	Known (deterministic)	*Updating/calibration*	Modal updating
Known (deterministic)	Unknown	Known (deterministic)	*System identification*	Kalman filter
Assumed (deterministic)	Unknown (deterministic)	Prescribed	*Design*	Design optimisation
Unknown	Partially Known	Known	*Structural Health Monitoring (SHM)*	SHM methods
Known (deterministic)	Known (deterministic)	Prescribed	*Control*	Modal control
Known (random)	Known (deterministic)	Unknown	*Random vibration*	Random vibration methods

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Problem-types in computational sciences

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (random)	Unknown	*Stochastic analysis (forward problem)*	SFEM/SEA/RMT
Known (random)	Incorrect (random)	Known (random)	*Probabilistic updating/calibration*	Bayesian calibration
Assumed (random/deterministic)	Unknown (random)	Prescribed (random)	*Probabilistic design*	RBOD
Known (random/deterministic)	Partially known (random)	Partially known (random)	*Joint state and parameter estimation*	Particle Kalman Filter/Ensemble Kalman Filter
Known (random/deterministic)	Known (random)	Known from experiment and model (random)	*Model validation*	Validation methods
Known (random/deterministic)	Known (random)	Known from different computations (random)	*Model verification*	verification methods

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Structural dynamics

The equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \qquad (1)$$

- Due to the presence of (parametric/nonparametric or both) uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objectives in the 'forward problem' are:
  - to quantify uncertainties in the system matrices
  - to predict the variability in the response vector $\mathbf{q}$
- Probabilistic solution of this problem is expected to have more credibility compared to a deterministic solution

UQ approaches: challenges

The main difficulties are due to:

- the computational time can be prohibitively high compared to a deterministic analysis for real problems,
- the volume of input data can be unrealistic to obtain for a credible probabilistic analysis,
- the predictive accuracy can be poor if considerable resources are not spend on the previous two items, and
- as the state-of-the art methodology stands now (such as the Stochastic Finite Element Method), only very few highly trained professionals (such as those with PhDs) can even attempt to apply the complex concepts (e.g., random fields) and methodologies to real-life problems.

Main objectives

Our work is aimed at developing methodologies [the 10-10-10 challenge] with the ambition that they should:

- not take more than 10 times the computational time required for the corresponding deterministic approach;
- result a predictive accuracy within 10% of direct Monte Carlo Simulation (MCS);
- use no more than 10 times of input data needed for the corresponding deterministic approach; and
- enable ‘normal’ engineering graduates to perform probabilistic structural dynamic analyses with a reasonable amount of training.

Current UQ approaches - 1

Two different approaches are currently available

- Parametric approaches : Such as the Stochastic Finite Element Method (SFEM):
  - aim to characterize parametric uncertainty (type 'a')
  - assumes that stochastic fields describing parametric uncertainties are known in details
  - suitable for low-frequency dynamic applications

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Current UQ approaches - 2

- Nonparametric approaches : Such as the Statistical Energy Analysis (SEA) and Wishart random matrix theory:
  - aim to characterize nonparametric uncertainty (types ‘b’ - ‘e’)
  - does not consider parametric uncertainties in details
  - suitable for high/mid-frequency dynamic applications
  - extensive works over the past decade $\longrightarrow$ general purpose commercial software is now available

Random Matrix Method (RMM)

- The objective: To have an unified method which will work across the frequency range.
- The methodology:
  - Derive the matrix variate probability density functions of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$
  - Propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \rightarrow \mathbb{R}$.

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Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}$, where $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$ and $\boldsymbol{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}\left(\mathbf{X}\right) = (2\pi)^{-np/2} \det\left\{\boldsymbol{\Sigma}\right\}^{-p/2} \det\left\{\boldsymbol{\Psi}\right\}^{-n/2} \operatorname{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (2)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}\left(\mathbf{M}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}\right)$.

Wishart matrix

A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}\left(\mathbf{S}\right) = \left\{2^{\frac{1}{2}np}\,\Gamma_n\left(\frac{1}{2}p\right)\det\left\{\mathbf{\Sigma}\right\}^{\frac{1}{2}p}\right\}^{-1}\left|\mathbf{S}\right|^{\frac{1}{2}(p-n-1)}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S}\right\} \tag{3}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$.

Note: If $p = n + 1$, then the matrix is non-negative definite.

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Matrix variate Gamma distribution

A $n \times n$ symmetric positive definite matrix random $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a) \det\{\mathbf{\Psi}\}^{-a}\right\}^{-1} \det\{\mathbf{W}\}^{a-\frac{1}{2}(n+1)} \operatorname{etr}\{-\mathbf{\Psi}\mathbf{W}\}; \quad \Re(a \tag{4}$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \text{ for } \Re(a) > (n-1)/2 \tag{5}$$

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Wishart random matrix approach

- The probability density function of the mass ($\mathbf{M}$), damping ($\mathbf{C}$) and stiffness ($\mathbf{K}$) matrices should be such that they are symmetric and non-negative matrices.
- Wishart random matrix (a non-Gaussian matrix) is the simplest mathematical model which can satisfy these two criteria: $[\mathbf{M}, \mathbf{C}, \mathbf{K}] \equiv \mathbf{G} \sim W_n(p, \mathbf{\Sigma})$.
- Suppose we 'know' (e.g, by measurement or stochastic modeling) the mean ($\mathbf{G}_0$) and the (normalized) standard deviation (σ_G) of the system matrices:

$$\sigma_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_\mathrm{F}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_\mathrm{F}^2}. \tag{6}$$

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Wishart parameter selection - 1

The parameters p and $\mathbf{\Sigma}$ can be obtained based on what criteria we select. We investigate four possible choices.

1. Criteria 1: $\mathrm{E}\left[\mathbf{G}\right] = \mathbf{G}_0$ and $\sigma_G = \widetilde{\sigma}_G$ which results

$$p = n + 1 + \theta \quad \text{and} \quad \mathbf{\Sigma} = \mathbf{G}_0/p \tag{7}$$

   where $\theta = (1+\beta)/\widetilde{\sigma}_G^2 - (n+1)$ and $\beta = \left\{\mathrm{Trace}\left(\mathbf{G}_0\right)\right\}^2 / \mathrm{Trace}\left({\mathbf{G}_0}^2\right)$.

2. Criteria 2: $\left\|\mathbf{G}_0 - \mathrm{E}\left[\mathbf{G}\right]\right\|_{\mathrm{F}}$ and $\left\|{\mathbf{G}_0}^{-1} - \mathrm{E}\left[\mathbf{G}^{-1}\right]\right\|_{\mathrm{F}}$ are minimum and $\sigma_G = \widetilde{\sigma}_G$. This results:

$$p = n + 1 + \theta \quad \text{and} \quad \mathbf{\Sigma} = \mathbf{G}_0/\alpha \tag{8}$$

   where $\alpha = \sqrt{\theta(n+1+\theta)}$.

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Wishart parameter selection - 2

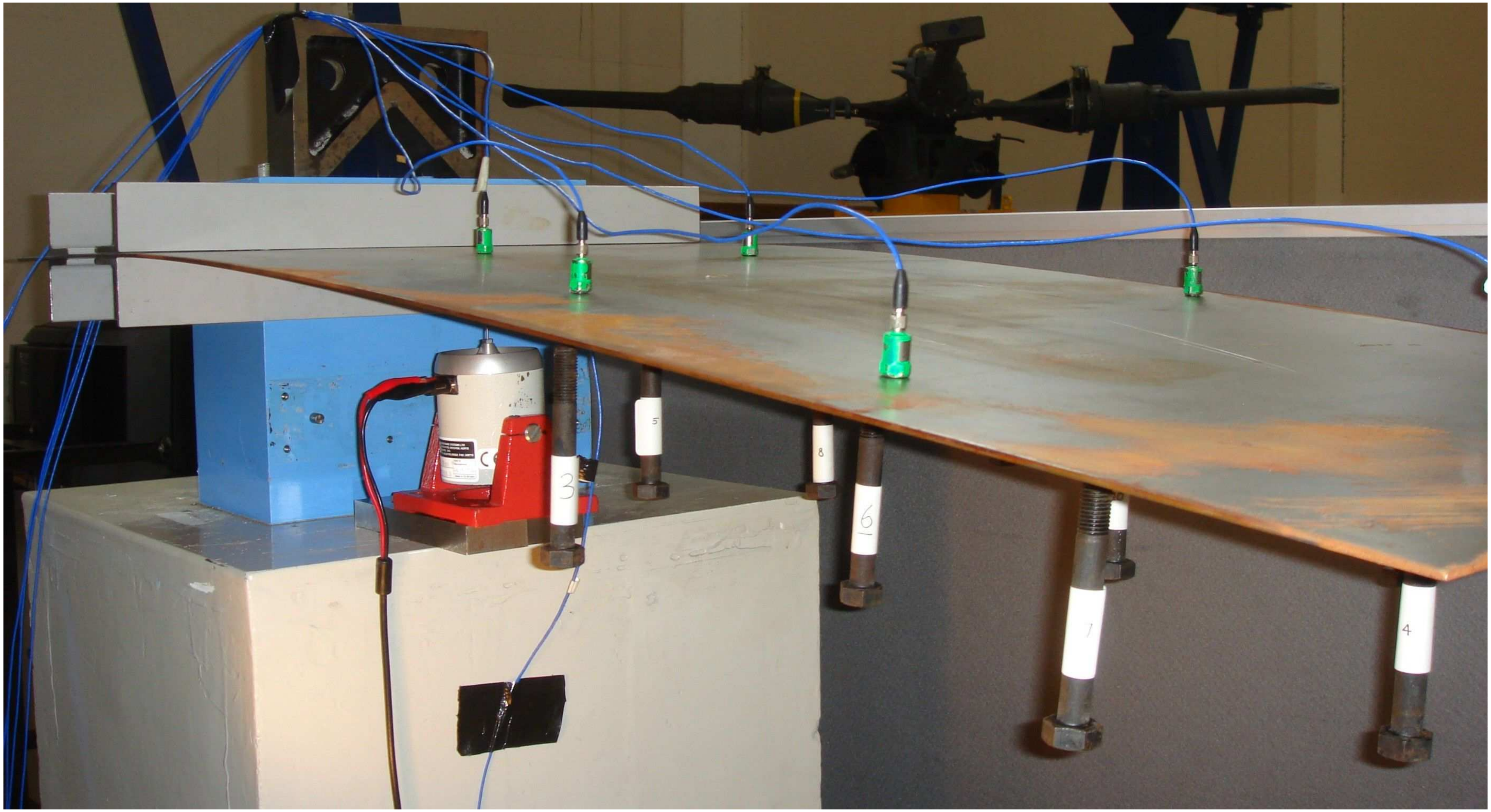
1. Criteria 3: $\mathrm{E}\left[\mathbf{G}^{-1}\right] = \mathbf{G}_0{}^{-1}$ and $\sigma_G = \widetilde{\sigma}_G$. This results:

$$p = n + 1 + \theta \quad \text{and} \quad \boldsymbol{\Sigma} = \mathbf{G}_0/\theta \tag{9}$$

2. Criteria 4: The mean of the eigenvalues of the distribution is same as the 'measured' eigenvalues of the mean matrix and the (normalized) standard deviation is same as the measured standard deviation:

$$\mathrm{E}\left[\mathbf{M}^{-1}\right] = \mathbf{M}_0{}^{-1},\ \mathrm{E}\left[\mathbf{K}\right] = \mathbf{K}_0,\ \sigma_M = \widetilde{\sigma}_M \quad \text{and} \quad \sigma_K = \widetilde{\sigma}_K. \tag{10}$$

A cantilever plate: front view



The test rig for the cantilever plate; front view.

A cantilever plate: side view

The test rig for the cantilever plate; side view.

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Physical properties

Plate Properties	Numerical values
Length (L_x)	998 mm
Width (L_y)	530 mm
Thickness (t_h)	3.0 mm
Mass density (ρ)	7860 kg/m^3
Young's modulus (E)	2.0×10^5 MPa
Poisson's ratio (μ)	0.3
Total weight	12.47 kg

Material and geometric properties of the cantilever plate considered for the experiment. The data presented here are available from http://engweb.swan.ac.uk/∼adhikaris/uq/.

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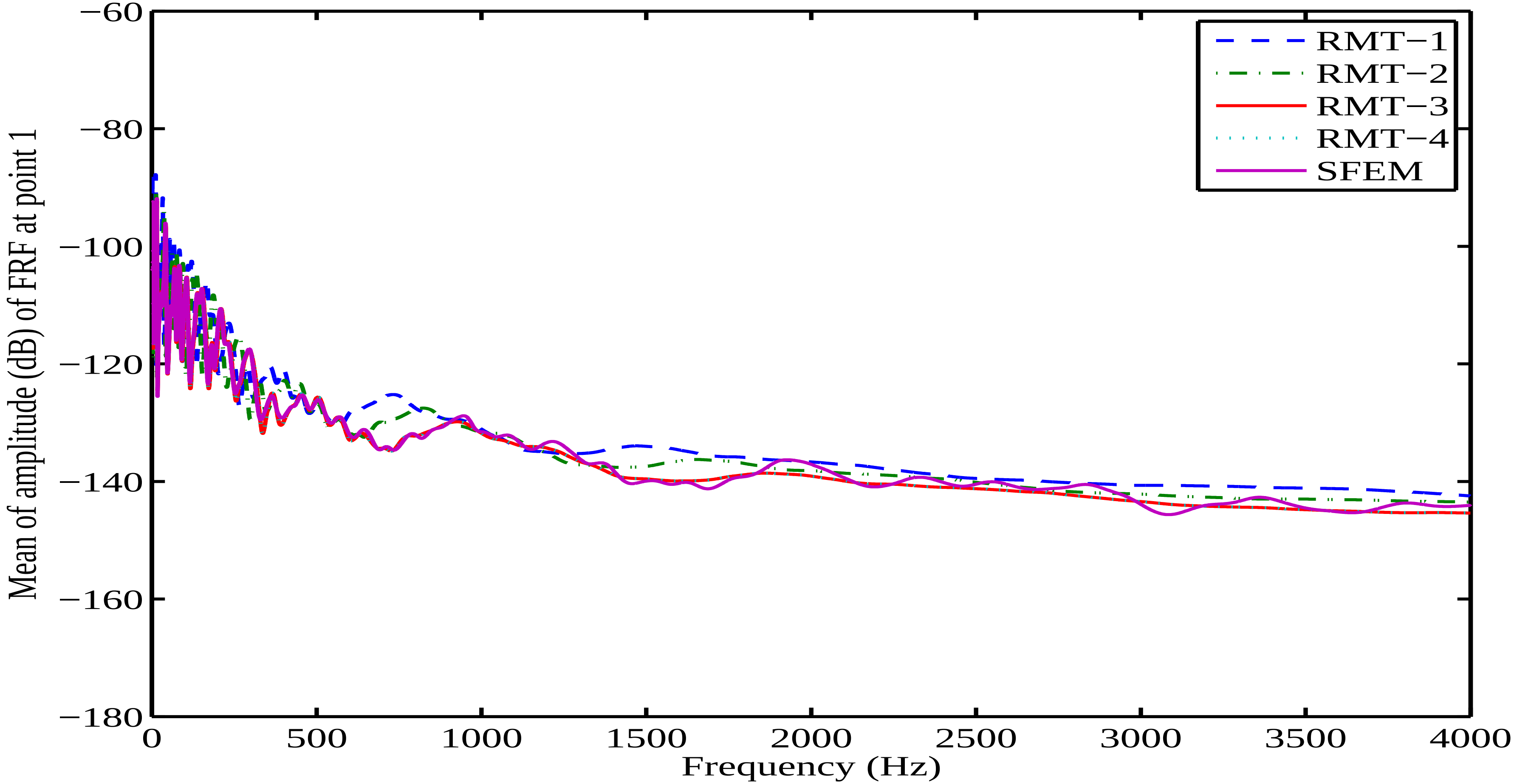
Mean of cross-FRF



Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

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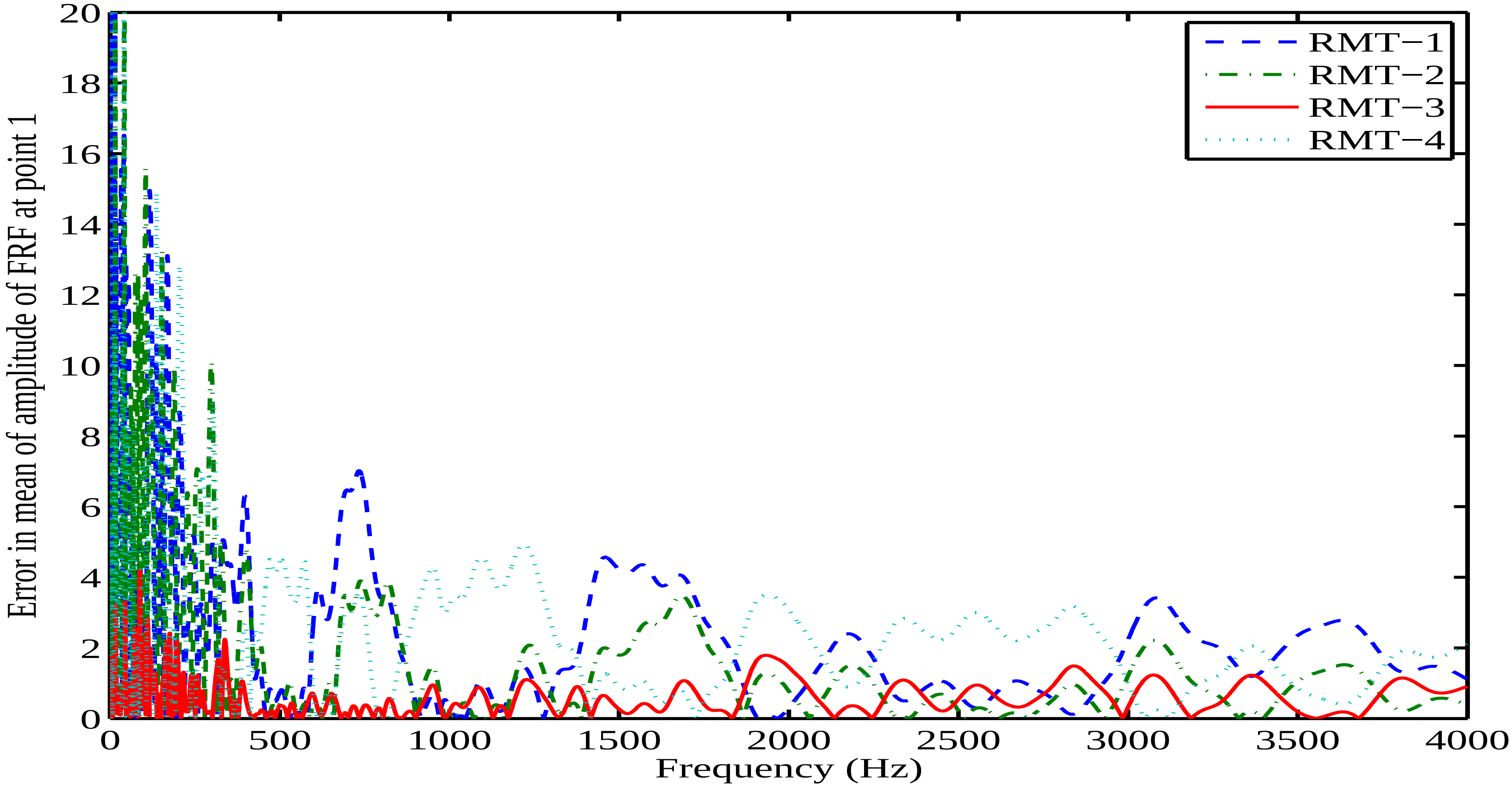
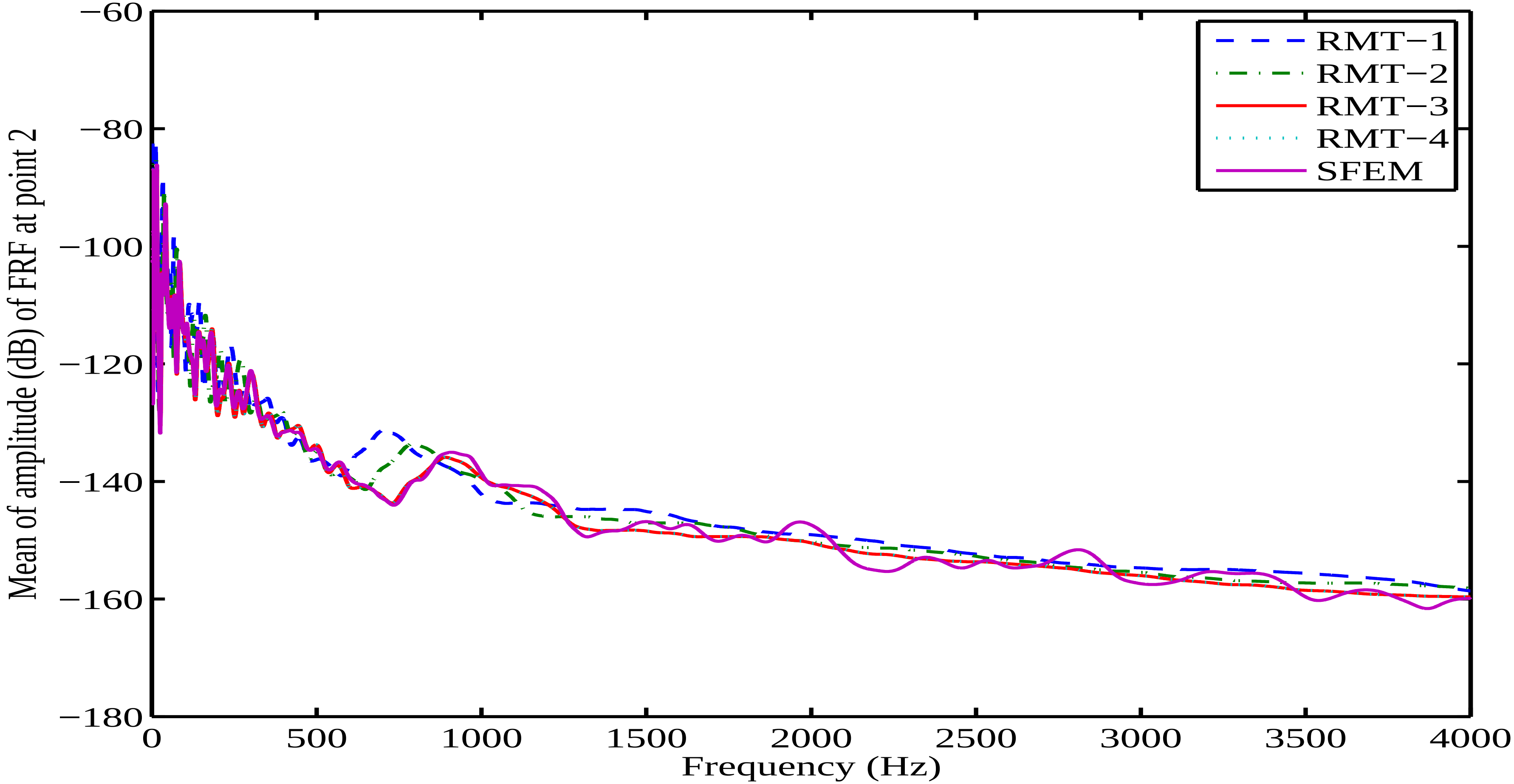
Error in the mean of cross-FRF



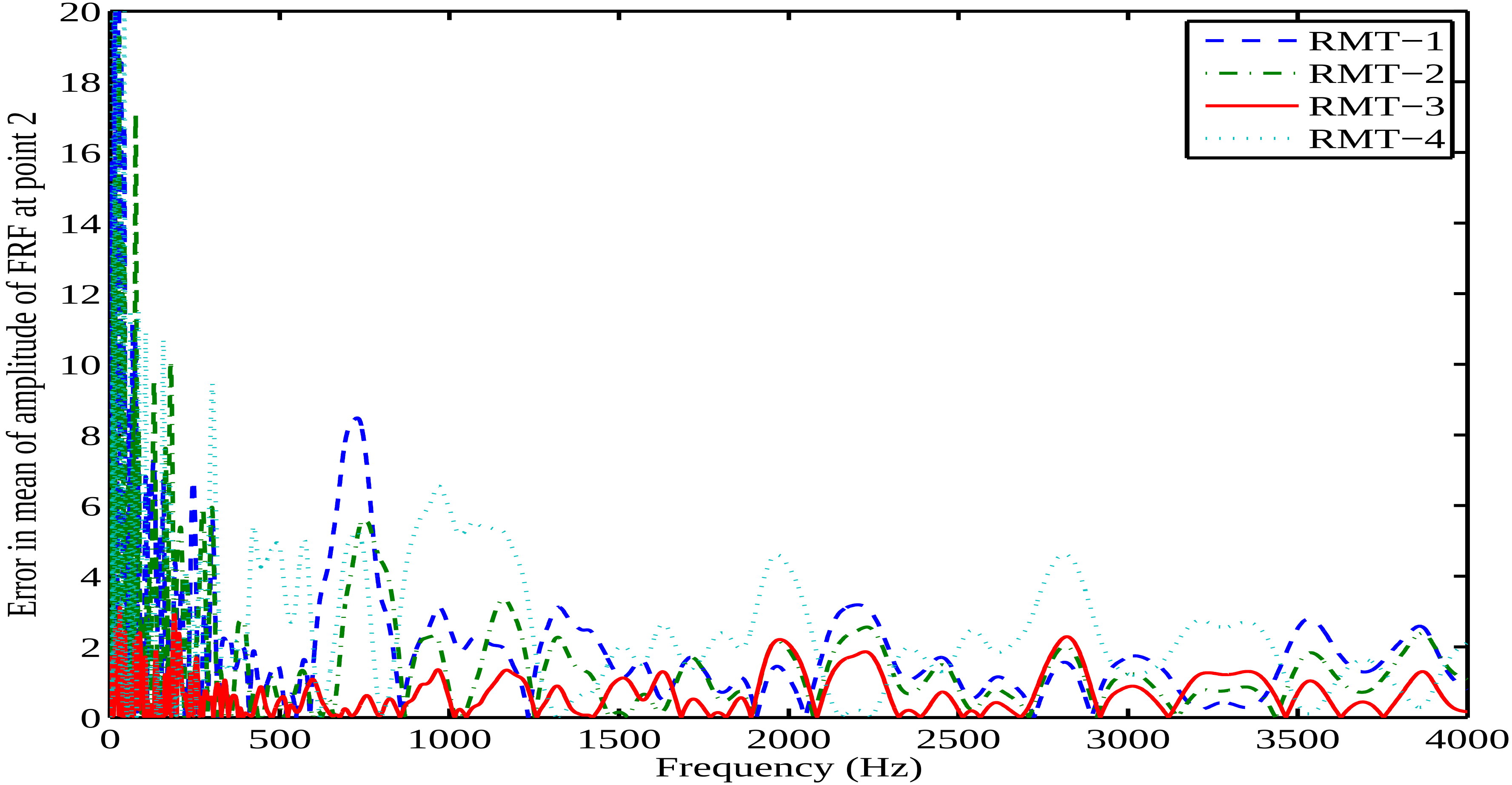
Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Mean of Driving-point-FRF



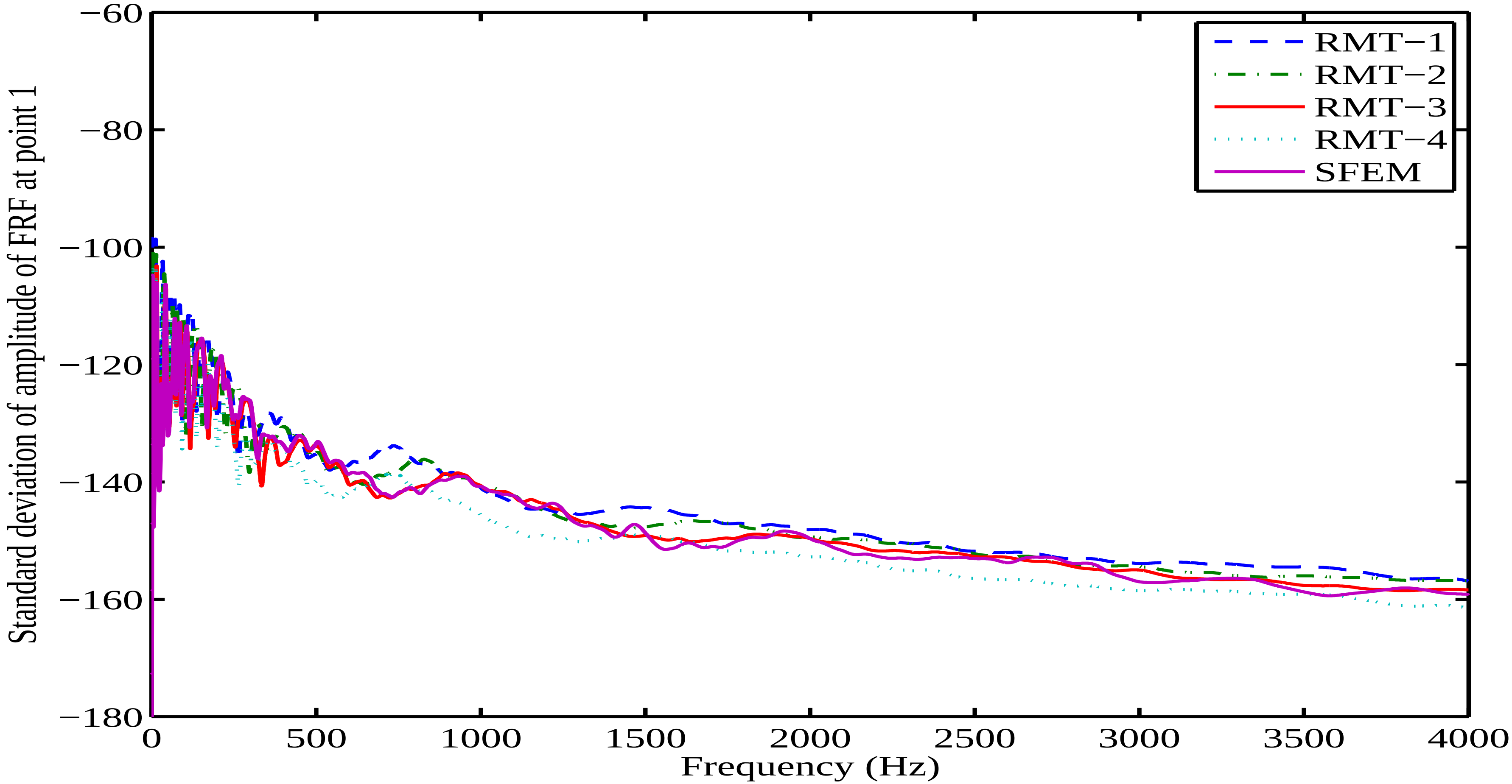
Mean of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Error in the Mean of Driving-point-FRF



Error in the mean of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Standard Deviation of Cross-FRF



Standard deviation of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

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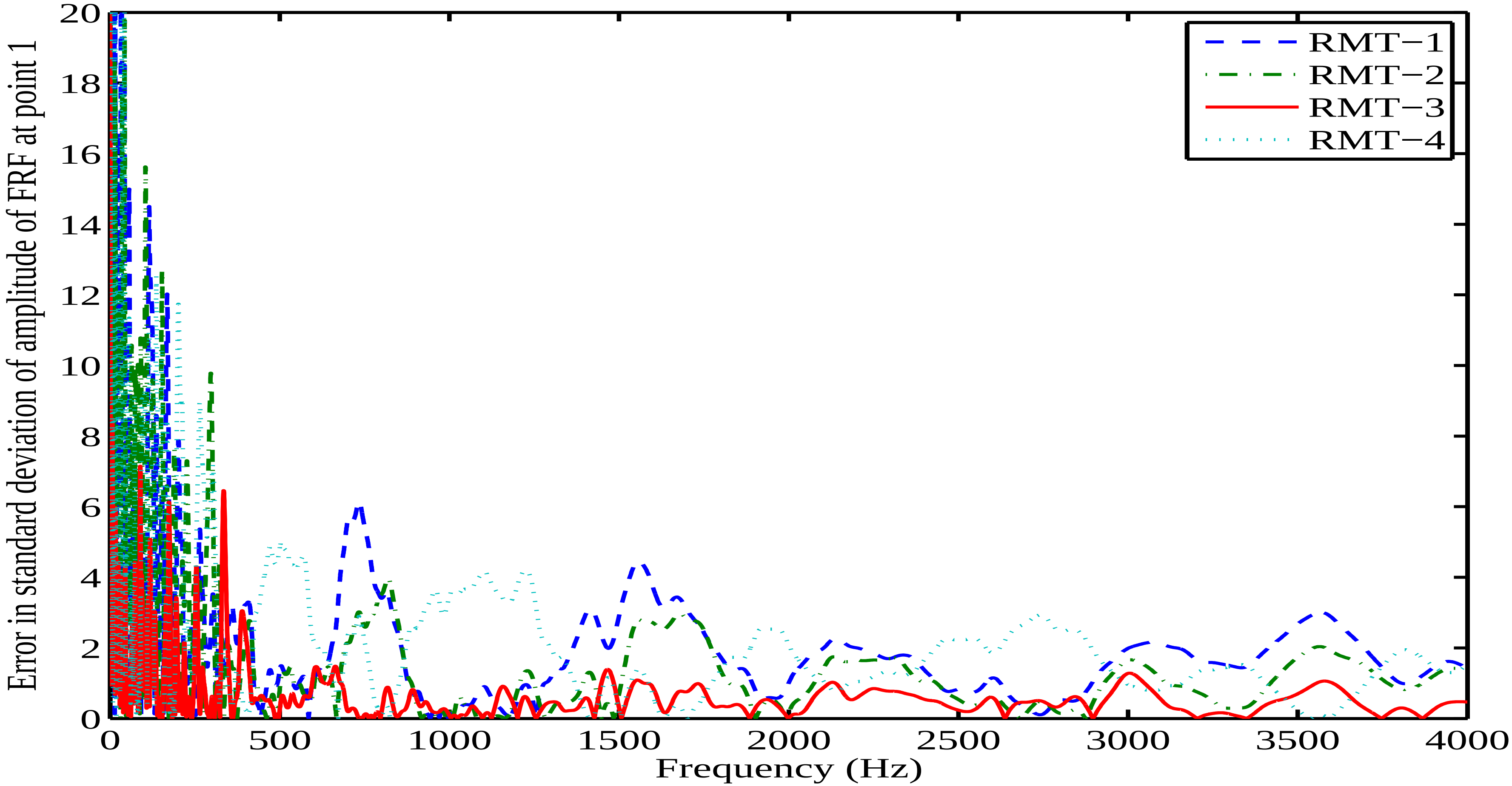
Error in the Standard Deviation of Cross-FRF



Error in the standard deviation of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

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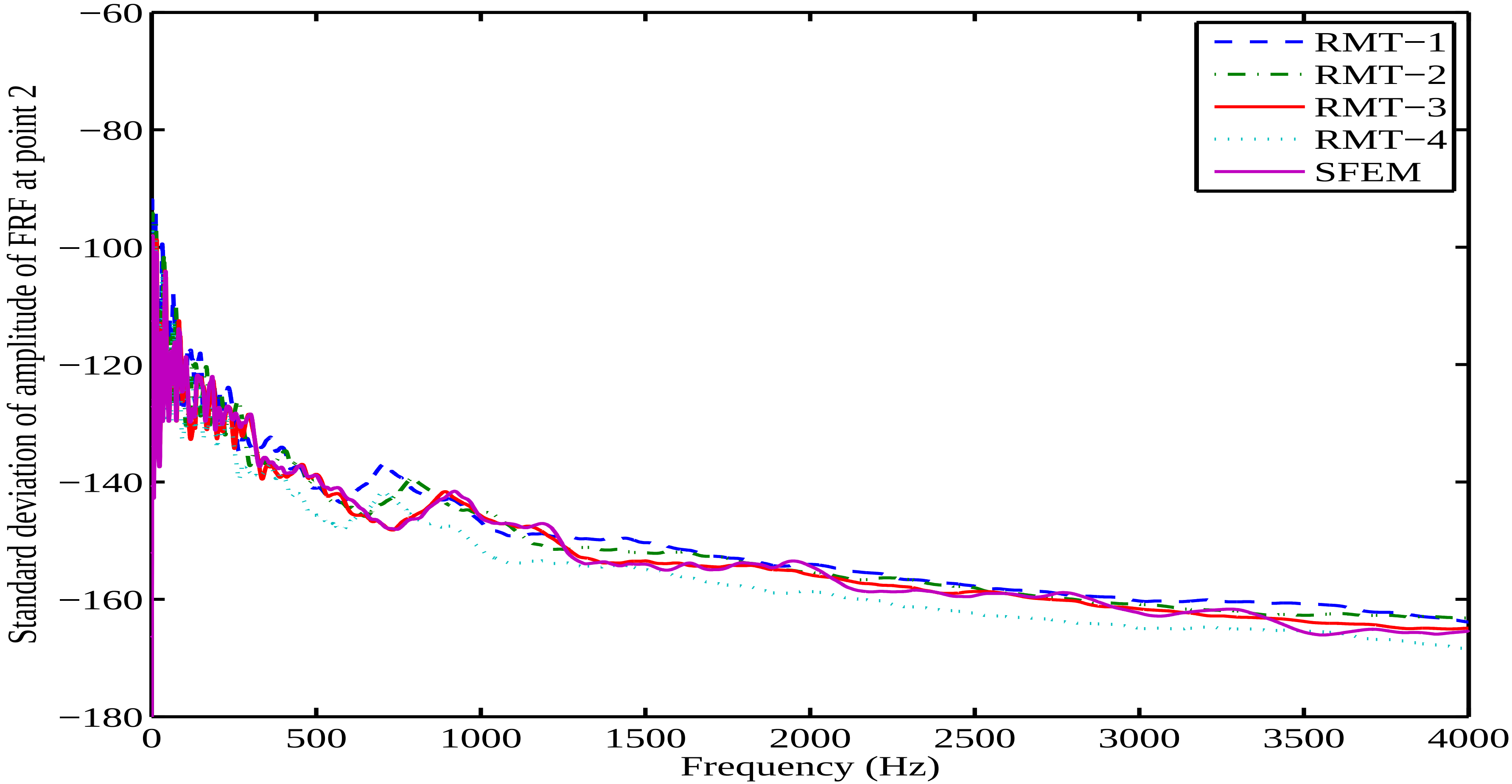
Standard deviation of driving-point-FRF



Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

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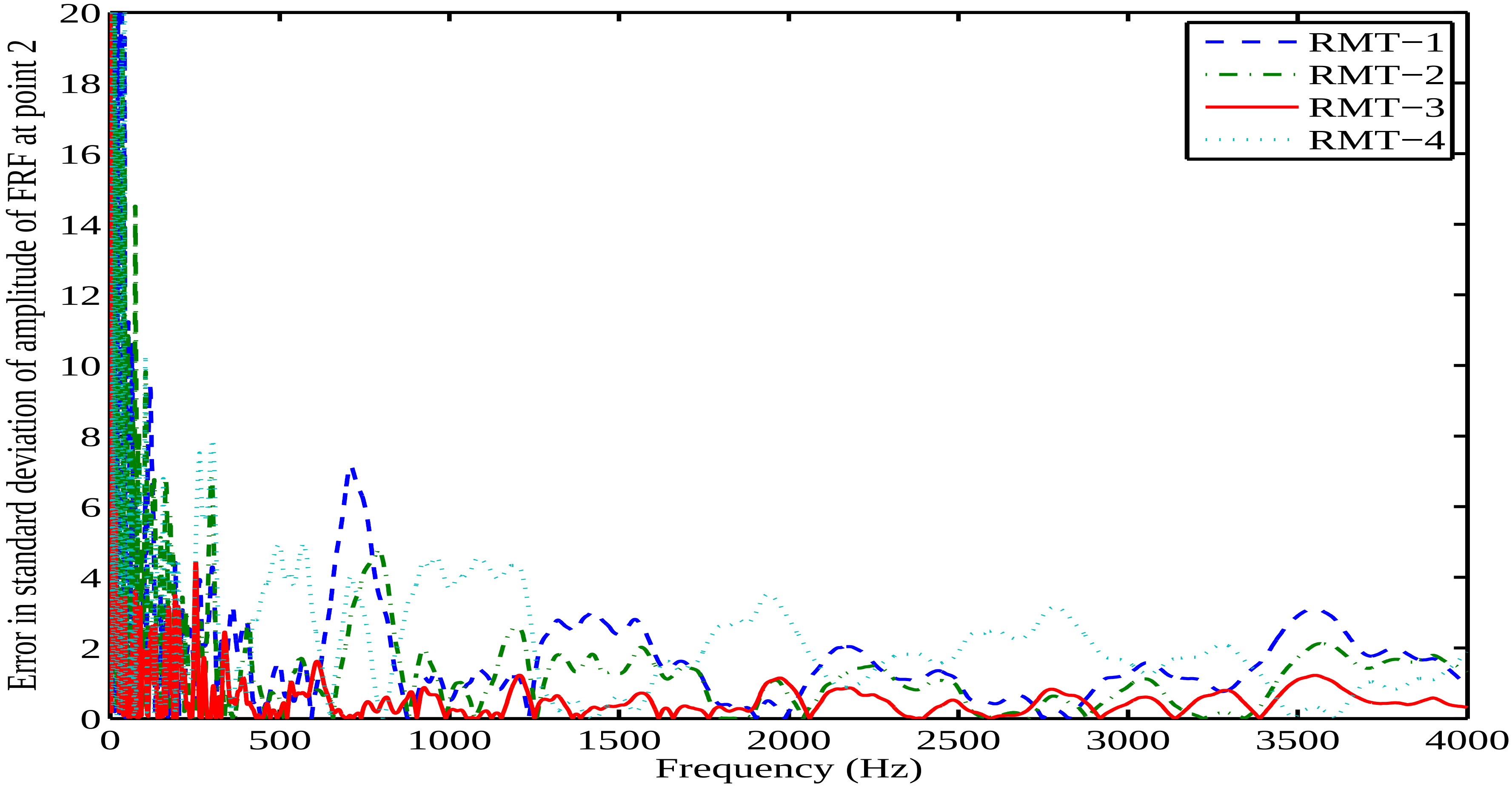
Error in the standard deviation of driving-point-FRF



Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

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Main observations

- Error in the low frequency region is higher than that in the higher frequencies
- In the high frequency region all methods are similar
- Overall, parameter selection 3 performs best; especially in the low frequency region.

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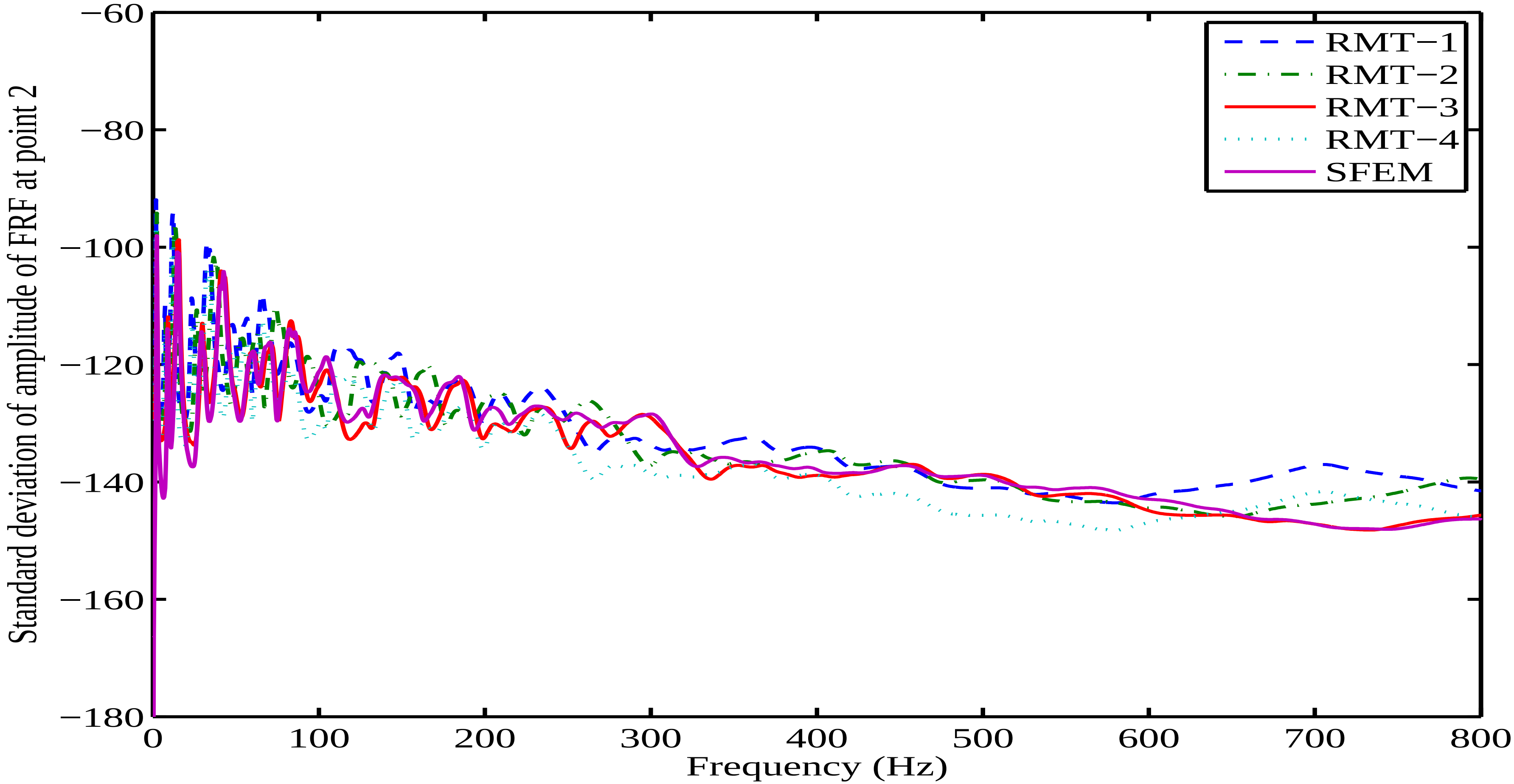
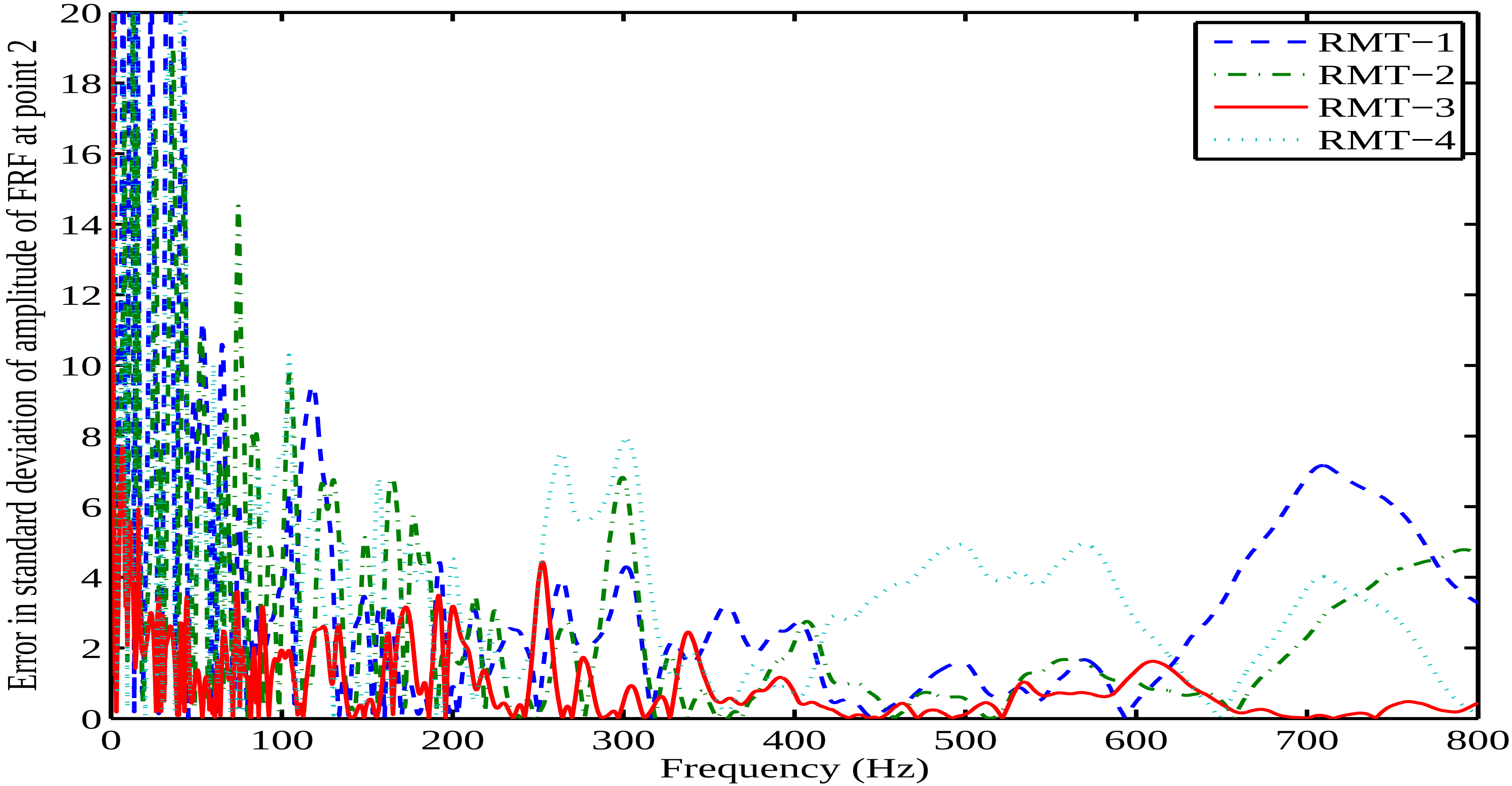
Standard deviation: low frequency



Standard deviation of the amplitude of the response of the driving-point-FRF of the plate in the low frequency region, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

Error in the standard deviation: low frequency



Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate in the low frequency region, $n = 1200$, $\sigma_M = 0.1326$ and $\sigma_K = 0.3335$.

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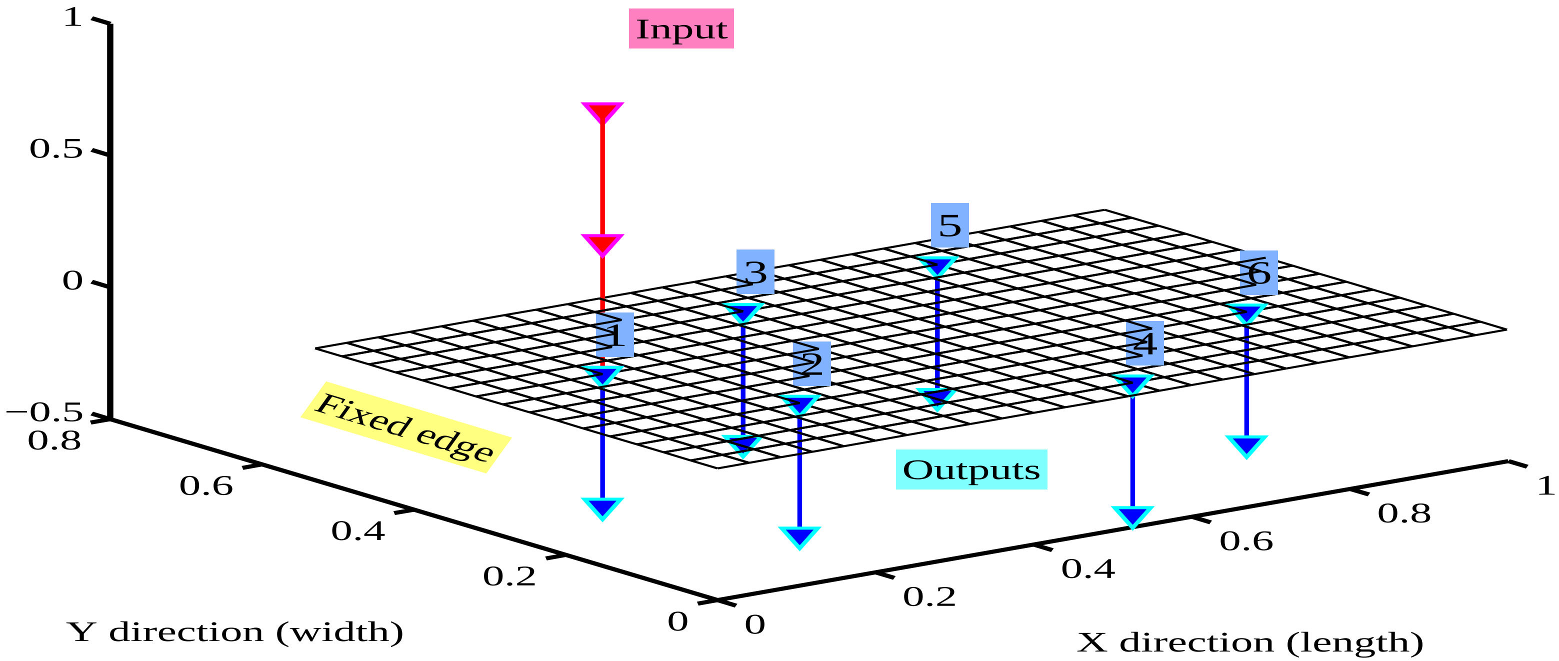
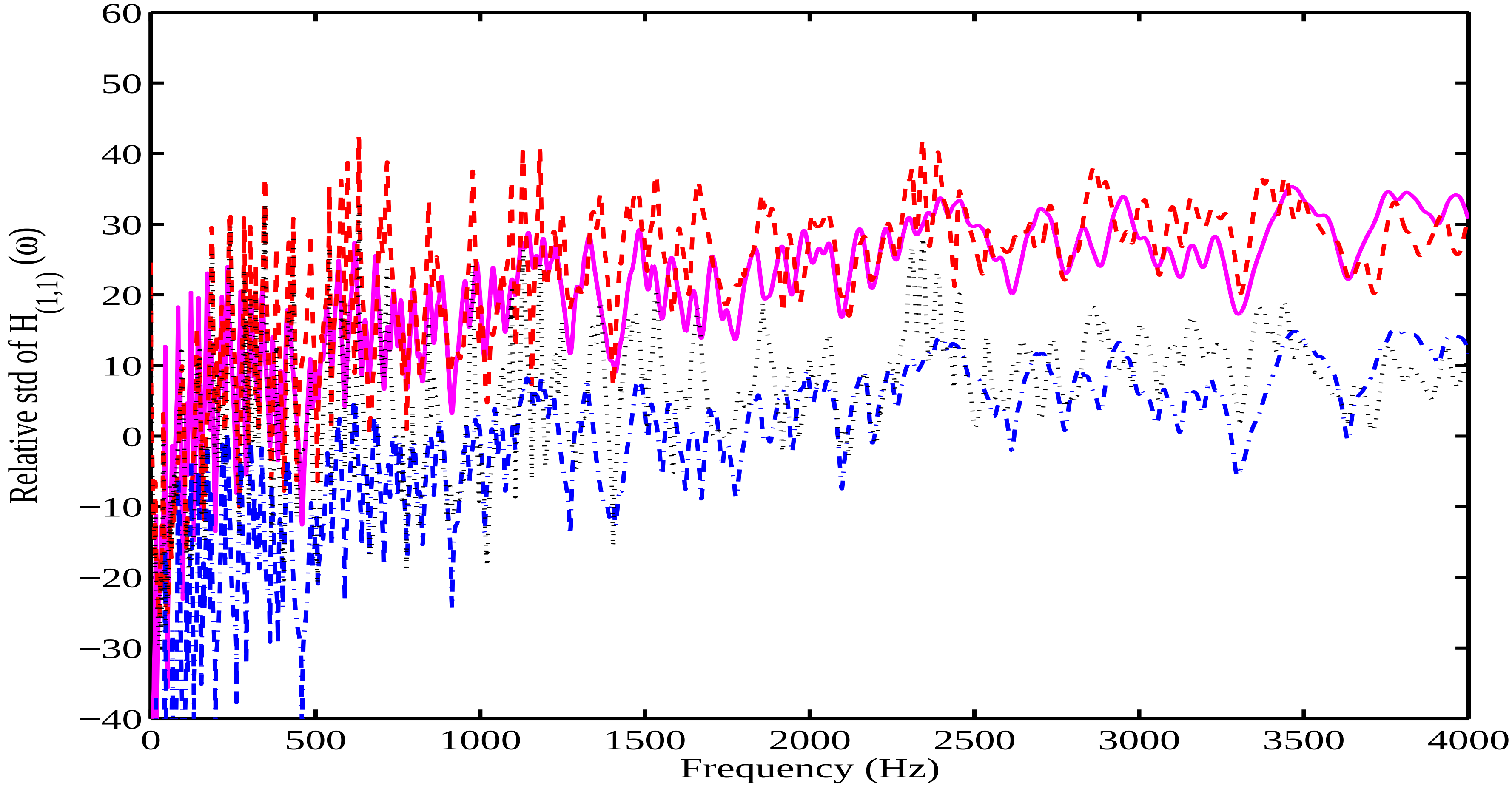
Finite element & Wishart matrix model



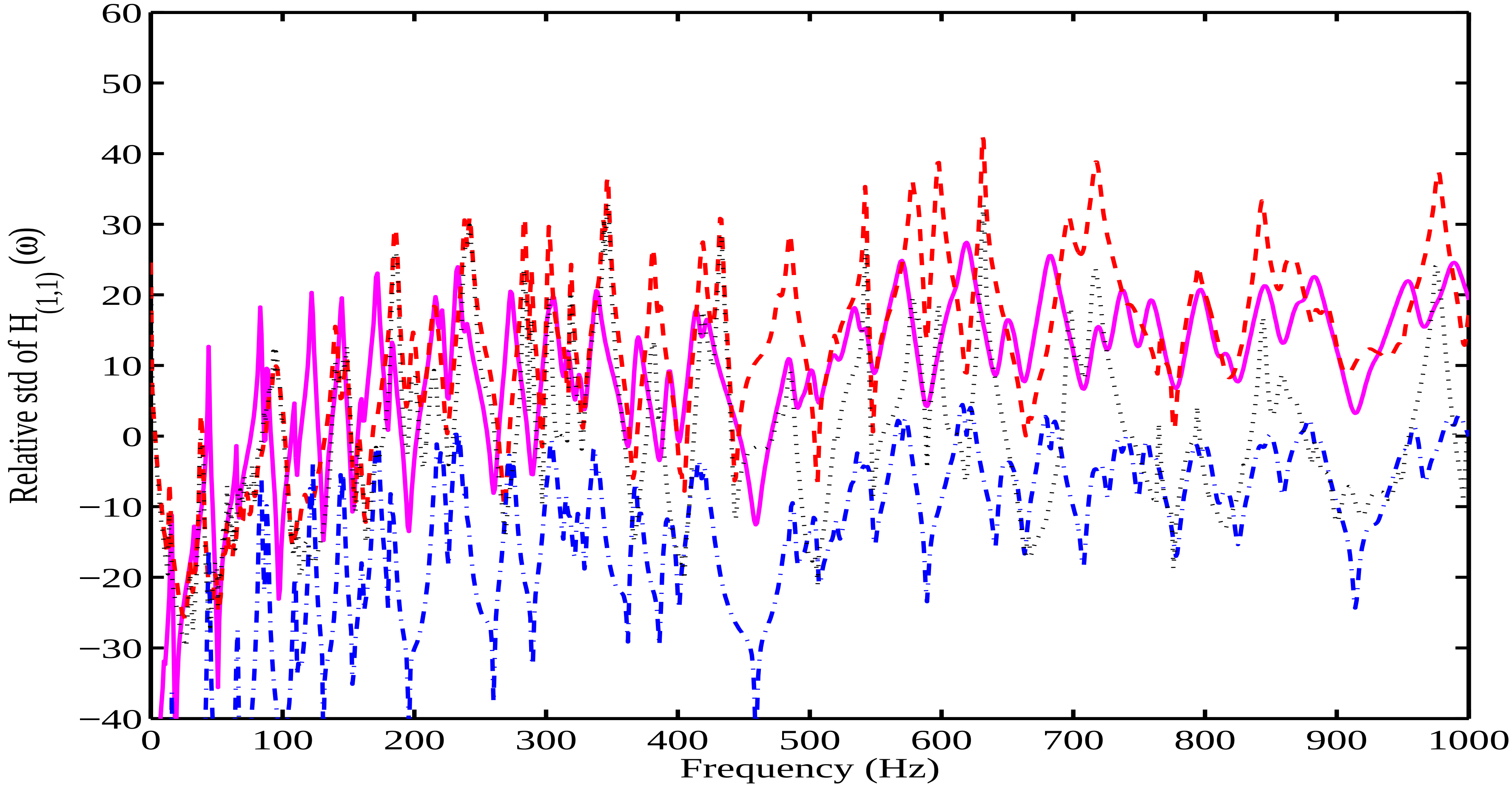
Baseline Model: 25×15 elements, 416 nodes, 1200 degrees-of-freedom. Input node number: 481, Output node numbers: 481, 877, 268, 1135, 211 and 844, 0.7% modal damping is assumed for all modes.

Comparison of driving-point-FRF



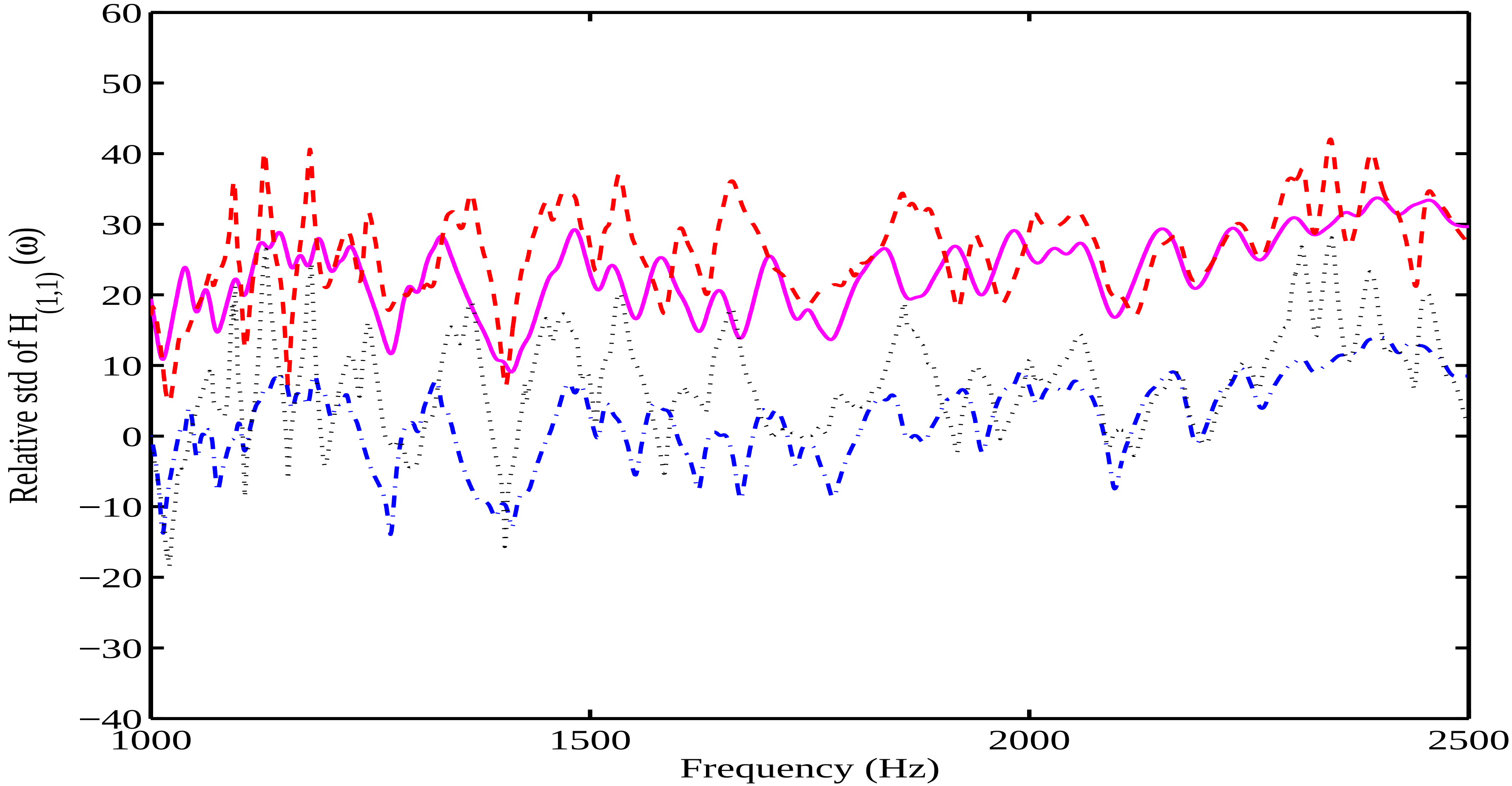
Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

Comparison of driving-point-FRF: Low Freq



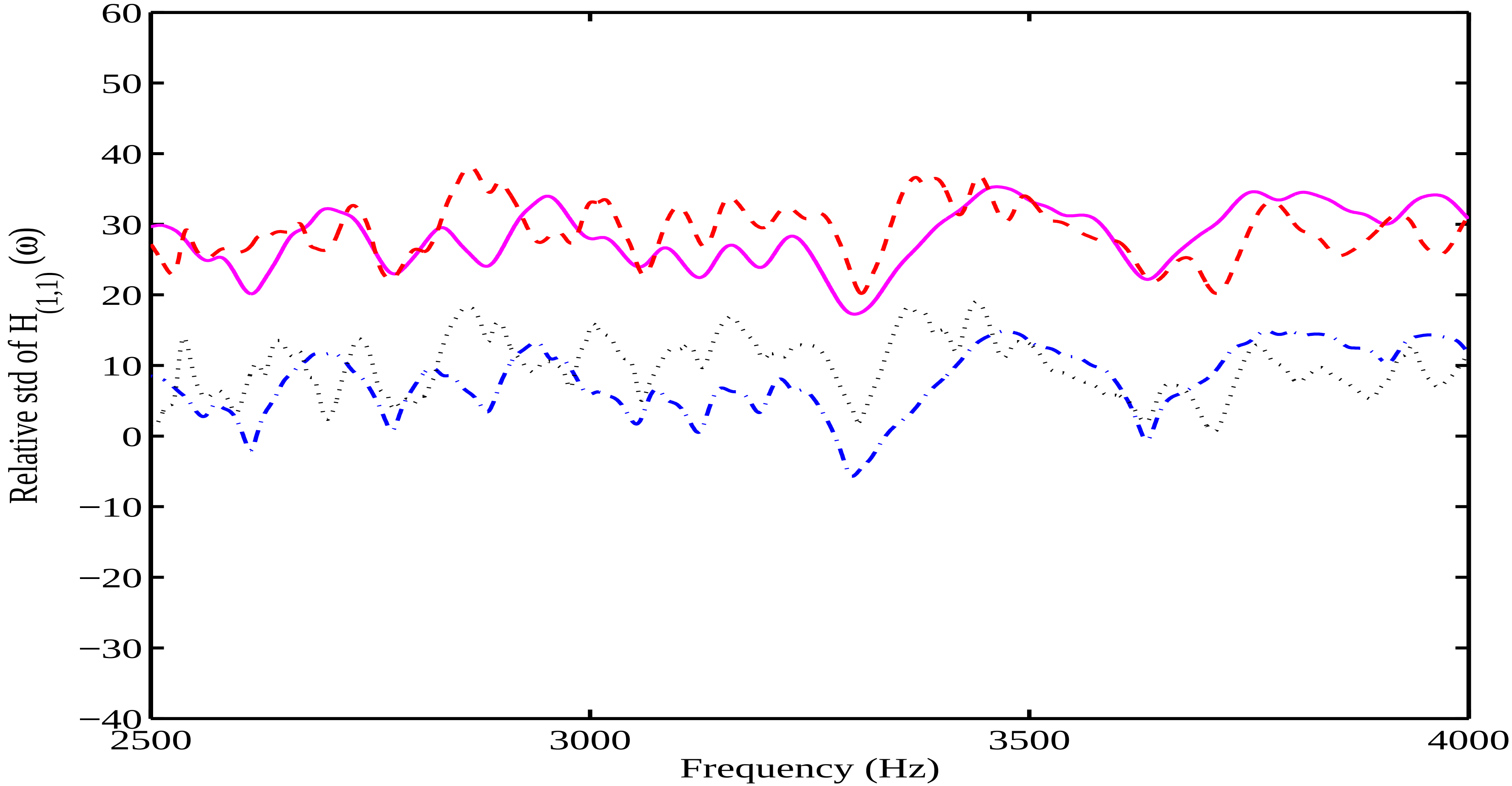
Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

Comparison of driving-point-FRF: Mid Freq



Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

Comparison of driving-point-FRF: High Freq



Comparison of the mean and standard deviation of the amplitude of the driving-point-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

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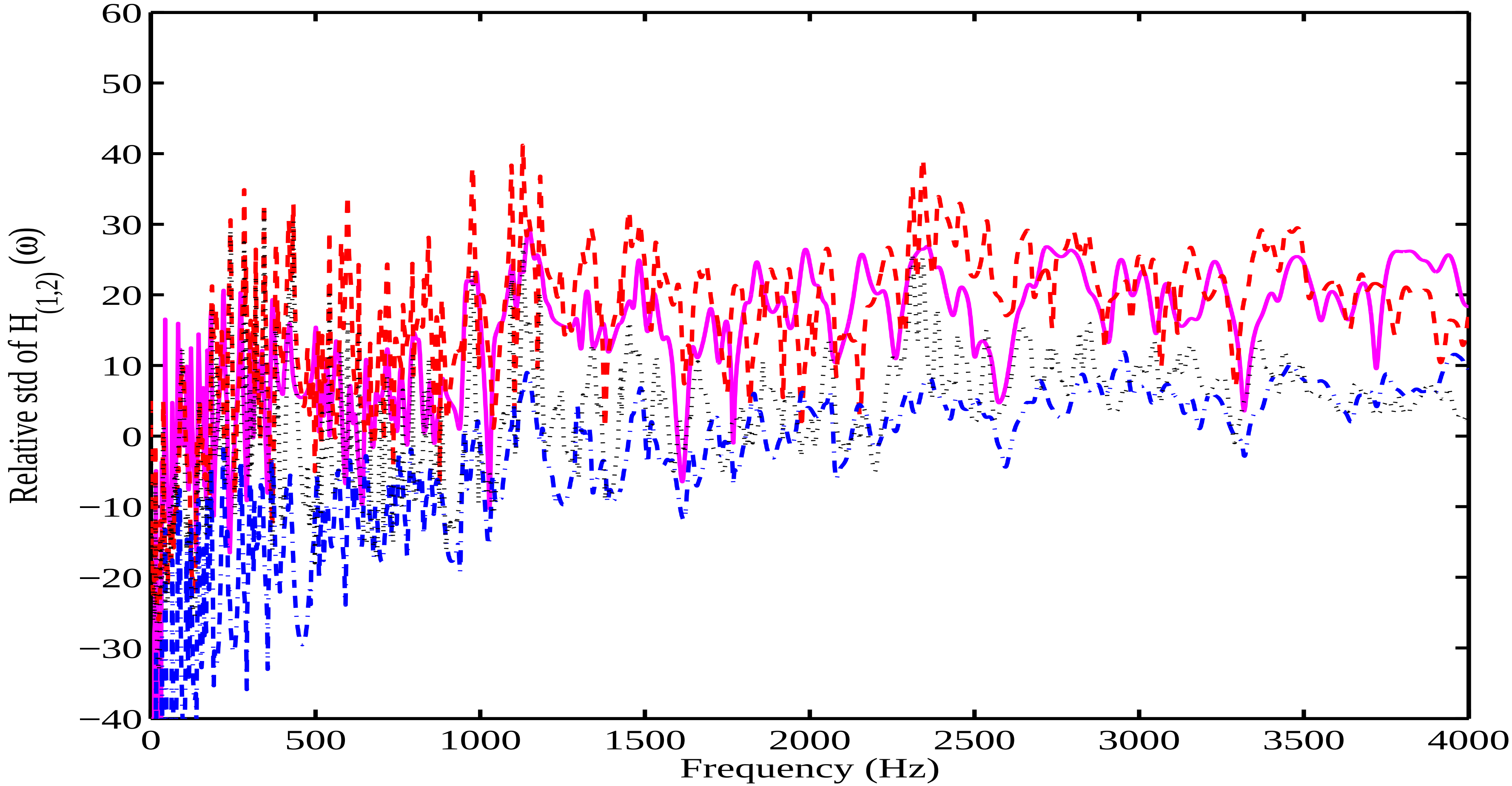
Comparison of cross-FRF



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

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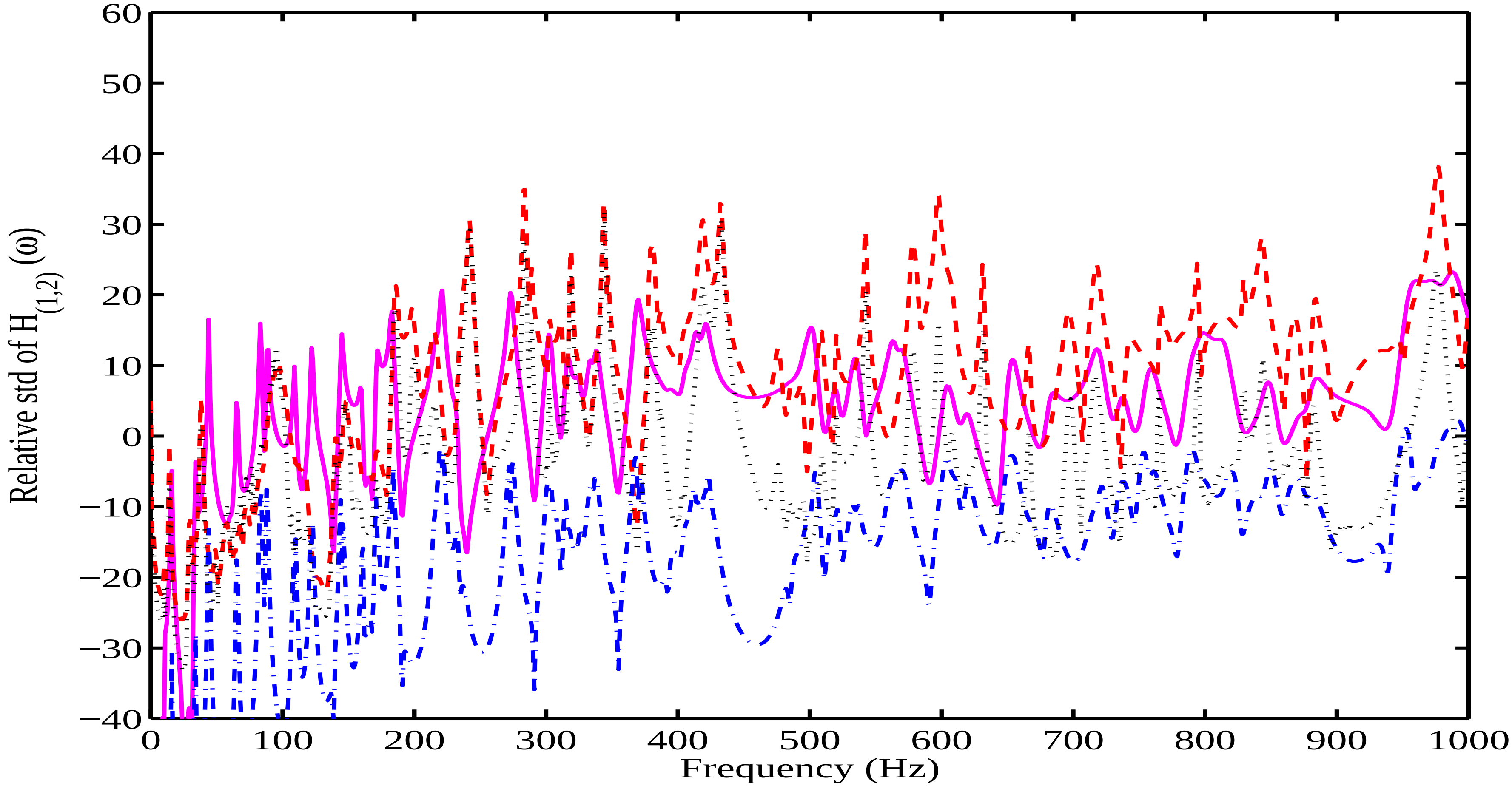
Comparison of cross-FRF: Low Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

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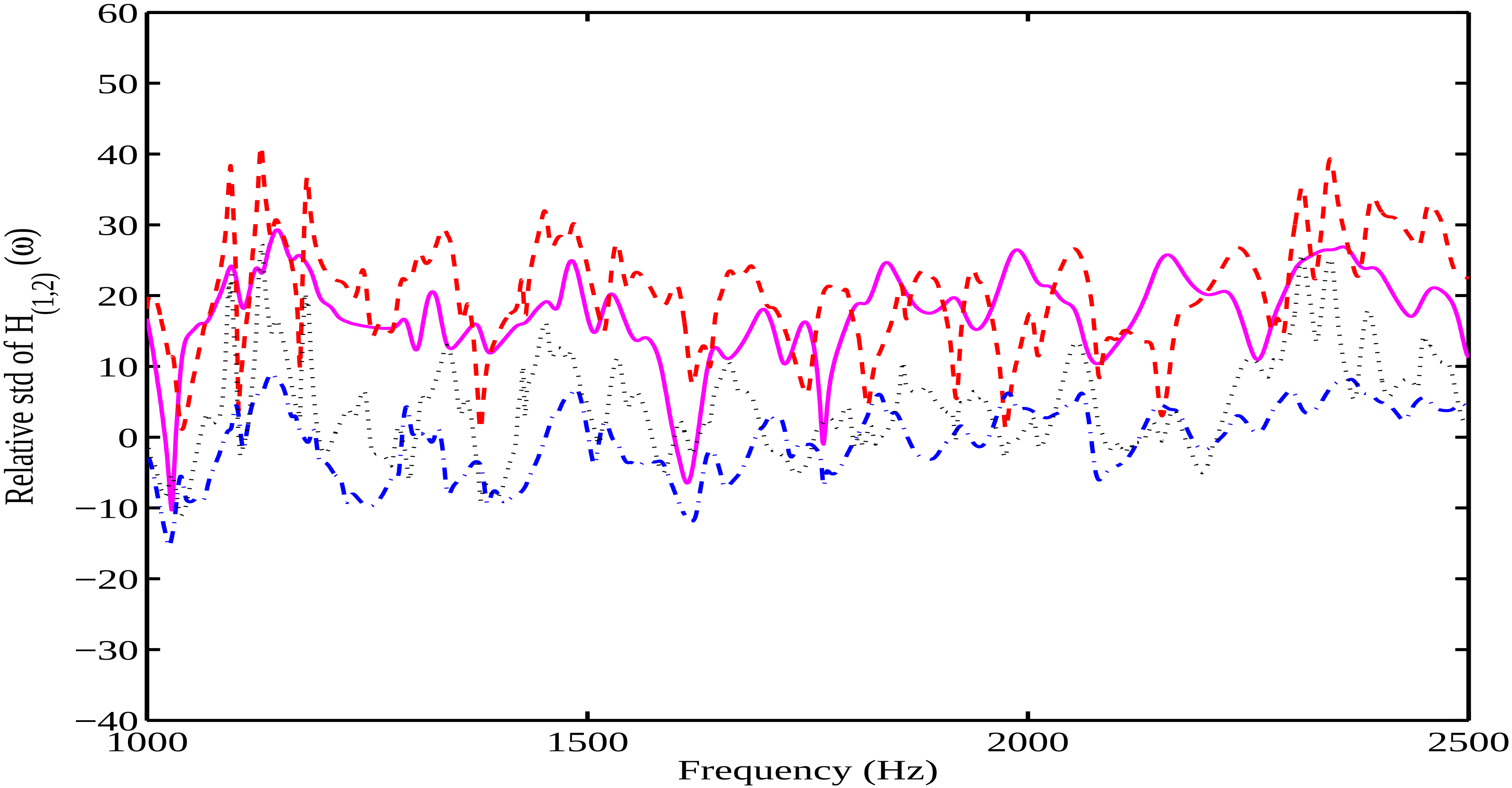
Comparison of cross-FRF: Mid Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

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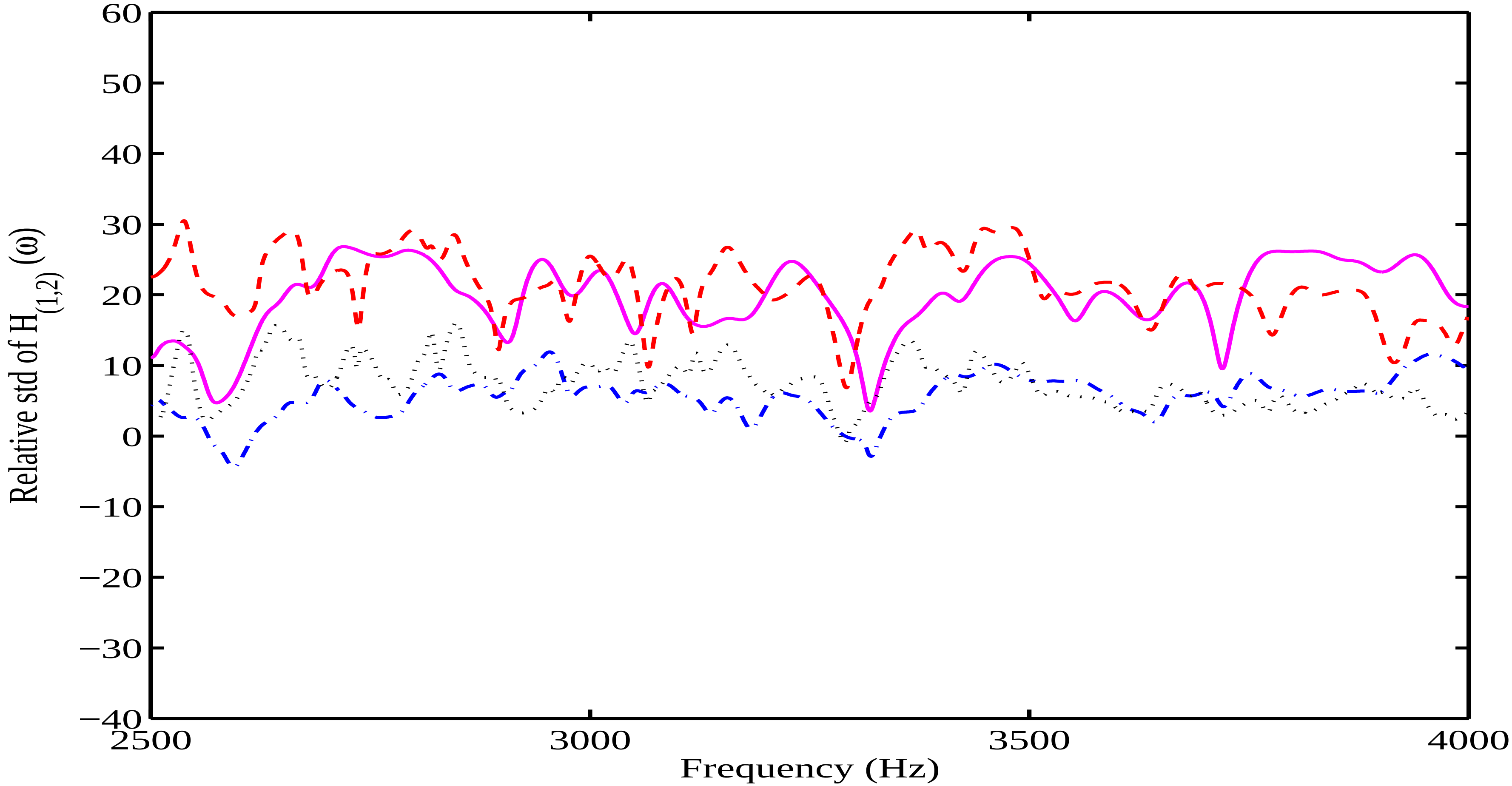
Comparison of cross-FRF: High Freq



Comparison of the mean and standard deviation of the amplitude of the cross-FRF, $n = 1200$, $\delta_M = 0.1166$ and $\delta_K = 0.2711$.

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Conclusions

- When uncertainties in the system parameters (parametric uncertainty) and modelling (nonparametric uncertainty) are considered, the discretized equation of motion of linear dynamical systems is characterized by random mass, stiffness and damping matrices.
- Wishart matrices may be used as the model for the random system matrices in structural dynamics.
- Only the mean matrix and normalized standard deviation is required to model the system.
- Our results show that experimental results and Wishart matrix based results match well in the mid and high frequency region.

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