Dynamic Response of Structures With Frequency Dependent Damping

BLANCA PASCUAL & S ADHIKARI

School of Engineering, Swansea University, Swansea, UK

Email: S.Adhikari@swansea.ac.uk

URL: http://engweb.swan.ac.uk/∼adhikaris

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Prifysgol Abertawe

Outline of the presentation

- Damping models in structural dynamics
- Review of current approaches
- Dynamic response of frequency dependent damped systems
- Non-linear eigenvalue problem
    - SDOF systems: real & complex solutions
    - MDOF systems: real & complex solutions
- Conclusions & discussions

Damping models

- In general a physically realistic model of damping may not be a viscous damping model.
- Damping models in which the dissipative forces depend on any quantity other than the instantaneous generalized velocities are non-viscous damping models.
- Possibly the most general way to model damping within the linear range is to use non-viscous damping models which depend on the past history of motion via convolution integrals over kernel functions.

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Equation of motion

The equations of motion of a N-DOF linear system:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\,\dot{\mathbf{u}}(\tau)\,\mathrm{d}\tau + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) \qquad (1)$$

together with the initial conditions

$$\mathbf{u}(t=0) = \mathbf{u_0} \in \mathbb{R}^N \quad \text{and} \quad \dot{\mathbf{u}}(t=0) = \dot{\mathbf{u}}_\mathbf{0} \in \mathbb{R}^N. \qquad (2)$$

$\mathbf{u}(t)$: displacement vector, $\mathbf{f}(t)$: forcing vector, $\mathbf{M}, \mathbf{K}$: mass and stiffness matrices.

In the limit when $\boldsymbol{\mathcal{G}}(t-\tau) = \mathbf{C}\,\delta(t-\tau)$, where $\delta(t)$ is the Dirac-delta function, this reduces to viscous damping.

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Damping functions - 1

Model Number	Damping function	Author and year of publication
1	$G(s) = \sum_{k=1}^{n} \frac{a_k s}{s + b_k}$	Biot[1] - 1955
2	$G(s) = \frac{E_1 s^\alpha - E_0 b s^\beta}{1 + b s^\beta} \quad (0 < \alpha, \beta < 1)$	Bagley and Torvik[2] - 1983
3	$sG(s) = G^\infty \left[1 + \sum_k \alpha_k \frac{s^2 + 2\xi_k \omega_k s}{s^2 + 2\xi_k \omega_k s + \omega_k^2}\right]$	Golla and Hughes[3] - 1985
		and McTavish and Hughes[4] - 1993
4	$G(s) = 1 + \sum_{k=1}^{n} \frac{\Delta_k s}{s + \beta_k}$	Lesieutre and Mingori[5] - 1990
5	$G(s) = c \frac{1 - e^{-s t_0}}{s t_0}$	Adhikari[6] - 1998
6	$G(s) = \frac{c}{s t_0} \frac{1 + 2(s t_0/\pi)^2 - e^{-s t_0}}{1 + 2(s t_0/\pi)^2}$	Adhikari[6] - 1998
7	$G(s) = c\, e^{s^2/4\mu} \left[1 - \mathrm{erf}\left(\frac{s}{2\sqrt{\mu}}\right)\right]$	Adhikari and Woodhouse[7] - 2001

Some damping functions in the Laplace domain.

Damping functions - 2

- We use a damping model for which the kernel function matrix:

$$\boldsymbol{\mathcal{G}}(t) = \sum_{k=1}^{n} \mu_k e^{-\mu_k t} \mathbf{C}_k \tag{3}$$

- The constants $\mu_k \in \mathbb{R}^+$ are known as the relaxation parameters and n denotes the number relaxation parameters.
- When $\mu_k \to \infty, \forall\, k$ this reduces to the viscous damping model:

$$\mathbf{C} = \sum_{k=1}^{n} \mathbf{C}_k. \tag{4}$$

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State-space Approach

Eq (1) can be transformed to

$$\mathbf{B}\,\dot{\mathbf{z}}(t) = \mathbf{A}\,\mathbf{z}(t) + \mathbf{r}(t) \tag{5}$$

where the m ($m = 2N + nN$) dimensional matrices and vectors are:

$$\mathbf{B} = \begin{bmatrix} \sum_{k=1}^{n} \mathbf{C}_k & \mathbf{M} & -\mathbf{C}_1/\mu_1 & \cdots & -\mathbf{C}_n/\mu_n \\ \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ -\mathbf{C}_1/\mu_1 & \mathbf{O} & \mathbf{C}_1/\mu_1^2 & \mathbf{O} & \mathbf{O} \\ \vdots & \mathbf{O} & \mathbf{O} & \ddots & \mathbf{O} \\ -\mathbf{C}_n/\mu_n & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{C}_n/\mu_n^2 \end{bmatrix}, \quad \mathbf{r}(t) = \begin{Bmatrix} \mathbf{f}(t) \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{Bmatrix} \tag{6}$$

$$\mathbf{A} = \begin{bmatrix} -\mathbf{K} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{M} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -\mathbf{C}_1/\mu_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & -\mathbf{C}_n/\mu_n \end{bmatrix}, \quad \mathbf{z}(t) = \begin{Bmatrix} \mathbf{u}(t) \\ \mathbf{v}(t) \\ \mathbf{y}_1(t) \\ \vdots \\ \mathbf{y}_n(t) \end{Bmatrix} \tag{7}$$

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Main issues

The main reasons for not using a frequency dependent non-viscous damping model include, but not limited to:

- although exact in nature, the state-space approach usually needed for this type of damped systems is computationally very intensive for real-life systems;
- the physical insights offered by methods in the original space (eg, the modal analysis) is lost in a state-space based approach
- the experimental identification of the parameters of a frequency dependent damping model is difficult.

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Dynamic Response - 1

Taking the Laplace transform of equation (1) and considering the initial conditions in (2) we have

$$s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q_0} - \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\,\mathbf{G}(s)\bar{\mathbf{q}} - \mathbf{G}(s)\mathbf{q_0} + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s)$$

$$\text{or} \quad \mathbf{D}(s)\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \left[s\mathbf{M} + \mathbf{G}(s)\right]\mathbf{q_0}.$$

The *dynamic stiffness matrix* is defined as

$$\mathbf{D}(s) = s^2\mathbf{M} + s\,\mathbf{G}(s) + \mathbf{K} \;\in \mathbb{C}^{N\times N}. \tag{8}$$

The inverse of the dynamics stiffness matrix, known as the transfer function matrix, is given by

$$\mathbf{H}(s) = \mathbf{D}^{-1}(s) \in \mathbb{C}^{N\times N}. \tag{9}$$

Dynamic Response - 2

Using the residue-calculus the transfer function matrix can be expressed like a viscously damped system as

$$\mathbf{H}(s) = \sum_{j=1}^{m} \frac{\mathbf{R}_j}{s - s_j}; \quad \mathbf{R}_j = \underset{s=s_j}{\text{res}} [\mathbf{H}(s)] = \frac{\mathbf{z}_j \mathbf{z}_j^T}{\mathbf{z}_j^T \frac{\partial \mathbf{D}(s_j)}{\partial s_j} \mathbf{z}_j} \qquad (10)$$

where m is the number of non-zero eigenvalues (order) of the system, s_j and $\mathbf{z}_j$ are respectively the eigenvalues and eigenvectors of the system, which are solutions of the non-linear eigenvalue problem

$$[s_j^2 \mathbf{M} + s_j \, \mathbf{G}(s_j) + \mathbf{K}] \mathbf{z}_j = \mathbf{0}, \quad \text{for } j = 1, \cdots, m \qquad (11)$$

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Dynamic Response - 3

The expression of $\mathbf{H}(s)$ allows the response to be expressed as modal summation as

$$\bar{\mathbf{q}}(s) = \sum_{j=1}^{m} \gamma_j \frac{\mathbf{z}_j^T \bar{\mathbf{f}}(s) + \mathbf{z}_j^T \mathbf{M} \dot{\mathbf{q}}_\mathbf{0} + s \mathbf{z}_j^T \mathbf{M} \mathbf{q}_\mathbf{0} + \mathbf{z}_j^T \mathbf{G}(s) \mathbf{q}_\mathbf{0}(s)}{s - s_j} \mathbf{z}_j \tag{12}$$

where

$$\gamma_j = \frac{1}{\mathbf{z}_j^T \frac{\partial \mathbf{D}(s_j)}{\partial s_j} \mathbf{z}_j}. \tag{13}$$

We aim to derive the eigensolutions in ‘N’-space by solving the non-linear eigenvalue problem.

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Non-linear Eigenvalue Problem

- The eigenvalue problem associated with a linear system with exponential damping model:

$$\left[s_j^2\mathbf{M} + s_j \sum_{k=1}^{n} \frac{\mu_k}{s_j+\mu_k}\mathbf{C}_k + \mathbf{K}\right]\mathbf{z}_j = \mathbf{0}, \quad \text{for } j = 1,\cdots,m. \tag{14}$$

- Two types of eigensolutions:
  - $2N$ complex conjugate solutions - underdamped/vibrating modes
  - p real solutions [$p = \sum_{k=1}^{n} \mathrm{rank}\left(\mathbf{C}_k\right)$] - overdamped modes

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Non-linear Eigenvalue Problem

The following four cases are considered:

- single-degree-of-freedom system with single exponential kernel ($N = 1, n = 1$)
- single-degree-of-freedom system with multiple exponential kernels ($N = 1, n > 1$)
- multiple-degree-of-freedom system with single exponential kernel ($N > 1, n = 1$)
- multiple-degree-of-freedom system with multiple exponential kernels ($N > 1, n > 1$)

SDOF systems

- Computational cost and other relevant issues identified before do not strictly affect the eigenvalue problem of a single-degree-of-freedom system (SDOF) with exponential damping.
- The main reason for considering a SDOF system is that in many cases the underlying approximation method can be extended to MDOF systems in a relatively straight-forward manner.

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Complex-conjugate solutions

- The main motivation of the approximations is that the approximate solution can be ‘constructed’ from the solution of equivalent viscously damped system.
- The solution of equivalent viscously damped system can in turn be expressed in terms of the undamped eigensolutions.
- Combining these together, one can therefore obtain the eigensolutions of frequency-dependent systems by simple ‘post-processing’ of undamped solutions only.

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Complex-conjugate solutions

- The eigenvalues of the equivalent viscously damped system:

$$s_0 = -\zeta_n \omega_n \pm \mathrm{i}\omega_n \sqrt{1-\zeta_n^2} \approx -\zeta_n \omega_n \pm \mathrm{i}\omega_n \qquad (15)$$

$\omega_n = \sqrt{k_u/m_u}$ and $\zeta_n = c/2\sqrt{k_u m_u}$.

- Viscous damped system is a special case when the function $g(s)$ is replaced by $g(s \to \infty)$. For that case this solution would have been the exact solution of the characteristic equation.
- The difference between the viscous solution and the true solution is essentially arising due to the 'varying' nature of the function $g(s)$.

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Complex-conjugate solutions

- The central idea here is that the actual solution can be obtained by expanding the solution in a Taylor series around s_0. We assume $s = s_0 + \delta$, (δ is small).
- Substituting this into the characteristic equation we have

$$(s_0 + \delta)^2 m_u + (s_0 + \delta) g(s_0 + \delta) + k_u = 0. \tag{16}$$

- First-order approximation

$$\delta^{(1)} = \frac{s_0(s_0 m_u + g(s_0)) + k_u}{s_0(2m_u + g'(s_0)) + g(s_0)} \tag{17}$$

Complex-conjugate solutions

- Second-order approximation:

$$\delta^{(2)} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{18}$$

where

$$A = (m_u + \frac{g''(s_0)}{2!} s_0 + g'(s_0)) \tag{19}$$

$$B = (2m_u s_0 + s_0 g'(s_0) + g(s_0)) \tag{20}$$

and

$$C = (s_0^2 m_u + s_0 g(s_0) + k). \tag{21}$$

- $g'(s_0)$ and $g''(s_0)$ are respectively the first and second order derivative of $g(s)$ evaluated at $s = s_0$.

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Real solutions

- When $N = 1, n = 1$ the eigenvalue equation:

$$s^2 m_u + sg(s) + k_u = 0 \quad \text{where} \quad g(s) = \frac{\mu}{s+\mu} c. \tag{22}$$

- While the complex-conjugate solution can be expected to be close to the solution of the equivalent viscously damped system, no such analogy can be made for the real solution as the equivalent viscously damped system doesn't have one.
- Rewrite the characteristic equation as

$$(s^2 m_u + k_u)(\mu + s) + sc\mu = 0. \tag{23}$$

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Real solutions

- Consider that the damping is small so that $sc\mu \approx 0$. Since $(s^2 m_u + k_u) \neq 0$ as we are considering the real solution only, the first guess is obtained as $\mu + s + 0 = 0 \quad$ or $\quad s_0 = -\mu$.
- We take the first approximation of the real root as $s = s_0 + \Delta = -\mu + \Delta$
- Substituting into the characteristic equation and neglecting all the higher-order terms:

$$\Delta \approx \frac{\mu^2 c}{\mu^2 m_u + k_u + \mu c} \tag{24}$$

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Real solutions: General case

- The characteristic equation:

$$s^2 m_u + s \sum_{k=1}^{n} \frac{\mu_k}{s+\mu_k} c_k + k_u = 0. \tag{25}$$

- This a polynomial in s of order $(n+2)$ - it has $(n+2)$ roots [2 complex conjugate and n real].
- Multiplying the characteristic equation by the product $\prod_j^n (s+\mu_j)$:

$$(s^2 m_u + k_u) \prod_{j=1}^{n} (s+\mu_j) + s \sum_{k=1}^{n} \left(\mu_k c_k \prod_{\substack{j=1 \\ j \neq k}}^{n} (\mu_j + s) \right) = 0. \tag{26}$$

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Real solutions: General case

- Use the approximation $s_k = \mu_k + \Delta_k$, $k = 1, 2, \cdots n$.
- Substituting into the characteristic equation and retaining only the first-order terms in Δ_k, after some simplifications:

$$\Delta_k \approx \frac{c_k \mu_k^2 \prod_{\substack{j=1 \\ j \neq k}}^{n} (\mu_j - \mu_k)}{(\mu_k^2 m + k) \prod_{\substack{r=1 \\ n \neq k}}^{n} (\mu_r - \mu_k) + \mu_k \theta_k} \text{ for } k = 1, 2, \cdots, n.$$

where $\theta_k =$

$$\left[-\mu_k c_k \sum_{\substack{j=1 \\ j \neq k}}^{n} \prod_{\substack{m=1 \\ m \neq j \\ m \neq k}}^{n} (\mu_m - \mu_k) - \sum_{\substack{r=1 \\ r \neq k}}^{n} \mu_r c_r \prod_{\substack{j=1 \\ j \neq n \\ j \neq k}}^{n} (\mu_j - \mu_k) + \prod_{\substack{j=1 \\ j \neq k}}^{n} c_k (\mu_j - \mu_k) \right]$$

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Numerical Results

SDOF system with eight exponential kernels: $m = 1$ kg, $k = 2$ N/m, μ_k for $k = 1, 2.., 8$ are 1.9442, 1.5231, 1.9317, 1.7657, 1.7454, 1.9558, 2.0677, 1.4973.

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Numerical Results

μ	Exact solution (state-space)	Proposed approximate solution	Percentage error
1.9442	-1.4649	-1.5135	3.3169
1.5231	-1.5136	-1.5185	0.3237
1.9317	-1.7123	-1.7579	2.6643
1.7657	-1.7576	-1.7613	0.2101
1.7454	-1.8954	-1.9253	1.5767
1.9558	-1.9380	-1.9375	0.0282
2.0677	-1.9517	-1.9527	0.0516
1.4973	-2.0560	-2.0592	0.1559
Complex Conjugate solution	-0.0619$\pm$ 1.4718i	-0.0619$\pm$1.4718i	0.0003$\pm$0i

Exact and approximate eigenvalues of the SDOF system.

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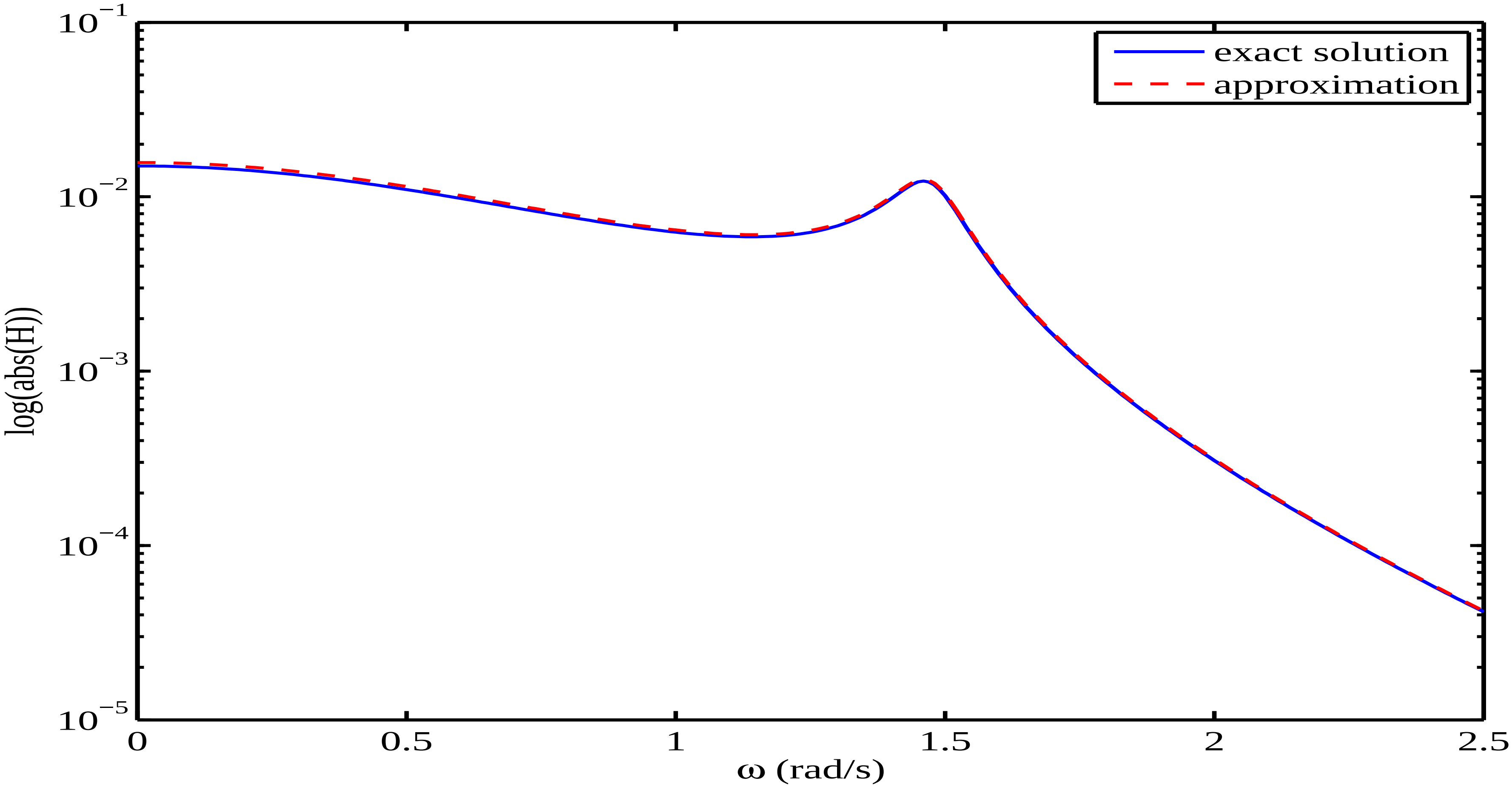
Frequency response function



Frequency response function of the SDOF system obtained using exact and approximate eigenvalues.

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Complex solutions: MDOF

- Complex modes can be expanded as a complex linear combination of undamped modes $\mathbf{z}_j = \sum_{l=1}^{N} \alpha_l^{(j)} \mathbf{x}_l$
- Substituting in to the eigenvalue equation (11):

$$\sum_{l=1}^{N} s_j^2 \alpha_l^{(j)} \mathbf{M}\mathbf{x}_l + s_j \alpha_l^{(j)} \mathbf{G}(s_j)\mathbf{x}_l + \alpha_l^{(j)} \mathbf{K}\mathbf{x}_l = \mathbf{0}. \qquad (27)$$

- Premultiplying by $\mathbf{x}_k^T$ and using the mass-orthogonality property of the undamped eigenvectors:

$$s_j^2 \alpha_k^{(j)} + s_j \sum_{l=1}^{N} \alpha_l^{(j)} G'_{kl}(s_j) + \omega_k^2 \alpha_k^{(j)} = 0, \;\; G'_{kl}(s_j) = \mathbf{x}_k^T \mathbf{G}(s_j)\mathbf{x}_l \qquad (28)$$

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Complex solutions: MDOF

- Considering the j-th set of equation and neglecting the second-order terms involving $\alpha_k^{(j)}$ and $G'_{kl}(s_j)$, $\forall k \neq l$:

$$s_j^2 + s_j G'_{jj}(s_j) + \omega_j^2 \approx 0 \tag{29}$$

- Similar to the SDOF case (replace m_u by 1, k_u by ω_j^2 and $g(s)$ by $G'_{jj}(s)$).
- To obtain the eigenvectors rewrite Eq. (28) for $j \neq k$ as

$$s_j^2 \alpha_k^{(j)} + s_j \left(G'_{kj}(s_j) + \alpha_k^{(j)} G'_{kk}(s_j) + \sum_{l \neq k \neq j}^{N} \alpha_l^{(j)} G'_{kl}(s_j) \right) + \omega_k^2 \alpha_k^{(j)} = 0, \quad \forall k = 1, \cdots, N; \neq j. \tag{30}$$

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Complex solutions: MDOF

- Retaining only the product terms $\alpha_l^{(j)} G'_{kl}$:

$$\mathbf{z}_j \approx \mathbf{x}_j - \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{s_j G'_{kj}(s_j) \mathbf{x}_k}{\omega_k^2 + s_j^2 + s_j G'_{kk}(s_j)}. \tag{31}$$

- A second-order expressions is given in the paper.
- The results derived here are based on 'small non-proportional damping'.

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Real solutions: MDOF

- For systems with single exponential kernel assume

$$s_l = -\mu + \Delta_l \tag{32}$$

- Substituting in the approximate characteristic Eq (29)

$$\Delta_l \approx \frac{\mu^2 C'_{ll}}{\mu^2 + \omega_l^2 + \mu C'_{ll}}; \quad \forall l = 1, 2, \cdots, N \tag{33}$$

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Real solutions: MDOF

- Assuming all coefficient matrices are of full rank, for systems with n kernels there are in general nN number of purely real eigenvalues.
- The approximate eigenvalues can be obtained as

$$s_{lk} = -\mu_k + \Delta_{lk} \tag{34}$$

where

$$\Delta_{lk} \approx \frac{C_{k_{ll}} \mu_k^2 \prod_{\substack{j=1 \\ j \neq k}}^{n} (\mu_j - \mu_k)}{(\mu_k^2 + \omega_l^2) \prod_{\substack{r=1 \\ n \neq k}}^{n} (\mu_r - \mu_k) + \mu_k \theta_{lk}} \forall k = 1, 2, \cdots, n; \; l = 1, 2, \cdots, N$$

$\theta_{lk} =$

$$-\mu_k C_{k_{ll}} \sum_{\substack{j=1 \\ j \neq k}}^{n} \prod_{\substack{m=1 \\ m \neq j \\ m \neq k}}^{n} (\mu_m - \mu_k) - \sum_{\substack{r=1 \\ r \neq k}}^{n} \mu_r c_r \prod_{\substack{j=1 \\ j \neq n \\ j \neq k}}^{n} (\mu_j - \mu_k) + \prod_{\substack{j=1 \\ j \neq k}}^{n} C_{k_{ll}} (\mu_j - \mu_k)$$

and $C_{k_{ll}} = \mathbf{x}_l^T \mathbf{C}_k \mathbf{x}_l$.

Numerical example

We consider a three degree-of-freedom system:

$$\mathbf{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_u & 0 \\ 0 & 0 & m_u \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2k_u & -k_u & 0 \\ -k_u & 2k_u & -k_u \\ 0 & -k_u & 2k_u \end{bmatrix} \tag{35}$$

$$\mathbf{G}(s) = \mathbf{C} \sum_{k=1}^{6} \frac{\mu_k}{s + \mu_k}, \quad \text{where} \quad \mathbf{C} = \begin{bmatrix} 0.30 & -0.15 & -0.05 \\ -0.15 & 0.30 & -0.15 \\ -0.05 & -0.15 & 0.30 \end{bmatrix}. \tag{36}$$

$m_u = 1$ kg, $k_u = 1$ N/m and μ_k are 1.4565, 1.0185, 1.8214, 1.4447, 1.6154, 1.7919.

Numerical results

Exact solution (state-space)	Proposed approximate solution	Percentage error
Real solutions		
-0.9380	-0.9425	0.4797
-1.2995	-1.3044	0.3771
-1.4507	-1.7413	20.0317
-1.5754	-1.8096	14.8661
-1.7405	-1.5761	9.4456
-1.8095	-1.4507	19.8287
-0.6301	-0.5614	10.89
-1.1276	-1.0760	4.57
-1.4505	-1.7096	17.8628
-1.5507	-1.8081	16.5990
-1.7096	-1.5507	9.2946

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Numerical results

Exact solution (state-space)	Proposed approximate solution	Percentage error
Real solutions		
-1.8081	-1.4505	19.7777
-0.6798	-0.6731	0.9856
-1.1295	-1.1289	0.0531
-1.4505	-1.7085	17.7870
-1.5501	-1.8080	16.6376
-1.7086	-1.5501	9.2766
-1.8080	-1.4505	19.7732
Complex Conjugate solutions		
-0.4109 $\pm$ 2.6579i	-0.4116 $\pm$ 2.6591i	0.1704 $\pm$ 0.0451i
-0.4359 $\pm$ 2.0939i	-0.4306 $\pm$ 2.0937i	1.2011 $\pm$ 0.0492i
-0.1674 $\pm$ 0.8523i	-0.1649 $\pm$ 0.8528i	1.4934 $\pm$ 0.0587i

Conclusions - 1

- Multiple degree-of-freedom linear systems with frequency depended damping kernels is considered.
- The transfer function matrix of the system was derived in terms of the eigenvalues and eigenvectors of the second-order system. The response can be expressed as a sum of two parts, one that arises in usual viscously damped systems and the other that occurs due to non-viscous damping.
- The calculation of the eigensolutions of frequency-depended damped systems requires the solution of a non-linear eigenvalue problem.

Conclusions - 2

- Approximate expressions are derived for the complex and real eigenvalues of the SDOF system with single and multiple exponential kernels. These results are extended to MDOF systems.
- These approximations allow one to obtain the dynamic response of general frequency-depended damped systems by simple post-processing of undamped eigensolutions.
- The accuracy of the proposed approximations were verified using numerical examples. The complex conjugate eigensolutions turn out to be more accurate compared to the real eigensolutions.

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