Response Variability of Linear Stochastic Systems: A General Solution Using Random Matrix Theory

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Outline of the presentation

- Uncertainty in structural dynamics
- Review of current UQ approaches
- Random matrix models
- Derivation of response statistics
- Numerical implementations and example
- Conclusions

Sources of Uncertainty - 1

(a) parametric uncertainty - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) model inadequacy - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results;

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Sources of Uncertainty - 2

(d) computational uncertainty - e.g, machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis, and
(e) model uncertainty - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.

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Structural dynamics

The equation of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objectives in the 'forward problem' are:
  - to quantify uncertainties in the system matrices
  - to predict the variability in the response vector $\mathbf{x}$

Current UQ approaches - 1

Two different approaches are currently available

- Parametric approaches : Such as the Stochastic Finite Element Method (SFEM):
  - aim to characterize parametric uncertainty (type 'a')
  - assumes that stochastic fields describing parametric uncertainties are known in details
  - suitable for low-frequency dynamic applications

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Current UQ approaches - 2

- Nonparametric approaches : Such as the Statistical Energy Analysis (SEA) and Wishart random matrix theory:
  - aim to characterize nonparametric uncertainty (types 'b' - 'e')
  - does not consider parametric uncertainties in details
  - suitable for high-frequency dynamic applications

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Current UQ approaches - 3

The reasons for not having a general purpose UQ code:

(a) the computational time can be prohibitively high compared to a deterministic analysis for real problems,

(b) the volume of input data can be unrealistic to obtain for a credible probabilistic analysis,

(c) the predictive accuracy can be poor if considerable resources are not spend on the previous two items, and

(d) as the state-of-the art methodology stands now (such as the Stochastic Finite Element Method), only very few highly trained professionals (such as those with PhDs) can even attempt to apply the complex concepts (e.g., random fields) and methodologies to real-life problems.

Main Objectives

This paper is aimed at developing an approach [the 10-10-10 challenge] with the ambition that it should:
(a) not take more than 10 times the computational time required for the corresponding deterministic approach;
(b) result a predictive accuracy within 10% of direct Monte Carlo Simulation (MCS);
(c) use no more than 10 times of input data needed for the corresponding deterministic approach; and
(d) enable 'normal' engineering graduates to perform probabilistic structural dynamic analyses with a reasonable amount of training.

Wishart Random Matrix Model

- The probability density function of the mass ($\mathbf{M}$), damping ($\mathbf{C}$) and stiffness ($\mathbf{K}$) matrices should be such that they are symmetric and non-negative matrices.
- Wishart random matrix (a non-Gaussian matrix) is the simplest mathematical model which can satisfy these two criteria: $[\mathbf{M}, \mathbf{C}, \mathbf{K}] \equiv \mathbf{G} \sim W_n(p, \mathbf{\Sigma})$.
- The parameters of the distribution can be fitted with 'measured' data, such as the mean ($\mathbf{G}_0$) and the standard deviation (σ_G) of the system matrices.

Parameters of the Wishart Distribution

Suppose we 'know' the mean (G_0) and the (normalized) standard deviation (σ_G) of the system matrices:

$$\sigma_G^2 = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|_{\mathrm{F}}^2}. \tag{2}$$

The parameters p and $\mathbf{\Sigma}$ can be obtained as

$$p = n + 1 + \theta, \quad \mathbf{\Sigma} = \mathbf{G}_0/\theta \tag{3}$$

$$\theta = (1+\beta)/\sigma_G^2 - (n+1); \;\; \beta = \left\{\mathrm{Trace}\left(\mathbf{G}_0\right)\right\}^2 / \mathrm{Trace}\left({\mathbf{G}_0}^2\right).$$

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Dynamic Response

- The dynamic response of the system can be expressed in the Frequency domain as

$$\mathbf{q}(\omega) = \mathbf{D}^{-1}(\omega)\mathbf{f}(\omega) \tag{4}$$

where the dynamic stiffness matrix is defined as

$$\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}. \tag{5}$$

This is a complex symmetric random matrix.

- The calculation of the response statistics requires the calculation of statistical moments of the inverse of this matrix.

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Main Assumptions

1. Damping matrix is ‘small’ compared to the mass and stiffness matrices.
2. The damping matrix is deterministic.
3. The mass and stiffness matrices are statistically independent Wishart matrices.
4. The input force is deterministic.

(no assumptions related to proportional damping, small randomness or Gaussianity).

Response Statistics

Since engineering interest often lies in the absolute value of the response, we are interested in the statistical moments of

$$|\mathbf{q}|(\omega) = \left|\mathbf{D}^{-1}(\omega)\right| |\mathbf{f}(\omega)| = |\mathbf{D}(\omega)|^{-1} |\mathbf{f}(\omega)| \qquad (6)$$

where the absolute of the dynamic stiffness matrix is given by

$$|\mathbf{D}(\omega)| = \left\{ [-\omega^2\mathbf{M} + \mathbf{K}]^2 + \omega^2\mathbf{C}^2 \right\}^{1/2}. \qquad (7)$$

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Response Moments - 1

- The first-order moment of the absolute of the response:

$$\bar{\mathbf{q}} = \mathrm{E}\left[|\mathbf{q}|\right] = \mathrm{E}\left[|\mathbf{D}|^{-1}\right]\bar{\mathbf{f}} \tag{8}$$

where $\bar{\mathbf{f}} = |\mathbf{f}|$.

- The second-order moment of the absolute of the response:

$$\begin{aligned} \mathrm{cov}_{|\mathbf{q}|} &= \mathrm{E}\left[(|\mathbf{q}| - \mathrm{E}\left[|\mathbf{q}|\right])(|\mathbf{q}| - \mathrm{E}\left[|\mathbf{q}|\right])^T\right] = \mathrm{E}\left[|\mathbf{q}|\,|\mathbf{q}|^T\right] - \bar{\mathbf{q}}\bar{\mathbf{q}}^T \\ &= \mathrm{E}\left[|\mathbf{D}|^{-1}\,\bar{\mathbf{f}}\bar{\mathbf{f}}^T\,|\mathbf{D}|^{-1}\right] - \bar{\mathbf{q}}\bar{\mathbf{q}}^T. \end{aligned} \tag{9}$$

Response Moments - 2

The dynamic response statistics is obtained in two steps:

- A Wishart distribution is fitted to $|\mathbf{D}|$ matrix, which is symmetric and non-negative definite. Note that $\mathbf{D}$ cannot be a Wishart matrix unless the system is undamped.
- Once the parameters of the Wishart distribution corresponding to $|\mathbf{D}|$ is identified, the inverse moments are obtained exactly in closed-from using the inverted Wishart distribution.

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Random Dynamic Stiffness Matrix

Consider that $|\mathbf{D}|(\omega) \sim W_n(p_D, \mathbf{\Sigma}_D) = \mathrm{S}$ where $p_D(\omega)$ and $\mathbf{\Sigma}_D(\omega)$ are the unknown parameters of the Wishart distribution to be identified. We employ the following criteria

- The square-root of the mean of $|\mathbf{D}|^2$ is same as the mean of S, that is

$$\sqrt{\mathrm{E}\left[|\mathbf{D}|^2\right]} = \mathrm{E}\left[\mathbf{S}\right]. \tag{10}$$

- The standard deviation of of $|\mathbf{D}|^2$ is same as the standard deviation of S^2, that is

$$\sigma_{|\mathbf{D}|^2} = \sigma_{\mathbf{S}^2}. \tag{11}$$

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Response Moments - 3

After some algebra we have

$$\bar{\mathbf{q}} = \frac{p_D(\omega)}{\theta_D(\omega)} \mathbf{q}_0(\omega) \qquad (12)$$

Here $\mathbf{q}_0(\omega)$ is the absolute value of the response for to the baseline or 'mean' system

$$\mathbf{q}_0(\omega) = |\mathbf{D}_0(\omega)|^{-1} |\mathbf{f}(\omega)| \qquad (13)$$

with $|\mathbf{D}_0(\omega)| = |-\omega^2 \mathbf{M}_0 + i\omega \mathbf{C} + \mathbf{K}_0|$
$\theta_D(\omega) = p_D(\omega) - n - 1$, $p_D(\omega) = \mathrm{Trace}\left(\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}\right) / \mathrm{Trace}\left(\boldsymbol{\mathcal{A}}^2\right)$
where
$\boldsymbol{\mathcal{A}} = \omega^4 p_M \left({\mathbf{M}_0}^2 + \mathbf{M}_0 \mathrm{Trace}\left(\mathbf{M}_0\right)\right) / \theta_M + p_K \left({\mathbf{K}_0}^2 + \mathbf{K}_0 \mathrm{Trace}\left(\mathbf{K}_0\right)\right) / \theta_K$
$\boldsymbol{\mathcal{B}} = |\mathbf{D}_0(\omega)|^2 + |\mathbf{D}\mathbf{D}_0| \, \mathrm{Trace}\left(|\mathbf{D}_0(\omega)|\right)$.

Response Moments - 4

The covariance of the absolute of the response can be obtained as

$$\mathrm{cov}_{|\mathbf{q}|}(\omega) = \frac{(\theta_D(\omega)+n+1)\mathrm{Trace}\left(\mathbf{q}_0(\omega)\bar{\mathbf{f}}(\omega)^T\right)\boldsymbol{\Sigma}_D^{-1}(\omega) + (\theta_D(\omega)+2)\mathbf{q}_0(\omega)\mathbf{q}_0^T(\omega)}{(\theta_D(\omega)+1)(\theta_D(\omega)-2)}. \tag{14}$$

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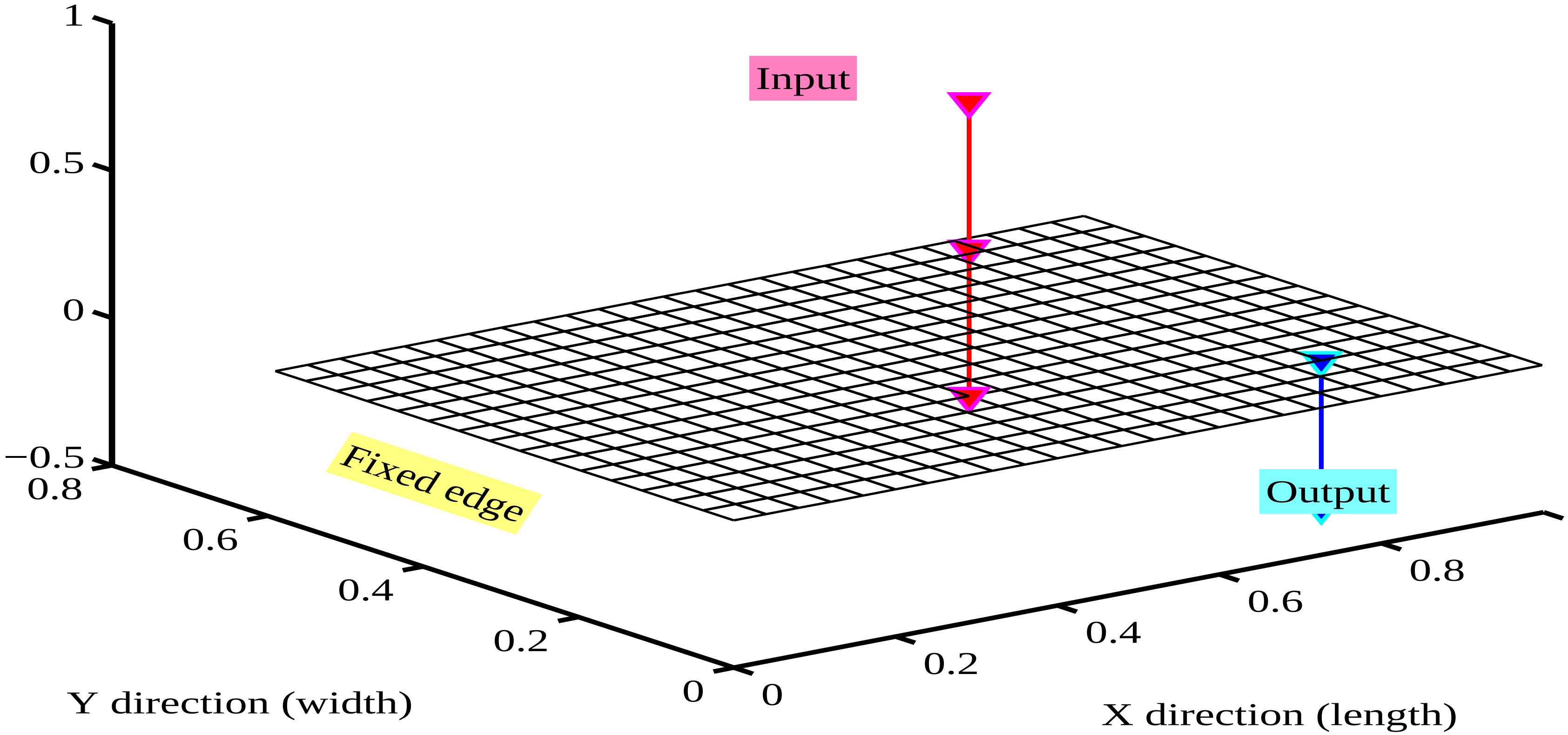
Example: A cantilever Plate



A steel cantilever plate with a slot; $\bar{E} = 200 \times 10^9$N/m^2, $\bar{\mu} = 0.3$, $\bar{t} = 7.5$mm, $L_x = 0.998$m, $L_y = 0.59$m; 25×15 elements resulting $n = 1200$.

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Stochastic properties

The Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\left(1 + \epsilon_E f_1(\mathbf{x})\right) \quad (15)$$

$$\mu(\mathbf{x}) = \bar{\mu}\left(1 + \epsilon_\mu f_2(\mathbf{x})\right) \quad (16)$$

$$\rho(\mathbf{x}) = \bar{\rho}\left(1 + \epsilon_\rho f_3(\mathbf{x})\right) \quad (17)$$

and $$t(\mathbf{x}) = \bar{t}\left(1 + \epsilon_t f_4(\mathbf{x})\right) \quad (18)$$

- The strength parameters: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.10$, $\epsilon_\rho = 0.15$ and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x}), i = 1, \cdots, 4$ are delta-correlated homogenous Gaussian random fields.

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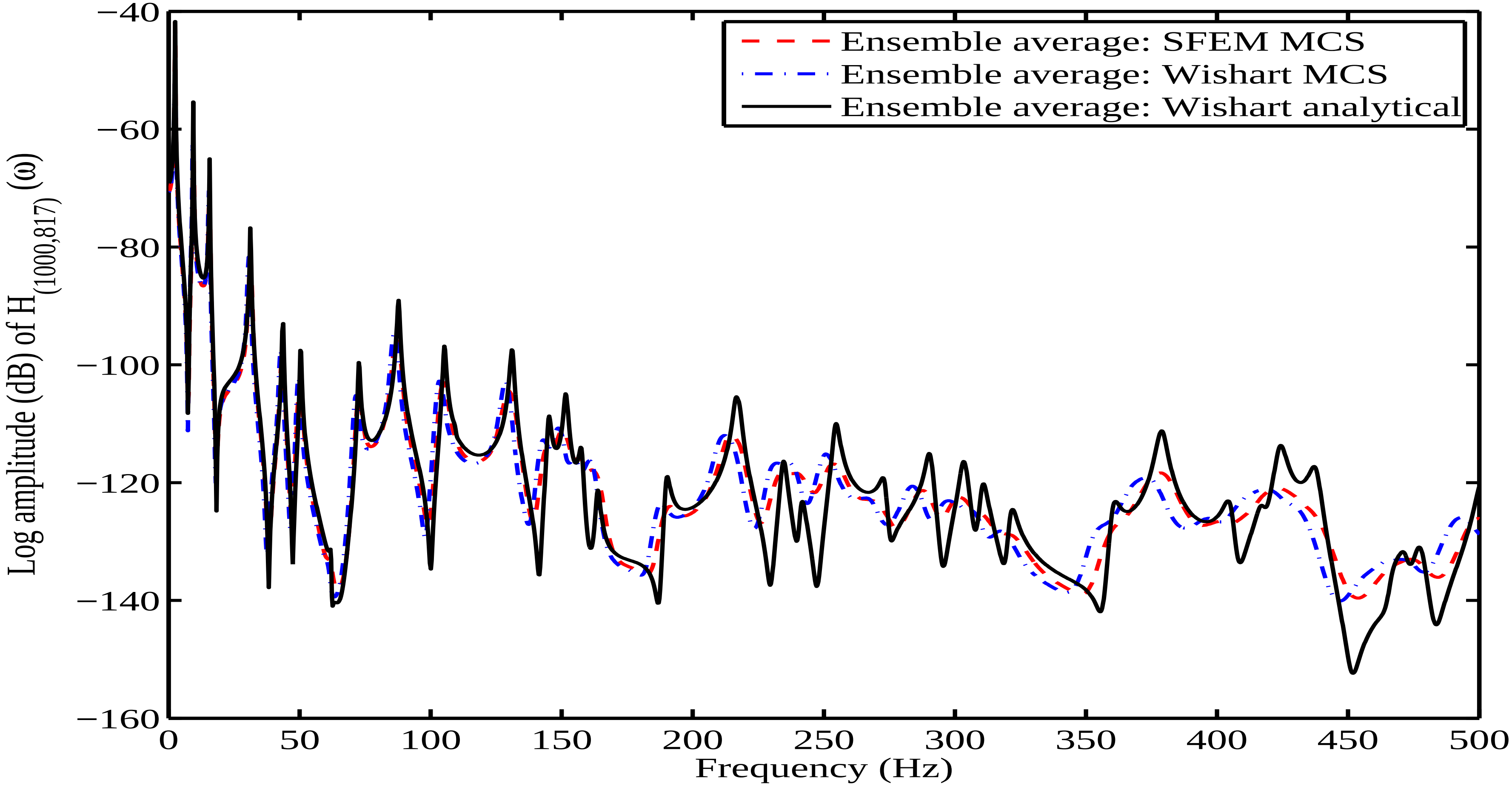
Compared Methods

- *Direct Monte Carlo Simulation:* Response statistics is obtained using 1000 samples (considered as the benchmark).
- *Monte Carlo Simulation using Wishart Matrices:* Response statistics is obtained using 1000 samples of the fitted Wishart matrices with $\sigma_M = 0.0999$ and $\sigma_K = 0.2151$.
- *Proposed Analytical Method using Wishart Matrices:* Closed-form expressions given by Eqs. (12) and (14) are used to obtain the mean and standard-deviation of the response.

Mean of Cross-FRF



Mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.0999$ and $\sigma_K = 0.2151$.

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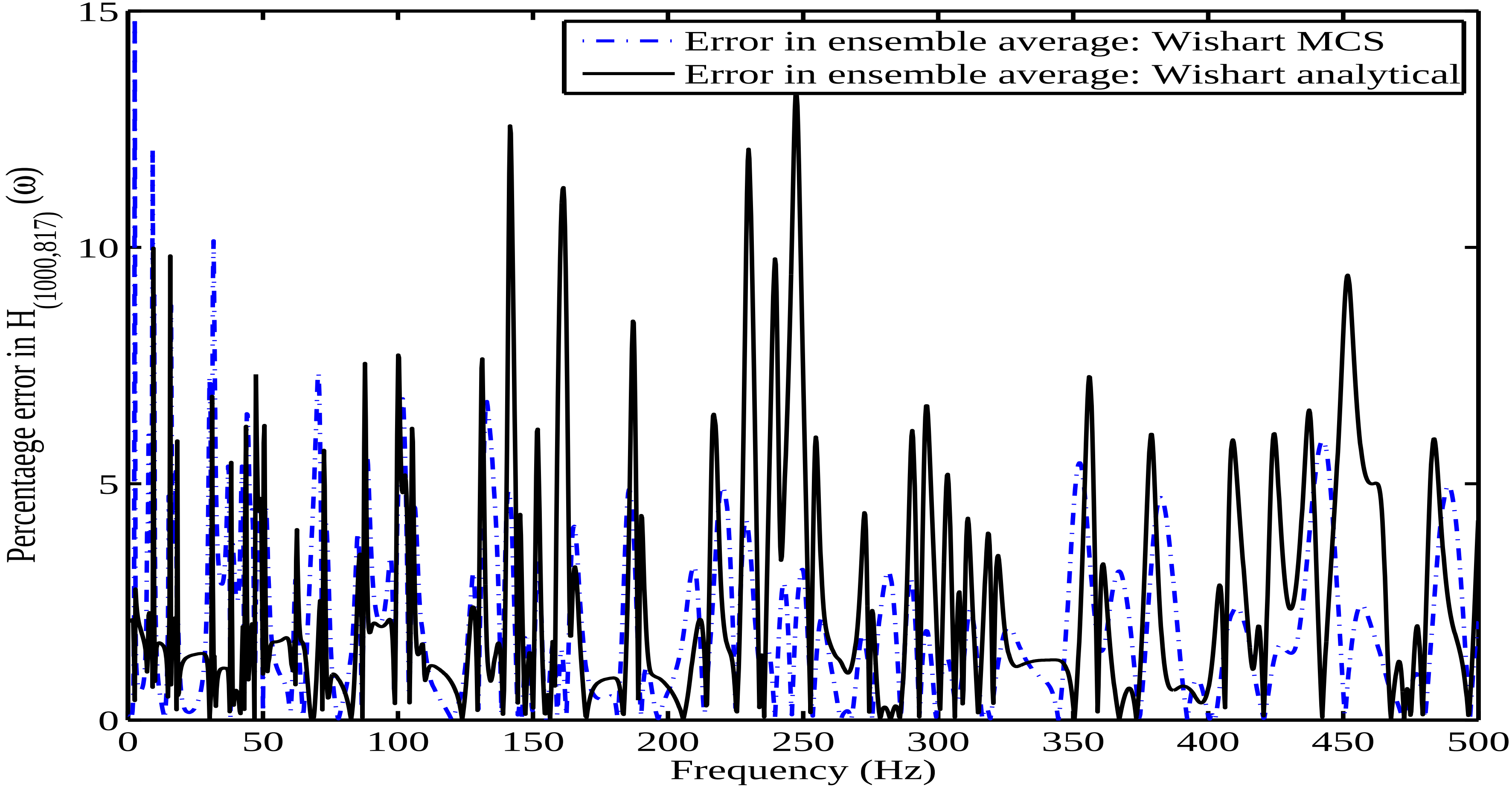
Error in the Mean of Cross-FRF



Error in the mean of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.0999$ and $\sigma_K = 0.2151$.

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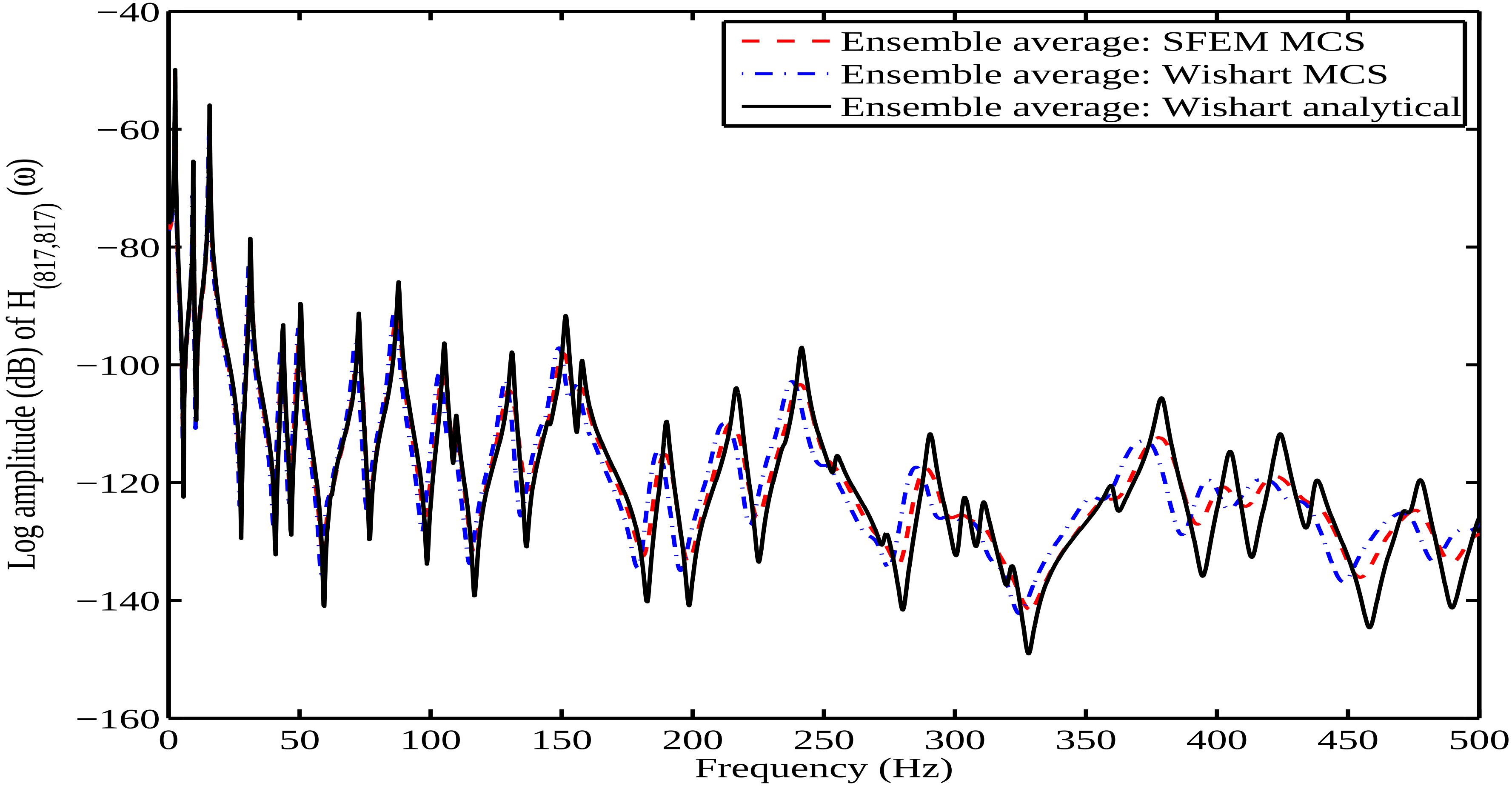
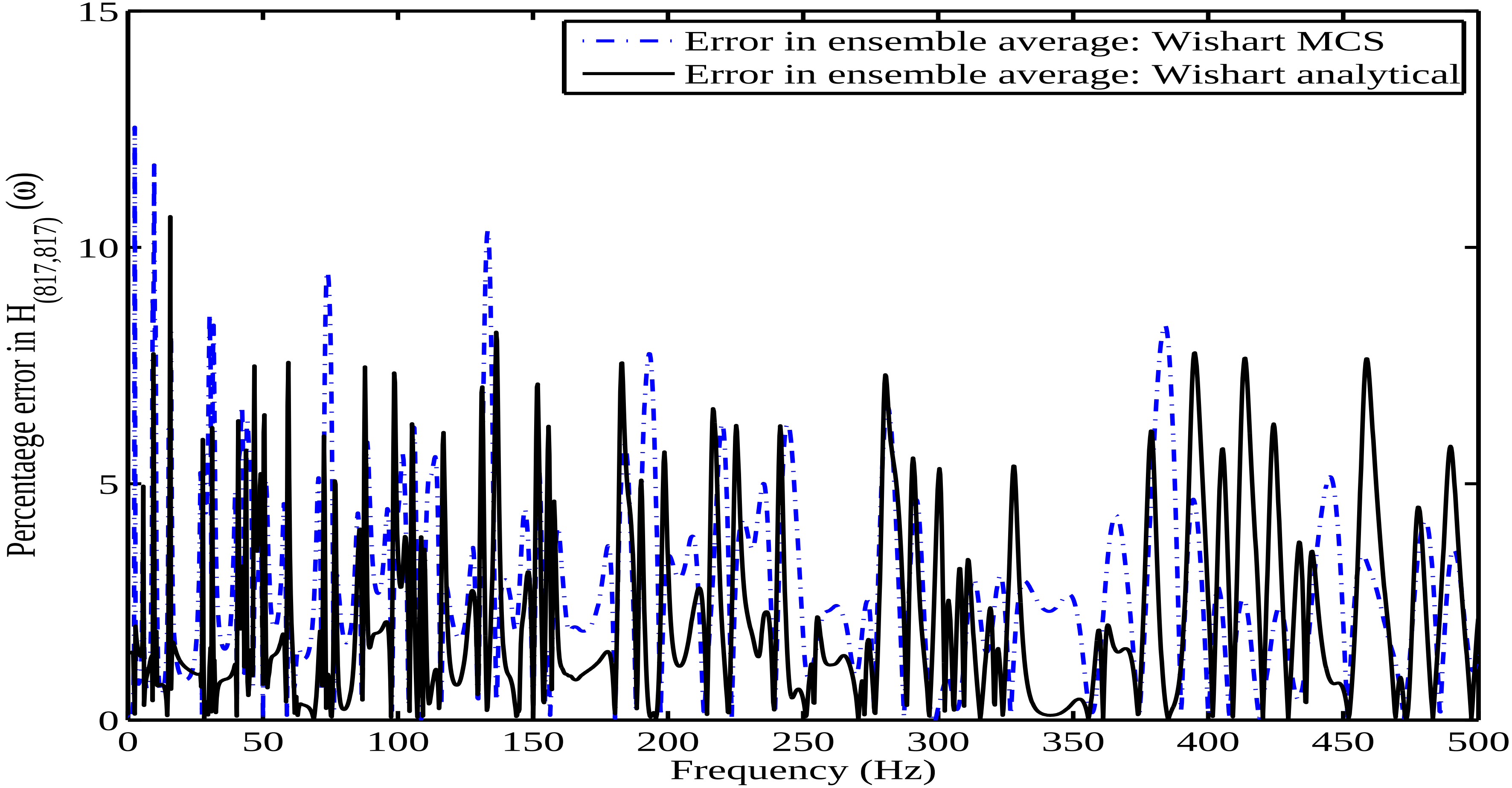
Mean of Driving-point-FRF



Mean of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.0999$ and $\sigma_K = 0.2151$.

Error in the Mean of Driving-point-FRF



Error in the mean of the amplitude of the response of the driving-point-FRF of the plate,

$n = 1200$, $\sigma_M = 0.0999$ and $\sigma_K = 0.2151$.

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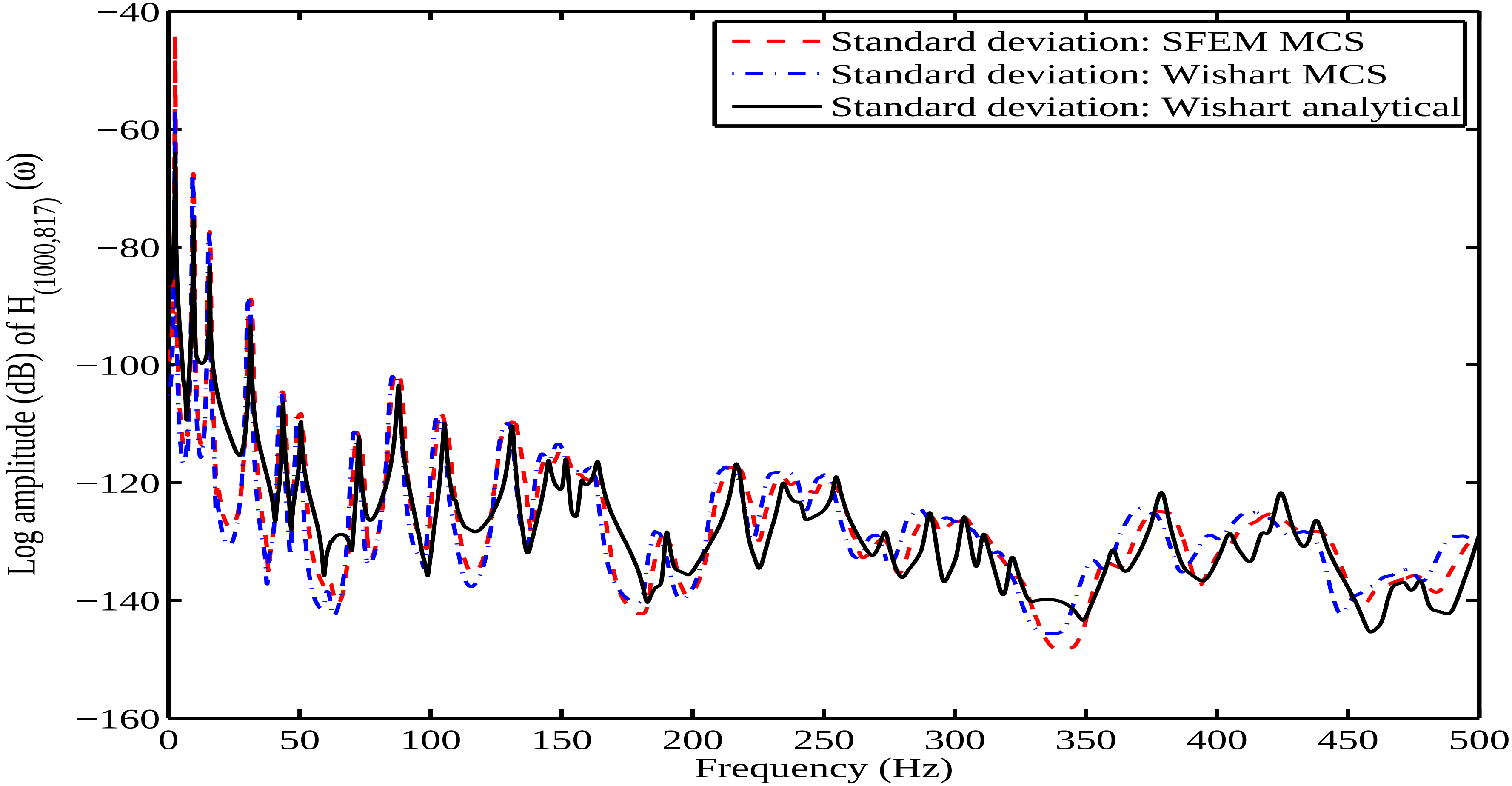
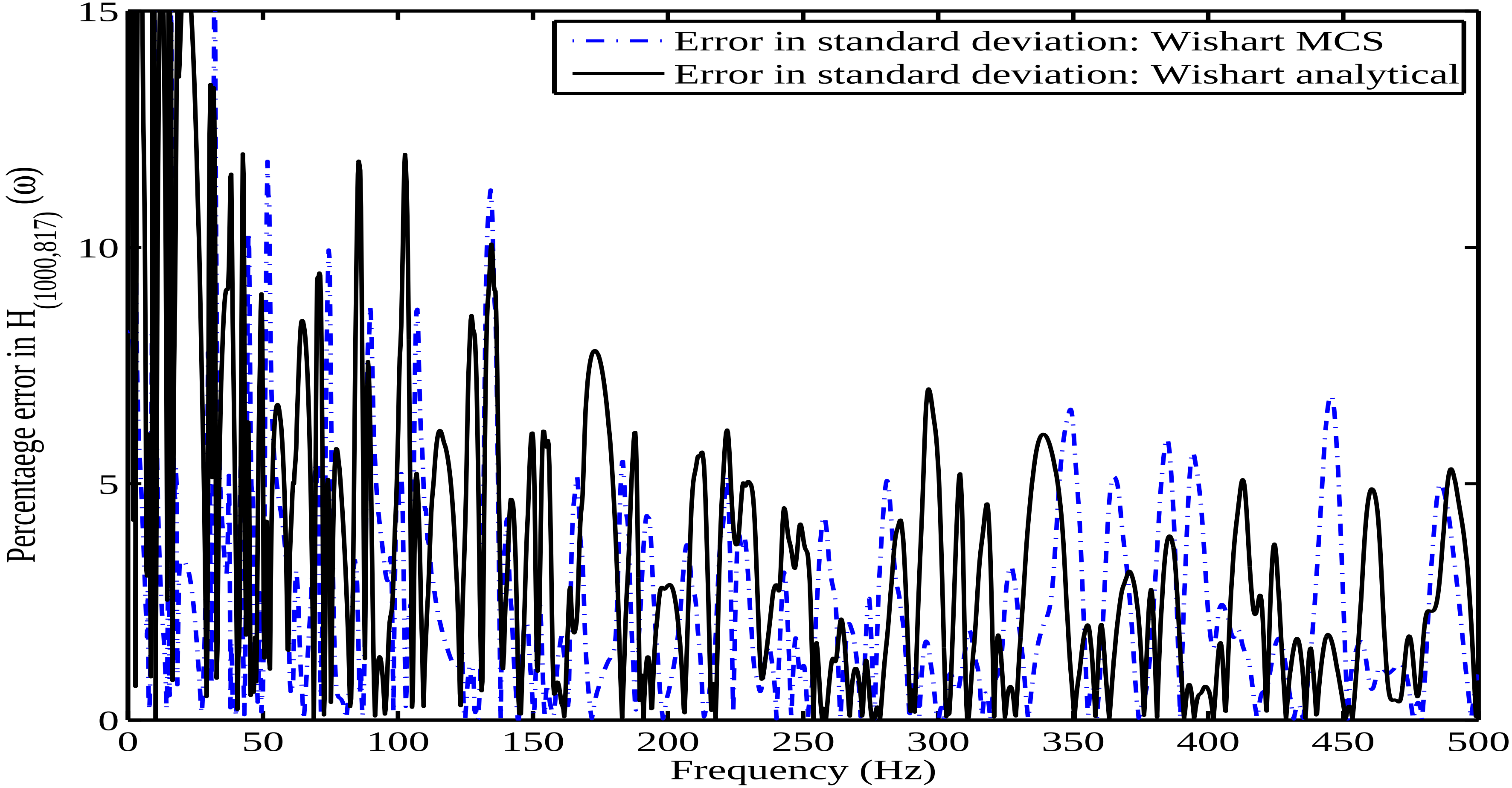
Standard Deviation of Cross-FRF



Standard deviation of the amplitude of the response of the cross-FRF of the plate, $n = 1200$, $\sigma_M = 0.0999$ and $\sigma_K = 0.2151$.

Error in the Standard Deviation of Cross-FRF



Error in the standard deviation of the amplitude of the response of the cross-FRF of the plate,

$n = 1200$, $\sigma_M = 0.0999$ and $\sigma_K = 0.2151$.

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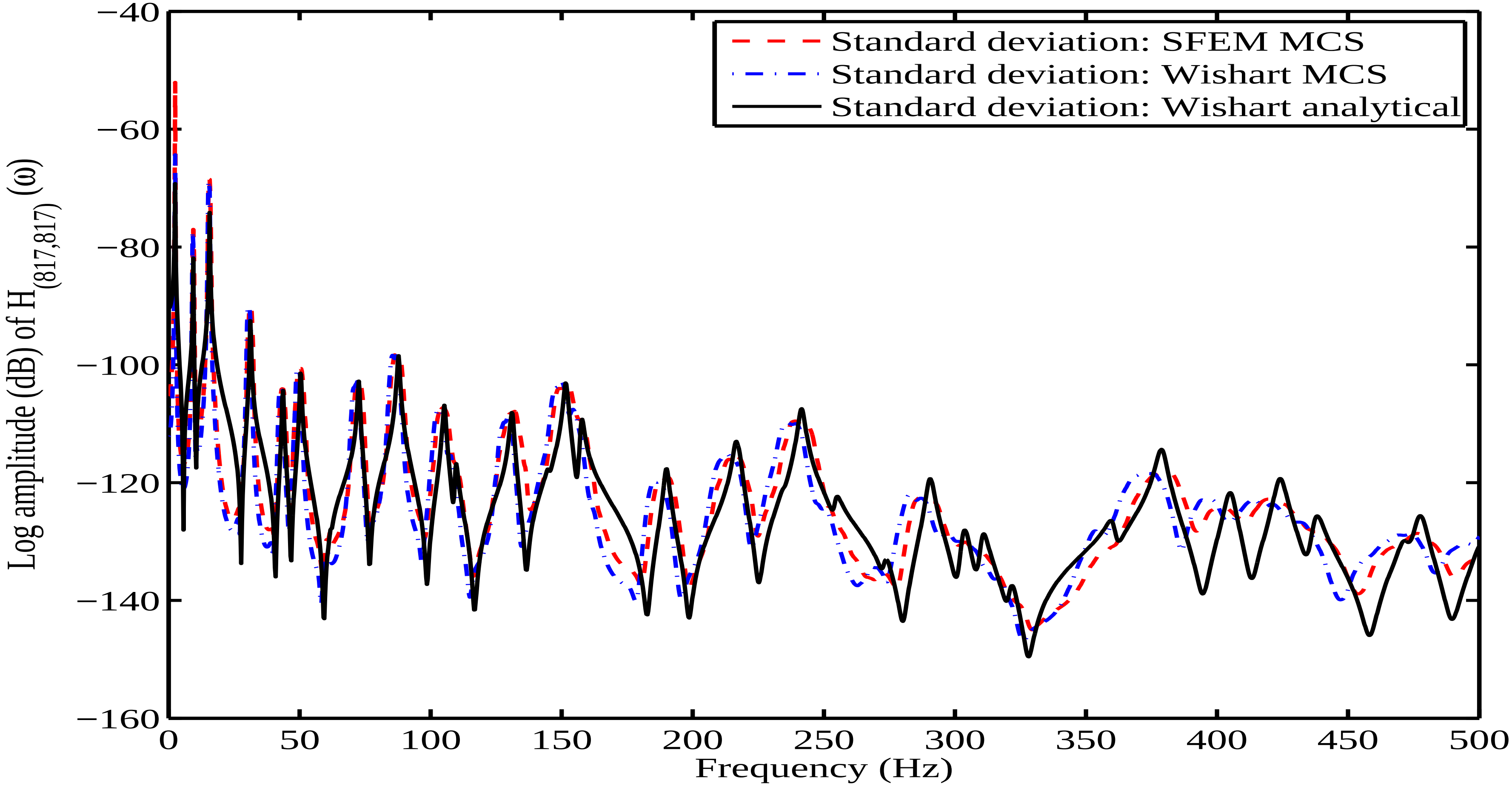
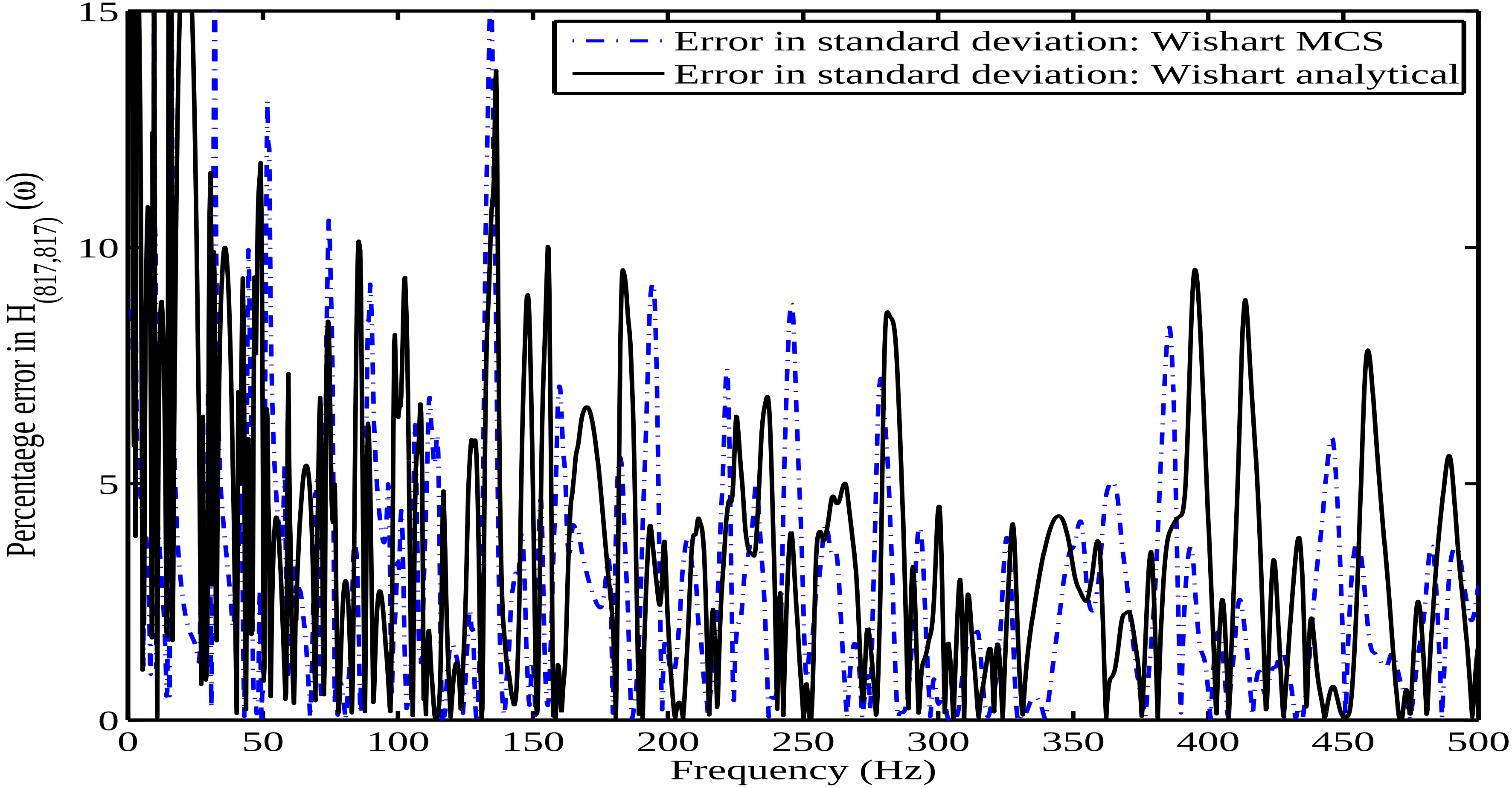
Standard Deviation of Driving-point-FRF



Standard deviation of the amplitude of the response of the driving-point-FRF of the plate,

$n = 1200$, $\sigma_M = 0.0999$ and $\sigma_K = 0.2151$.

Error in the Standard Deviation of Driving-point-FRF



Error in the standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.0999$ and $\sigma_K = 0.2151$.

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Conclusions - 1

- The probabilistic characterization of the response of linear stochastic dynamical systems is considered. This problem requires the solution of a set of coupled complex algebraic equations, which in turn involves the inverse of a complex symmetric random matrix.
- Assuming the damping is small and not random, a Wishart distribution is fitted to the amplitude of the dynamic stiffness matrix for every frequency point.
- Properties of the inverted Wishart distribution are used to obtain the statistics of the amplitude of the dynamic response.

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Conclusions - 2

- Approximate closed-form expressions of the mean and covariance of the amplitude of the dynamic response in the frequency domain is derived. These expressions are simple post-processing of the results corresponding to the baseline system.
- The method is applied to frequency response analysis of a cantilever plate with uncertainties. Error obtained using the proposed analytical method is less than 10% when compared with the results obtained from direct Monte Carlo simulation.
- Proposed method has the potential to solve the 10-10-10 challenge problem.

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