Stochastic response analysis using a random eigenfunction expansion

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A mathematical form for the response of the stochastic finite element analysis of elliptical partial differential equations has been established through summing products of random scalars and random vectors. The method is based upon the eigendecomposition of a systems stiffness matrix. Computational reduction is achieved by approximating the random eigenvalues and the random eigenvectors, and by only summing terms that include dominant eigenvalues. A novel error minimization technique has been applied through the Galerkin error minimization approach. The proposed method is used to analyse the static bending of a stochastic beam. The results obtained through the proposed eigenfunction approach are compared with those obtained by using direct Monte Carlo Simulations and by using polynomial chaos.

Elliptic Stochastic Partial Differential Equation

Uncertainties can substantially affect the analysis of physical structures. These uncertainties can occur in the properties of the material, in the geometry or boundary conditions of the structure or in the applied loads. In order to represent the uncertainties, systems can be discretized through the stochastic finite element method [SFEM]. This method has been applied to numerous static and dynamic problems in the fields of structural mechanics, fluid mechanics and heat transfer. In this work, static systems described by the stochastic elliptic partial differential equation with the suitable Dirichlet boundary conditions have been considered

$$-\nabla[a(x,\omega)\nabla u(x,\omega)] = p(x) \qquad x \text{ in } \mathcal{D} \tag{1}$$

In Equation (**??**) $a : \mathbb{R}^d \times \Omega \to \mathbb{R}(d \leq 3)$ is a random field which we assume to be stationary and square integrable. $\omega \in \Omega$ is a sample point from the sampling space Ω. The random process $a(x,\omega)$ can be expanded by a generalised Fourier expansion known as the Karhunen-Loève expansion

$$a(x,\omega) = a_0(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\widetilde{\xi}_i(\omega)\boldsymbol{\phi}_i(x) \tag{2}$$

where a_0 is the mean function, $\widetilde{\xi}_i(\omega)$ uncorrelated standard Gaussian random variables, and λ_i and $\boldsymbol{\phi}_i(x)$ the eigenvalues and eigenvectors satisfying the autocorrelation function. After truncating the series seen in Equation (**??**) to the Mth term, the resulting equation can be substituted into the original stochastic elliptical partial differential equation. By applying appropriate boundary conditions, the discretized equation takes the form

$$\underbrace{\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\omega)\mathbf{A}_i\right]}_{A(\omega)} \mathbf{u}(\omega) = \mathbf{f} \tag{3}$$

where $\mathbf{A}_0 \in \mathbb{R}^{n\times n}$ represents a deterministic, positive definite, symmetric matrix. $\mathbf{A}_i \in \mathbb{R}^{n\times n}$ are random symmetric matrices for $i = 1, 2, ...M$, $\mathbf{u}(\omega) \in \mathbb{R}^n$ the response vector and $\mathbf{f} \in \mathbb{R}^n$ the deterministic input force vector. Our aim is to propose a new solution approach for this equation.

Polynomial Chaos

After a finite truncation, the polynomial chaos expansion for the solution of Equation (**??**) can be written as a summation of the product of random scalars and deterministic vectors.

$$\mathbf{u}(\omega) = \sum_{k=1}^{P} H_k(\xi(\omega))\mathbf{u}_k \quad \text{where} \quad P = \sum_{j=0}^{p} \frac{(M+j-1)!}{j!(M-1)!} \tag{4}$$

where $H_k(\xi(\omega))$ represents the stochastic scalar polynomial chaoses, and $\mathbf{u}_k$ the deterministic vectors. The value of P is determined by a basic random variable M and by the order of the Polynomial Chaos expansion (p). In this instance, M corresponds to the order of the Karhunen-Loève expansion. However it is evident that P increases rapidly when either M or p are increased.

Random Eigenfunction Approach

The aim of our work is to obtain an expression for the response of Equation (**??**) that can be written as a summation of the product of stochastic scalars and stochastic vectors.
The random eigenvalue problem is initially considered

$$\mathbf{A}(\omega)\boldsymbol{\phi}_k(\omega) = \lambda_k(\omega)\boldsymbol{\phi}_k(\omega); \quad k = 1, 2, \ldots n \tag{5}$$

For convenience, the matrices of the random eigenvalues and eigenvectors of $\mathbf{A}(\omega)$ are defined as follows

$$\begin{aligned}\boldsymbol{\Lambda}(\omega) &= \text{diag}\left[\lambda_1(\omega), \lambda_2(\omega), \ldots, \lambda_n(\omega)\right] \in \mathbb{R}^{n\times n} \quad \text{and} \\ \boldsymbol{\Phi}(\omega) &= \left[\boldsymbol{\phi}_1(\omega), \boldsymbol{\phi}_2(\omega), \ldots, \boldsymbol{\phi}_n(\omega)\right] \in \mathbb{R}^{n\times n}\end{aligned} \tag{6}$$

The random eigenvalues are arranged in ascending order so $\lambda_1(\omega) < \lambda_2(\omega) < \ldots < \lambda_n(\omega)$ and the corresponding eigenvectors are arranged in the same order. Due to the orthogonal property of $\boldsymbol{\Phi}(\omega)$ it is apparent that $\boldsymbol{\Phi}(\omega)^{-1} = \boldsymbol{\Phi}(\omega)^T$. Thus the following identities can be defined (ω has been omitted for notational convenience)

$$\boldsymbol{\Phi}^T\mathbf{A}\boldsymbol{\Phi} = \boldsymbol{\Lambda}; \quad \mathbf{A} = \boldsymbol{\Phi}^{-T}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} \quad \text{and} \quad \mathbf{A}^{-1} = \boldsymbol{\Phi}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Phi}^T \tag{7}$$

Using these identities, the response of Equation (**??**) can be expressed as

$$\mathbf{u}(\omega) = \mathbf{A}(\omega)^{-1}\mathbf{f} \tag{8}$$

$$\mathbf{u}(\omega) = \boldsymbol{\Phi}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Phi}^T\mathbf{f} = \sum_{j=1}^{n} \underbrace{\frac{\boldsymbol{\phi}_j^T(\omega)\mathbf{f}}{\lambda_j(\omega)}}_{a_j(\omega)} \underbrace{\boldsymbol{\phi}_j(\omega)}_{\mathbf{g}_j(\omega)} \tag{9}$$

where $\frac{\boldsymbol{\phi}_j^T(\omega)\mathbf{f}}{\lambda_j(\omega)}$ corresponds to the stochastic scalars and $\boldsymbol{\phi}_j(\omega)$ corresponds to the stochastic vectors. However, calculating the exact values of random scalars ($a_j(\omega)$) and random vectors ($\mathbf{g}_j(\omega)$) is computationally expensive. This has stimulated a reduced method to reduce the computational cost. This can be achieved in two ways:

Truncation: The series given in (**??**) could be truncated after a certain number of terms (t). The high terms of the summation have a relatively low value due to the eigenvalues being ordered ascendingly; this allows the low valued terms to be discarded whilst retaining the dominant terms in the series. Hence Equation (**??**) can be truncated as follows

$$\mathbf{u}(\omega) \approx \sum_{j=1}^{t} \frac{\boldsymbol{\phi}_j^T(\omega)\mathbf{f}}{\lambda_j(\omega)}\boldsymbol{\phi}_j(\omega) \tag{10}$$

where $\lambda_j(\omega)$ and $\boldsymbol{\phi}_j(\omega)$ represent the random eigenvalues and the random eigenvectors.

Approximating the random eigensolutions: Approximating the random eigenvalues and eigenvectors also improves the calculation cost. Due to its efficiency and ease, the first order perturbation method has been used. The jth random eigenvalue and its corresponding random eigenvector is given by

$$\lambda_j = \lambda_{j_0} + \sum_{k=1}^{M} \left(\frac{\partial\lambda_j}{\partial\xi_k}\right)\xi_k(\omega) \tag{11}$$

$$\text{and} \quad \boldsymbol{\phi}_j = \boldsymbol{\phi}_{j_0} + \sum_{k=1}^{M} \left(\frac{\partial\boldsymbol{\phi}_j}{\partial\xi_k}\right)\xi_k(\omega) \tag{12}$$

where $\xi_k(\omega)$ is a set of Gaussian random variables with mean zero and unit variance, and λ_{j_0} and $\boldsymbol{\phi}_{j_0}$ the jth deterministic eigenvalue and eigenvector. The derivatives of the random eigenvalues and eigenvectors are defined by

$$\frac{\partial\lambda_j}{\partial\xi_k} = \boldsymbol{\phi}_j^T\frac{\partial\mathbf{A}}{\partial\xi_k}\boldsymbol{\phi}_j \tag{13}$$

$$\frac{\partial\boldsymbol{\phi}_j}{\partial\xi_k} = \sum_{i=1\neq j}^{N} \alpha_{jki}\boldsymbol{\phi}_k \quad \text{where} \quad \alpha_{jki} = \frac{\boldsymbol{\phi}_{k_0}^T\mathbf{A}_k\boldsymbol{\phi}_{j_0}}{\lambda_{j_0} - \lambda_{k_0}} \tag{14}$$

Galerkin error minimisation

A Galerkin approach is proposed in order to minimise the error. For this new approach, the solution vector has been modified to take the following form

$$\widetilde{\mathbf{u}}(\omega) = \sum_{j=1}^{M} \left(\frac{\boldsymbol{\phi}_j^T(\omega)\mathbf{f}}{\lambda_j(\omega)} + c_j\right)\boldsymbol{\phi}_j(\omega) \tag{15}$$

where $c_j \in \mathbb{R}^M$ are unknown constants to be determined. The error vector for a single realisation is given by

$$\widetilde{\epsilon_\zeta}(\omega) = \mathbf{A}(\omega)\widetilde{\mathbf{u}}(\omega) - \mathbf{f} \tag{16}$$

The unknown constants c_j can be obtained through using a Galerkin approach where the error is made orthogonal to the random eigenvectors

$$< \boldsymbol{\phi}_k(\omega), \widetilde{\epsilon_\zeta}(\omega) >= 0 \quad \forall \quad k = 1, 2, ...t \tag{17}$$

where $< \mathbf{u}, \mathbf{v} >= \mathbb{E}\{\mathbf{u}^T\mathbf{v}\}$ is the inner product. By using this condition, it is possible to obtain a closed form expression for the unknown constants c_j.

$$c_j = \frac{1}{\mathbb{E}\left\{\sum_{j=1}^{t} b_{jk}\right\}}\left[\mathbb{E}\left\{\boldsymbol{\phi}_k^T\mathbf{f}\right\} - \mathbb{E}\left\{\sum_{j=1}^{t}\frac{(b_{jk})(\boldsymbol{\phi}_k^T\mathbf{f})}{\lambda_j}\right\}\right] \quad \forall \quad j = 1, 2, ...t \quad \text{and} \quad k = 1, 2, ...t \tag{18}$$

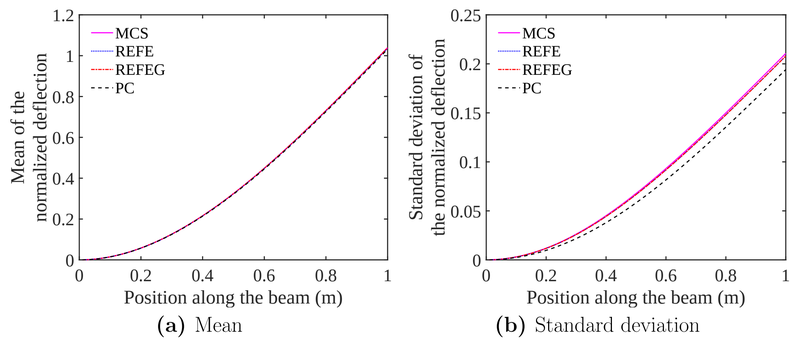
where $b_{jk} = \boldsymbol{\phi}_k^T\mathbf{A}\boldsymbol{\phi}_j$. The arising expected values can be computed using low-order fast Monte Carlo Simulations.

Application

A 1.00 m cantilever beam has been discritized into a 100 elements. A deterministic vertical point load of 1.00 N is applied at the free end of the beam. For the deterministic case, the Young's modulus is $E = 69 \times 10^9$ Nm^{-2} thus corresponding to an aluminium beam and the deterministic second moment of area (moment of inertia) of the beam is $I = 6.75 \times 10^{-11}$ m^4. The bending rigidity of the beam, EI, can be assumed to be a stationary Gaussian random field with a coefficient of variation of 0.2. The KL expansion of the stiffness matrix, given by Equation (**??**), has been truncated and two terms have been kept. The solution for the vertical displacement of the beam has been obtained through four different methods:

- Direct Monte Carlo Simulation applied through directly solving Equation (**??**) (MCS)
- Random eigenvalue eigenfunction expansion (REFE)
- Random eigenvalue eigenfunction expansion including the Galerkin error minimising method (REFEG)
- Polynomial Chaos of order four (PC)

All methods have been simulated 10,000 times. For both the REFE and REFEG methods, Equations (**??**) and (**??**) have been truncated to include the first 5 terms.



(a) Mean **(b)** Standard deviation

Figure 1: Mean and standard deviation of the normalised vertical displacement of the beam

Method	MCS	REFE	REFEG	PC
CPU time (sec)	9.59	0.86	4.07	1.07

Table 1: The CPU times for calculating the response of the beam problem by using the MCS, REFE, REFEG and PC methods

Figures (**??**) and (**??**) illustrate the mean and standard deviation of the normalised vertical displacement at all the nodes of the beam, whilst Table (**??**) contains the CPU times for the four methods attempted.

Conclusion

A mathematical form for the solution of discretized stochastic equations has been presented where $\mathbf{u}(\omega) = \sum_{j=1}^{t} a_j(\omega)\mathbf{g}_j(\omega)$. $a_j(\omega)$ are random scalars and $\mathbf{g}_j(\omega)$ are random vectors, and both $a_j(\omega)$ and $\mathbf{g}_j(\omega)$ can be calculated through a random eigenfunction approach. Through approximating and truncating, the computational time of the random eigenfunction approach has been reduced. The error arising due to the approximation and truncation has been addressed by using a Galerkin approach. In comparison to direct Monte Carlo Simulations, the new approach produces accurate results in a faster computational time. Further work will be carried out to adapt the proposed method to dynamic structures.

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