Cantilever Biosensors: Mass and Rotary Inertia Identification

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Accurate and conclusive identification of bacteria and foreign bodies in cells is a vital part of correctly diagnosing disease and infection. In the past decade the process has been further developed in order to calculate this more accurately. This poster provides an accurate method to identify bacteria to further speed diagnosis. This has been done using cantilever beam theory in order to calculate the mass and rotary inertia which will then be used to find the size and shape of a mass on the end of the cantilever beam. This has been researched theoretically using equations of motion and energy conservation equations, and computationally using finite element method to model the beam and mass.

I. Analytical Analyses

a. Equations of Motion

The first step in identifying the natural frequencies of the cantilever beam analytically is to assemble the equations of motion[1]. This enables us to find the first mode shape and natural frequency (1) for differing values of mass ratio and rotational inertia (2-3).

(1) $\Omega^2 = \omega^2 \frac{mL^4}{EI}$ (non-dimensional frequency parameter)

(2) $\alpha = \frac{M}{mL^4}$ (mass ratio)

(3) $\beta = \frac{J}{mL^3}$ (non-dimensional rotary inertia)

Here E is the Young's Modulus, I is the inertia of the mass, L is the length of the beam in the x-direction, m is the mass per unit length, ω is the rotational speed, M is the mass of the object at the end of the beam and J is rotary inertia. By assuming that the solution will be harmonic it is possible to define ξ (4), the damping ratio. By assuming the equations of motion take a specified form (5) we are able to calculate the equation of motion (6) and hence express the motion (7) in terms of rotary inertia and natural frequency.

(4) $\omega(x,t) = W(\xi)e^{i\omega t}$ where $\xi = \frac{x}{L}$

(5) $W(\xi) = e^{\lambda\xi}$

(6) $W^{\mathrm{IV}}(\xi) - \frac{mL^4}{EI}\omega^2 W(\xi) = 0$

(7) $W^{\mathrm{IV}}(\xi) = \lambda^4 e^{\lambda\xi}$

Where:

(8) $\lambda^4 - \Omega^2 = 0$

$\lambda = \sqrt{\Omega}, -\sqrt{\Omega}, i\sqrt{\Omega}, -i\sqrt{\Omega}$

(9) $W(\xi) = \boldsymbol{S^T}(\xi)\boldsymbol{a}$

(10) $\boldsymbol{S^T} = [\sin(\lambda_1\xi), \cos(\lambda_1\xi), \sinh(\lambda_1\xi), \cosh(\lambda_1\xi)]^{\mathrm{T}}$

(11) $\boldsymbol{a} = [a_1, a_2, a_3, a_4]^{\mathrm{T}}$.

By applying boundary conditions to the equation of motion at x=0 (12-13) and x=L (14-15) we can conclude something about the equation of motion (16).

(12) $W''(0) = 0$

(13) $W^{\mathrm{III}}(0) = 0$

(14) $W^{\mathrm{II}}(1) - \beta\Omega^2 W'(1) = 0$

(15) $W^{\mathrm{III}}(1) + \alpha\Omega^2 W(1) = 0$

(16) $R\boldsymbol{a} = \boldsymbol{0}.$

Where

(17) $R = \Omega^4\beta\alpha + \alpha\beta\Omega^4\cos(\sqrt{\Omega})\cosh(\sqrt{\Omega}) + \alpha\Omega^{5/2}\sin(\sqrt{\Omega})\cosh(\sqrt{\Omega}) - \alpha\Omega^{5/2}\cos(\sqrt{\Omega})\sinh(\sqrt{\Omega}) + \beta\Omega^{7/2}\cos(\sqrt{\Omega})\sinh(\sqrt{\Omega}) + \beta\Omega^{7/2}\sin(\sqrt{\Omega})\cosh(\sqrt{\Omega}) + \Omega^2 - \Omega^2\cos(\sqrt{\Omega})\cosh(\sqrt{\Omega})$.

From these equations it is possible to see that the most vital terms that help to identify the length of the attached mass were the non-dimensional rotary inertia (3), the mass ratio (2) and the non-dimensional frequency parameter (1). Once these variables have been isolated and formed into an equation (17), it can be solved to find the natural frequency for different values of mass ratio and rotary inertia. The natural frequencies of the beam can be calculated by solving (17) for Ω. In order to make this simpler it is acceptable to model α and β at zero values (18).

(18) $R|_{\alpha=0} = \beta\Omega^{7/2}\cos(\sqrt{\Omega})\sinh(\sqrt{\Omega}) + \beta\Omega^{7/2}\sin(\sqrt{\Omega})\cosh(\sqrt{\Omega}) + \Omega^2 - \Omega^2\cos(\sqrt{\Omega})\cosh(\sqrt{\Omega})$

As $\boldsymbol{a}$ (11) is a constant value, it can't be zero, therefore for Eqn (16) to be correct the following (21) must be true; which results in the governing natural frequency (22).

(19) $|R| = 0$

(20) $0 = \Omega^2 - \Omega^2\cos(\sqrt{\Omega})\cosh(\sqrt{\Omega})$

From the graph in **Error! Reference source not found.** it can be clearly perceived that the frequency decreases as the mass ratio increases. It can also be seen that the frequency is most greatly affected by the mass ratio change initially and then the cumulative effect lessens as the mass ratio increases. As there are multiple values of beta used, it was possible to examine the change of frequency with different rotary inertias. It can be observed that for a greater value of rotary inertia the frequency was smaller.

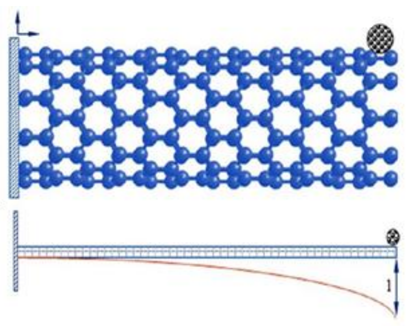

Fig.1: SWNT cantilever beam with mass attached[2]

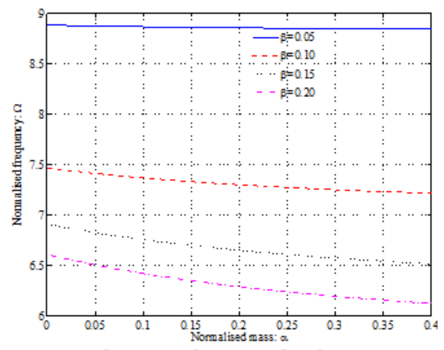

Fig. 1: Change in the normalized resonance frequency of first mode of a nano-cantilever due to variation in the mass and inertia of an attached object

b. Energy Methods

By using the equations of motion in the previous section it is possible to find the first natural frequency, however for this project it is important to investigate into the second natural frequency in addition to this. The energy method equations [3] required finding the overall equivalent mass (21) and the equivalent stiffness (22), in order to make the process of finding the natural frequency simpler.

(21) $m_{eq} = \rho AL[I_1 + \alpha I_2 + \beta I_3]$

(22) $k_{eq} = \frac{EI}{L^3} I_4$

Where

(23) $I_1 = \int_0^1 {y_j}^2(\xi)d\xi$

(24) $I_2 = \left[y_{j\,(x=L,\xi=1)}\right]$

(25) $I_3 = \left[{y_j}'_{(\xi=1)}\right]^2$

(26) $I_4 = \int_0^1 {y_j}''^2(\xi)d\xi$

(27) $\rho AL = m$

(28) $y_j = \left(\cosh(\lambda_j\xi) - \cos(\lambda_j\xi)\right) - \left(\frac{\sinh\lambda_j - \sin\lambda_j}{\cosh\lambda_j - \cos\lambda_j}\right)\left(\sinh(\lambda_j\xi) - \sin(\lambda_j\xi)\right)$

Where y_j is the vibration mode shape, λ_j is the mode of vibration value and j = 1,2. When mass is added to the end of the cantilever beam the overall equivalent mass of the system will change which will in turn change the natural frequency of the beam. The natural frequency of a SWNT can be found using the following equation:

(29) $f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m_{eq}}}$.

The mass ratio (2) and non-dimensional rotary inertia (3) have been previously defined for the expression. Therefore it is possible to display the natural frequency (30) in the form of I_x as below.

(30) $f_n = \frac{c_0}{2\pi}\frac{c_k}{\sqrt{1+\alpha c_m + \beta c_r}}$

The resonant frequency of the beam with no additional mass can then be calculated by substituting in the fact that the mass ratio will be zero.

(31) $f_{0_n} = \frac{c_0}{2\pi}\frac{c_k}{\sqrt{1+\beta c_r}}$

Where

(32) $c_k = \sqrt{\frac{I_4}{I_1}}$ (stiffness calibration constant)

(33) $c_m = \frac{I_2}{I_1}$ (mass calibration constant)

(34) $c_r = \frac{I_3}{I_1}$ (rotary calibration constant).

c. Sensor Equations

The natural frequency of the beam can be expressed in the form of the natural frequency of the beam with no additional mass (54). The frequency change can then be expressed as a function of the natural frequency with no additional mass (55).

(35) $f_n = \sqrt{\frac{1+\beta c_r}{1+\alpha c_m + \beta c_r}} f_{0_n}$

(36) $\Delta f = f_{0_n} - f_n = f_{0_n} - f_{0_n}\sqrt{\frac{1+\beta c_r}{1+\alpha c_m + \beta c_r}}$.

It is now possible to acquire the following expression

(37) $\frac{\Delta f}{f_{0_n}} = 1 - \sqrt{\frac{1+\beta c_r}{1+\alpha c_m + \beta c_r}}$.

Rearranging this equation we obtain the expression

(38) $\alpha = \frac{1+\beta c_r}{\left(1 - \Delta f / f_{0_n}\right)^2} - 1 - \beta c_r$

Hence

(39) $\alpha = \frac{(1+\beta c_r)(2-x)x}{c_m(1-x)^2}$ where $x = \frac{\Delta f}{f_{0_n}}$

From these equations the actual value of the mass added to the end of the SWNT can be calculated as

(40) $M = \frac{mL^4(1+\beta c_r)(2-x)x}{c_m(1-x)^2}$ (60)

This equation relates the mass added to the end of the SWNT to the frequency shift with the calibration constants. However these calibration constants can change depending on the size and shape of the mass added as well as the boundary conditions.

The next step is to calculate the value of the calibration constants, however to do this we need to evaluate the value of the integrals I_{1-4} and the calibration constants (Table 1). This can be done by using the vibration mode shape (28) and the mode of vibration (28).

		Mode Shape	
	Mode shpae	1	2
	Mode of vibration	1.87510407	4.69409113
Integral	1	0.9816621314	0.9922188006
	2	2.2697318239	1.962067694
	3	12.8013923239	88.10845968
	4	3.5779033419	3096.760908
Calibration Constant	Stiffness	1.9091202443	55.8663261
	Mass	2.3121313854	1.977454911
	Rotary	13.0405278093	88.79942572

Table 1: Calculated constants for each mode shape

From the values that have been calculated in Table 1 it is possible to calculate the normalized frequency of the beam at two different mode shapes.

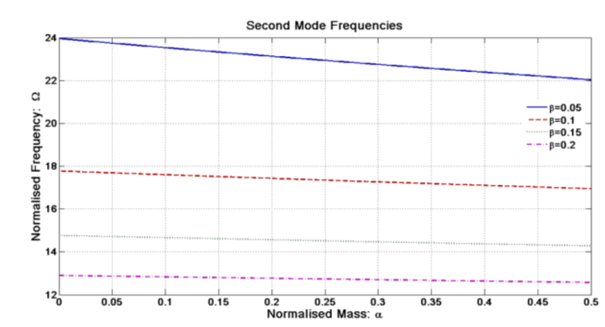

Fig. 2: Change in the normalized resonance frequency of second mode of a nano-cantilever due to variation in the mass and inertia of an attached object

From the graph in Fig. 3 it can be clearly perceived that for the second mode, in addition to the first mode as shown in Fig. 2 the frequency decreases as the mass ratio increases. It can also be seen that the natural frequencies do change with rotary inertia. This means that the frequency will change with a different shape on the end of the mass. Therefore as we can see that as the frequency changes with the volume and shape of the mass it is possible to identify different sizes and shapes of masses on the end of cantilever beams.

d. Conclusion

- The natural frequency of a beam changes with the size and shape of the mass on the end of it
- The natural frequency of the beam is at its lowest when the mass on the end is in the form of the smallest shape of the lowest density
- The natural frequency of the beam is at its highest when the mass on the end of the cantilever beam is in the form of the largest shape and highest density

e. References

[1] S. Adhikari and S. Bhattacharya, Shock and Vibration 19, 2012, Dynamic analysis of wind turbine towers on flexible foundations, 37-56

[2] Picture obtained from "Vibrating Carbon nanotube based biosensors" by R.Chowdhury, S.Adhikari & J.Mitchell

[3] S. Adhikari & R. Chowdhury, Journal of Applied Physics 107, 2010, The calibration of carbon nanotube based bionanosensors, 124322-2