Uncertainty Quantification in Computational Mechanics

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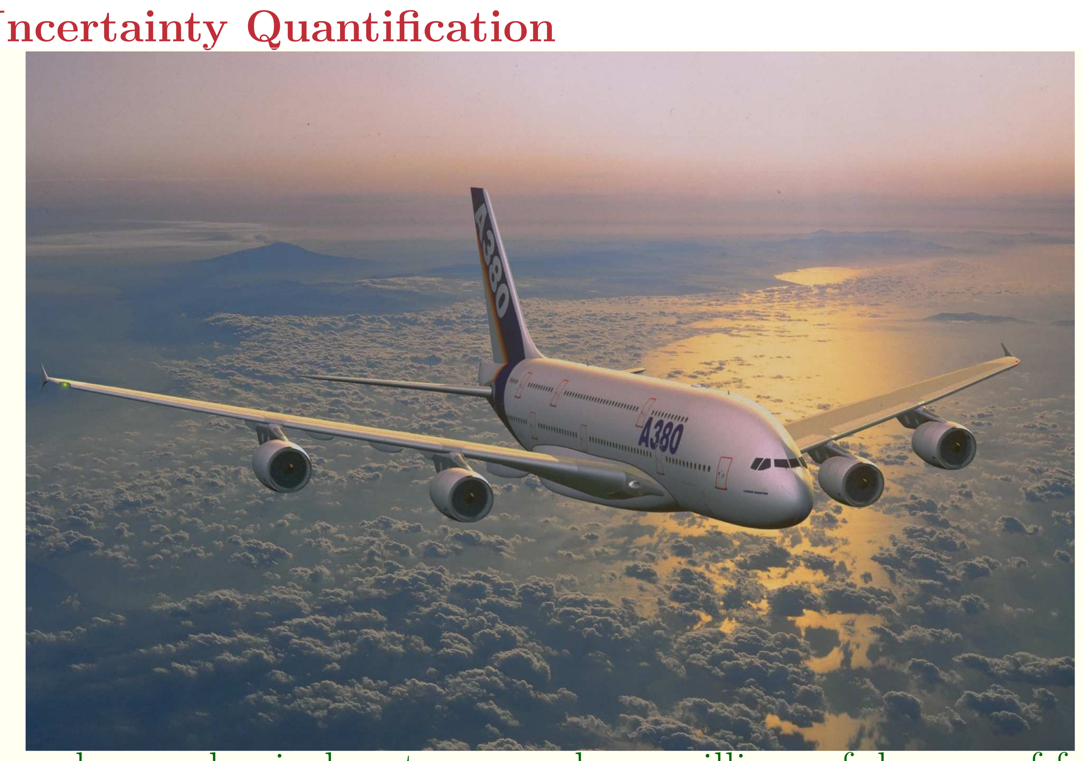
Uncertainties need to be taken into account for credible predictions of the response of complex mechanical systems. Such uncertainties should include uncertainties in the system parameters and those arising due to the modelling of a complex system. In spite of extensive research over the past four decades a general purpose probabilistic predictive code for real-life mechanical systems is still not available. The reasons behind this include: (a) the computational time can be prohibitively high compared to a deterministic analysis, and (b) the detailed and complete information regarding parametric and model uncertainties are in general not available. In this work various methods are investigated to address these two problems in the context of computational mechanics. The proposed methods can be broadly categorised as (a) parametric methods and (b) non-parametric methods. Under the parametric approaches, we have developed (1) doubly spectral stochastic finite elements method, and (2) Gaussian Process (GP) emulator approach. Under the nonparametric approaches we have developed a random matrix based approach. Several numerical and analytical techniques have been proposed and the results were validated against experimental results. This poster is aimed at summarise these works and present some representative results.

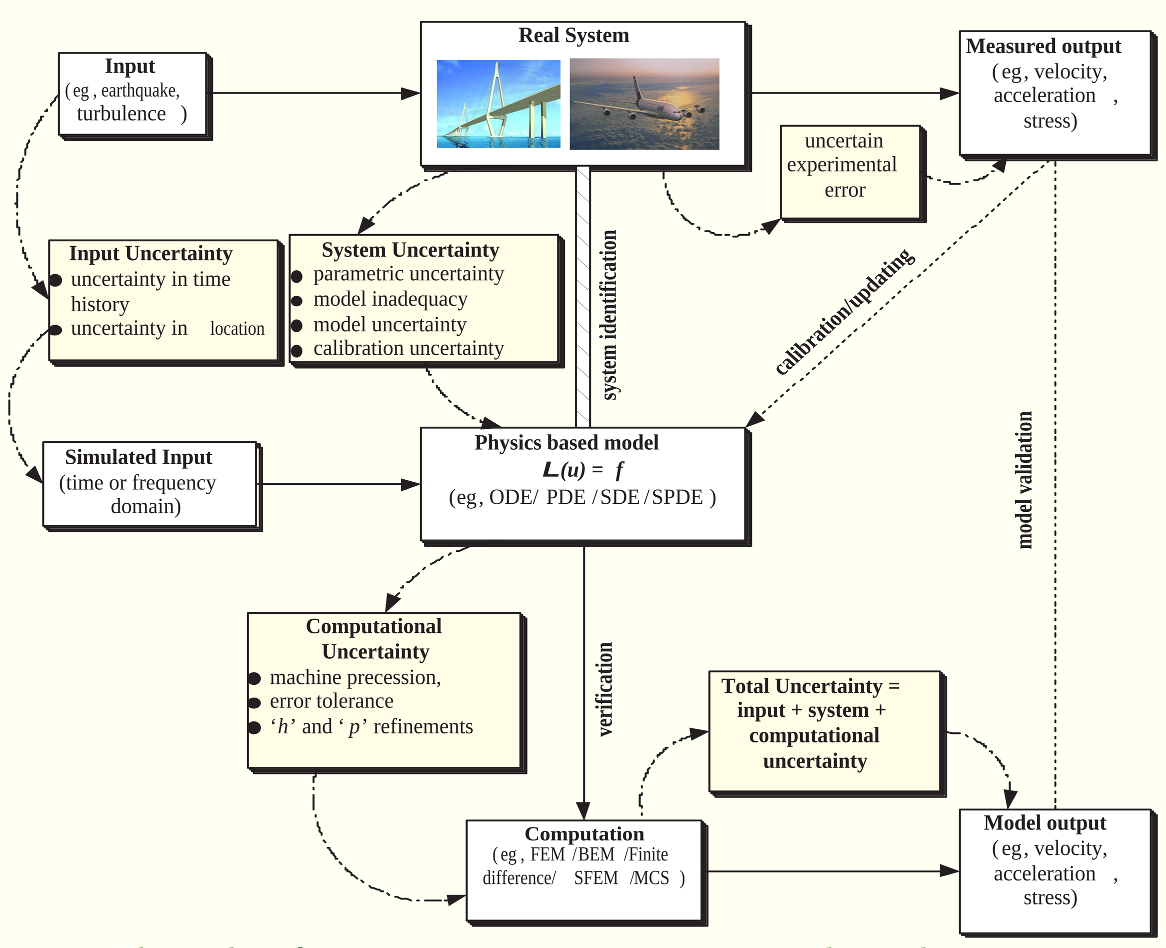
Uncertainty Quantification



Complex mechanical systems can have millions of degrees of freedom and significant uncertainty in their computational models.

The Sources of uncertainty include:

(a) parametric uncertainty - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) model inadequacy - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(d) computational uncertainty - e.g, machine precession, error tolerance and the so called 'h' and 'p' refinements in finite element analysis, and
(e) model uncertainty - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.



The role of uncertainty in computational mechanics.

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (deterministic)	Unknown	*Analysis (forward problem)*	FEM/BEM
Known (deterministic)	Incorrect (deterministic)	Known (deterministic)	*Updating /calibration*	Modal updating
Known (deterministic)	Unknown	Known (deterministic)	*System identification*	Kalman filter
Assumed (deterministic)	Unknown (deterministic)	Prescribed	*Design*	Design optimisation
Unknown	Partially Known	Known	*Structural Health Monitoring (SHM)*	SHM methods
Known (deterministic)	Known (deterministic)	Prescribed	*Control*	Modal control
Known (random)	Known (deterministic)	Unknown	*Random vibration*	Random vibration methods

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (random)	Unknown	*Stochastic analysis (forward problem)*	SFEM /SEA /RMT
Known (random)	Incorrect (random)	Known (random)	*Probabilistic updating /calibration*	Bayesian calibration
Assumed (random/deterministic)	Unknown (random)	Prescribed (random)	*Probabilistic design*	RBOD
Known (random /deterministic)	Partially known (random)	Partially known (random)	*Joint state and parameter estimation*	Particle Kalman Filter/Ensemble Kalman Filter
Known (random /deterministic)	Known (random)	Known (experiment and model)	*Model validation*	Validation methods
Known (random /deterministic)	Known (random)	Known (computations)	*Model verification*	verification methods

Problem-types in computational mechanics.

There are two approaches to quantify uncertainties in a model:

- **Parametric approach**: This is suitable to quantify aleatoric uncertainties. Here the uncertainties associated with the system parameters are quantified and propagated, for example, using the stochastic finite element method[1–7].
- **Nonparametric approach**: This is aimed at quantifying epistemic uncertainty which do not explicitly depend on the system parameters. For example, there can be unquantified errors associated with the equation of motion. Random matrix theory based on central and non-central Wishart distribution[8–10] has been proposed for this purpose.

Doubly Spectral Finite Element Method for Stochastic Field Problems in Structural Dynamics

For distributed parameter systems, parametric uncertainties can be represented by random fields leading to stochastic partial differential equations. A linear damped distributed parameter dynamical system in which the displacement variable $U(\mathbf{r}, t)$, where $\mathbf{r}$ is the spatial position vector and t is time, specified in some domain $\mathcal{D}$, is governed by a linear partial differential equation

$$\rho(\mathbf{r}, \theta)\frac{\partial^2 U(\mathbf{r}, t)}{\partial t^2} + L_1 \frac{\partial U(\mathbf{r}, t)}{\partial t} + L_2 U(\mathbf{r}, t) = p(\mathbf{r}, t); \quad \mathbf{r} \in \mathcal{D}. \quad (1)$$

Here $\rho(\mathbf{r}, \theta)$ is the random mass distribution of the system, $p(\mathbf{r}, t)$ is the distributed time-varying forcing function, L_1 is the random spatial self-adjoint damping operator and L_2 is the random spatial self-adjoint stiffness operator. When parametric uncertainties are considered, the mass density $\rho(\mathbf{r}, \theta)$ as well as the damping and stiffness operators involve random processes. A random process $H(\mathbf{r}, \theta)$ can be expressed in a spectral decomposition as

$$H(\mathbf{r}, \theta) = H_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \xi_i(\theta) \varphi_i(\mathbf{r}) \quad (2)$$

where $\xi_i(\theta)$ are uncorrelated random variables, λ_i and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying an integral equation over the auto-correlation function. Over the past two decades spectral stochastic finite element method has been developed to discretise the random fields based on this decomposition. On the other hand, for deterministic distributed parameter linear dynamical systems, spectral finite element method has been developed to efficiently solve the problem in the frequency domain. In spite of the fact that both approaches use spectral decomposition (one for the random fields and while the other for the dynamic displacement fields), there has been very little overlap between them in literature. In this work these two spectral techniques have been unified with the aim that the unified approach would outperform any of the spectral methods considered on its own.

Random Matrix Theory

The equation of motion of a damped n-degree-of-freedom linear dynamic system can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \quad (3)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices respectively. In order to completely quantify the uncertainties associated with system (3), we need the probability density functions of the random matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. We have shown[8–11] that these matrices can be expresses as central or non-central Wishart matrices:

Wishart matrix: An $n \times n$ random symmetric positive definite matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{2^{\frac{1}{2}np}\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1}|\mathbf{S}|^{\frac{1}{2}(p-n-1)}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S}\right\} \quad (4)$$

Noncentral Wishart matrix: A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a noncentral Wishart distribution with parameters $p \geq n$, $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Theta} \in \mathbb{R}_n^+$, if its pdf is given by

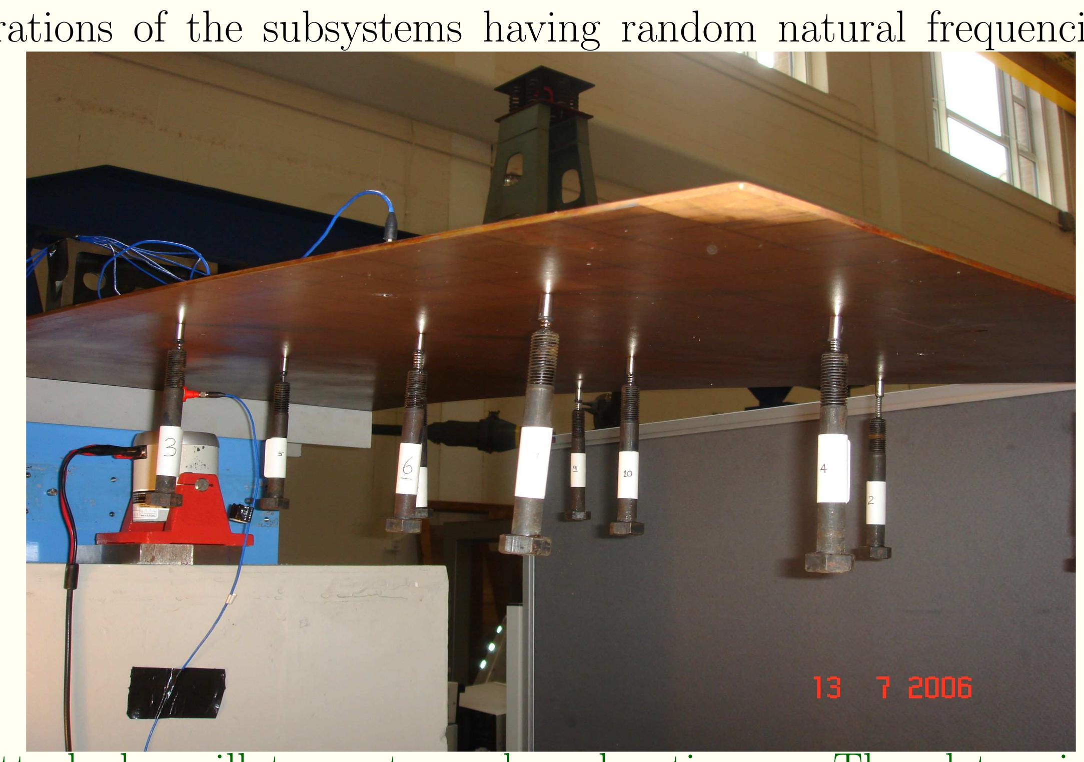
$$p_{\mathbf{S}}(\mathbf{S}) = \left\{2^{\frac{1}{2}np}\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Theta}\right\}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S}\right\} |\mathbf{S}|^{\frac{1}{2}(p-n-1)}\,{}_0F_1(p/2, \mathbf{\Theta}\mathbf{\Sigma}^{-1}\mathbf{S}/4). \quad (5)$$

where ${}_0F_1$ the hypergeometric function (Bessel function) of matrix argument.

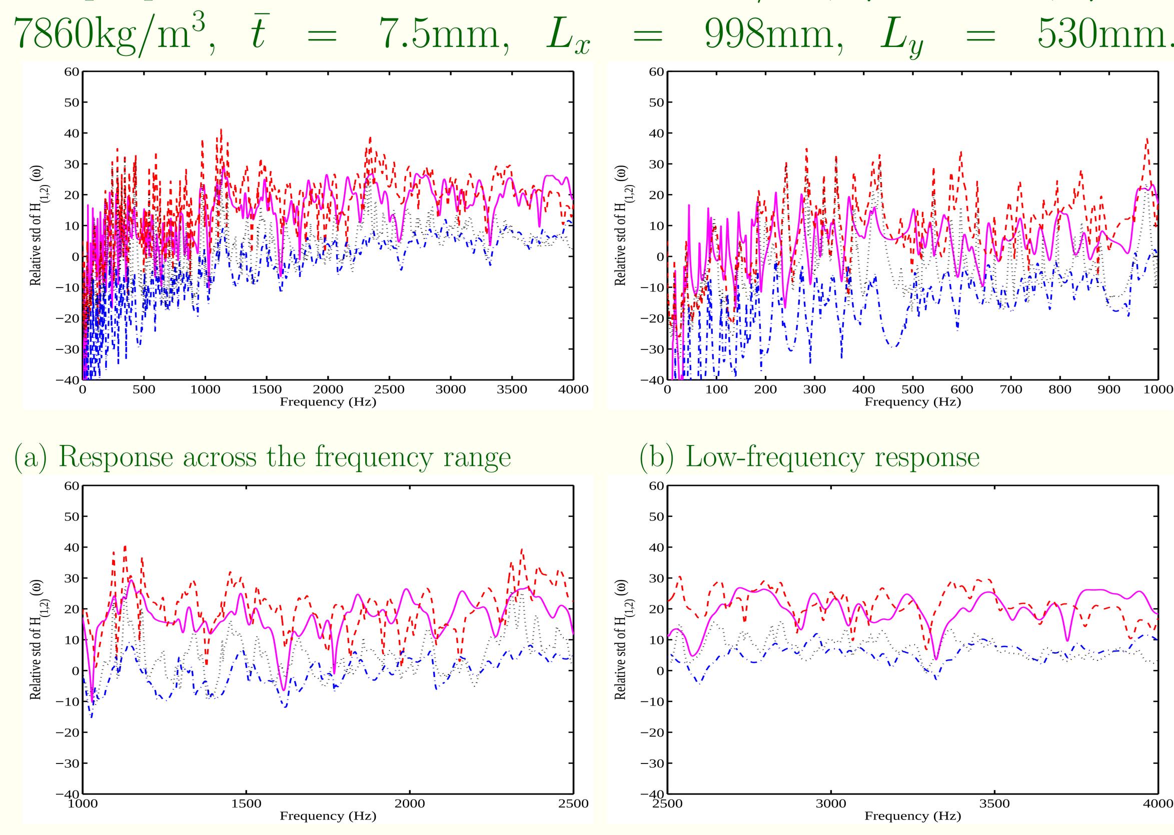
Here $\mathrm{etr}\{\bullet\} \equiv \exp\{\mathrm{Tr}(\bullet)\}$ and $|\bullet| \equiv$ determinant of a matrix. The function $\Gamma_n(a)$ is the multivariate gamma function, which can be expressed as $\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)}\prod_{k=1}^{n}\Gamma\left[a - \frac{1}{2}(k-1)\right]$; for $\Re(a) > \frac{1}{2}(n-1)$.

Experimental Validation

The uncertain dynamics is realized by 10 sprung-mass oscillators with randomly distributed stiffness properties attached at random locations. One hundred nominally identical dynamical systems are created and individually tested. The probabilistic characteristics of the frequency response functions are obtained in the low, medium and high frequency ranges. Special measures were taken so that the uncertainty in the response of the main structure primarily emerges from the random attachment configurations of the subsystems having random natural frequencies.



Attached oscillators at random locations. The deterministic properties are: $\bar{E} = 200 \times 10^9 \mathrm{N/m}^2$, $\bar{\mu} = 0.3$, $\bar{\rho} = 7860 \mathrm{kg/m}^3$, $\bar{t} = 7.5\mathrm{mm}$, $L_x = 998\mathrm{mm}$, $L_y = 530\mathrm{mm}$.



(a) Response across the frequency range (b) Low-frequency response
(c) Medium-frequency response (d) High-frequency response

Comparison of the mean and standard deviation of the amplitude of the near-field cross-FRF of the plate at point 2 (nodal coordinate: (6,11)) with 10 randomly placed oscillators; — ensemble mean from Wishart model; - - - ensemble mean from experiment; -. -. standard deviation from Wishart model; .... standard deviation from experiment.

Selected Publications

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[6] Adhikari, S., "Asymptotic distribution method for structural reliability analysis in high dimensions," *Proceedings of the Royal Society of London, Series- A*, Vol. 461, No. 2062, 2005, pp. 3141–3158.
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[9] Adhikari, S., "On the quantification of damping model uncertainty," *Journal of Sound and Vibration*, Vol. 305, No. 1-2, September 2007, pp. 153–171.
[10] Adhikari, S., "Wishart random matrices in probabilistic structural mechanics," *ASCE Journal of Engineering Mechanics*, 2008, accepted for publication.
[11] Adhikari, S., "An unified parametric-nonparametric uncertainty quantification approach for linear dynamical systems," *48th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics & Materials Conference*, AIAA, Waikiki, Hawaii, USA, April 2007.

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