An Unified Uncertainty Quantification Approach for Structural Dynamic Models

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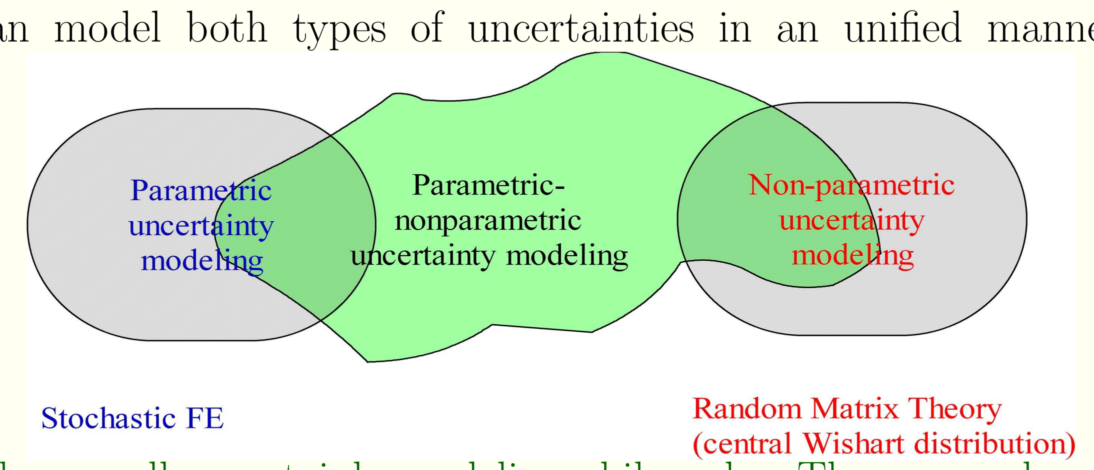
Uncertainties are unavoidable in the description of real-life engineering systems. The quantification of uncertainties plays a crucial role in establishing the credibility of a numerical model. Uncertainties can be broadly divided into two categories. The first type is due to the inherent variability in the system parameters, often referred to as aleatoric uncertainty. If enough samples are present, it is possible to characterize the variability using well established statistical methods and consequently the probably density functions (pdf) of the parameters can be obtained. The second type of uncertainty is due to the lack of knowledge regarding a system, often referred to as epistemic uncertainty. This kind of uncertainty generally arise in the modelling of complex systems, for example, in the modeling of cabin noise in helicopters. Due to its very nature, it is comparatively difficult to quantify or model this type of uncertainties. In this work a new parametric-nonparametric uncertainty quantification tool, which can model both types of uncertainties in an unified manner, is developed. The new method is based on random matrix theory and entropy optimization method.

Uncertainty Quantification

There are two approaches to quantify uncertainties in a model:

- **Parametric approach**: This is suitable to quantify aleatoric uncertainties. Here the uncertainties associated with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficient and geometric parameters are quantified and propagated, for example, using the stochastic finite element method[1;2].
- **Nonparametric approach**: This is aimed at quantifying epistemic uncertainty which do not explicitly depend on the system parameters. For example, there can be unquantified errors associated with the equation of motion (linear on non-linear), in the damping model (viscous or non-viscous), in the model of structural joints, and also in the numerical methods. Random matrix theory based on central Wishart distribution[3-5] has been proposed for this purpose.

Although we made differences between the two different types of uncertainties, in practical problems it is in general very difficult, if not impossible, to distinguish them. To come up with a credible numerical model of complex systems, we need to quantify and model both types of uncertainties *simultaneously*. The aim of this work is to develop a parametric-nonparametric uncertainty quantification tool which can model both types of uncertainties in an unified manner.



The overall uncertainly modeling philosophy. The approach proposed here is based on the random matrix theory and the maximum entropy method.

Random Matrix Theory

The probability density function of a random matrix[6;7] can be defined in a manner similar to that of a random variable. If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$.

Wishart matrix: An $n \times n$ random symmetric positive definite matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{2^{\frac{1}{2}np}\,\Gamma_n\left(\frac{1}{2}p\right)|\boldsymbol{\Sigma}|^{\frac{1}{2}p}\right\}^{-1}|\mathbf{S}|^{\frac{1}{2}(p-n-1)}\text{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right\} \quad (1)$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma})$.

Noncentral Wishart matrix: A $n \times n$ symmetric positive definite random matrix $\mathbf{S}$ is said to have a noncentral Wishart distribution with parameters $p \geq n$, $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$ and $\boldsymbol{\Theta} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{2^{\frac{1}{2}np}\,\Gamma_n\left(\frac{1}{2}p\right)|\boldsymbol{\Sigma}|^{\frac{1}{2}p}\right\}^{-1}\text{etr}\left\{-\frac{1}{2}\boldsymbol{\Theta}\right\}\text{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{S}\right\} |\mathbf{S}|^{\frac{1}{2}(p-n-1)}\,{}_0F_1(p/2, \boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\mathbf{S}/4). \quad (2)$$

where ${}_0F_1$ the hypergeometric function (Bessel function) of matrix argument. This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$. If the noncentrality parameter $\boldsymbol{\Theta}$ is a null matrix, then noncentral Wishart distribution in Eq. (2) reduces to the Wishart distribution in Eq. (1).

Here $\text{etr}\{\bullet\} \equiv \exp\{\text{Tr}(\bullet)\}$ and $|\bullet| \equiv$ determinant of a matrix. The function $\Gamma_n(a)$ is the multivariate gamma function, which can be expressed as $\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)}\prod_{k=1}^{n}\Gamma\left[a - \frac{1}{2}(k-1)\right]$; for $\Re(a) > \frac{1}{2}(n-1)$.

The Unified Distribution of the System Matrices

The equation of motion of a damped n-degree-of-freedom linear dynamic system can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \quad (3)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices respectively. In order to completely quantify the uncertainties associated with system (3), we need the probability density functions of the random matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. Once this is done, the uncertainty in the response $\mathbf{q}(t)$ can be obtained using uncertainty propagation methods (such as the Monte Carlo simulation).

The Limitations of the Nonparametric Distribution:

We will use the notation $\mathbf{G}$ which stands for any one the system matrices. Suppose the matrix variate density function of $\mathbf{G} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{G}}(\mathbf{G}) : \mathbb{R}_n^+ \to \mathbb{R}$. According to the nonparametric distribution[3-5], $\mathbf{G}$ has the central Wishart distribution, i.e., $\mathbf{G} \sim W_n(p, \boldsymbol{\Sigma})$. There is only one parameter (p) which controls the uncertainty. Whereas any $n \times n$ symmetric matrix $\mathbf{G}$ can have $N = n(n+1)/2$ number of independent elements. Which implies that it can have $N(N+1)/2 = n(n+1)(n(n+1)+2)/8$ number of independent elements in its covariance matrix. The nonparametric approach therefore is a gross oversimplification of the naturae of the uncertainty.

Derivation of the Unified Distribution:

The distribution of random system matrices should be such that they are (a) symmetric and (b) positive-definite. Because $\mathbf{G}$ is a symmetric and positive-definite random matrix, it can be always be factorized as

$$\mathbf{G} = \mathbf{X}\mathbf{X}^T \quad (4)$$

where $\mathbf{X} \in \mathbb{R}^{n \times m}, m \geq n$ is in general a rectangular matrix. Since the factorization in Eq. (4) will always guarantee the satisfaction of the the symmetry and the positive-definiteness condition, we assume that this is *the* form of the random system matrices arising in structural dynamics. Now we need to study the probabilistic nature of the random matrix $\mathbf{X}$. Once the pdf of $\mathbf{X}$ is known, the pdf $\mathbf{G}$ will be derived using the non-linear matrix transformation in Eq. (4).

Suppose the mean of $\mathbf{X}$ is $\mathbf{M} \in \mathbb{R}^{n \times m}, m \geq n$ and the covariance tensor of $\mathbf{X}$ is given by $\boldsymbol{\Sigma} \otimes \mathbf{I}_m \in \mathbb{R}^{nm \times nm}$ where $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$. How to obtain m, $\mathbf{M}$ and $\boldsymbol{\Sigma}$ will be discussed later. At this point it is suffice to assume that the mean and covariance of the random matrix $\mathbf{X}$ exist. Suppose the matrix variate probability density function of $\mathbf{X} \in \mathbb{R}^{n \times m}$ is given by $p_{\mathbf{X}}(\mathbf{X}) : \mathbb{R}^{n \times m} \to \mathbb{R}$. We have the following information and constrains to obtain $p_{\mathbf{X}}(\mathbf{X})$:

$$\int_{\mathbf{X} \in \mathbb{R}^{n \times m}} p_{\mathbf{X}}(\mathbf{X})\, d\mathbf{X} = 1 \quad \text{(normalization)}$$

$$\int_{\mathbf{X} \in \mathbb{R}^{n \times m}} \mathbf{X}\, p_{\mathbf{X}}(\mathbf{X})\, d\mathbf{X} = \mathbf{M} \quad \text{(the mean matrix)}$$

$$\text{and} \quad \int_{\mathbf{X} \in \mathbb{R}^{n \times m}} [\mathbf{X} - \mathbf{M}] \otimes [\mathbf{X} - \mathbf{M}]^T p_{\mathbf{X}}(\mathbf{X})\, d\mathbf{X} = \boldsymbol{\Sigma} \otimes \mathbf{I}_m.$$

Using the maximum entropy approach, the probability density function of $\mathbf{X}$ (a Gaussian random matrix) can be obtained as

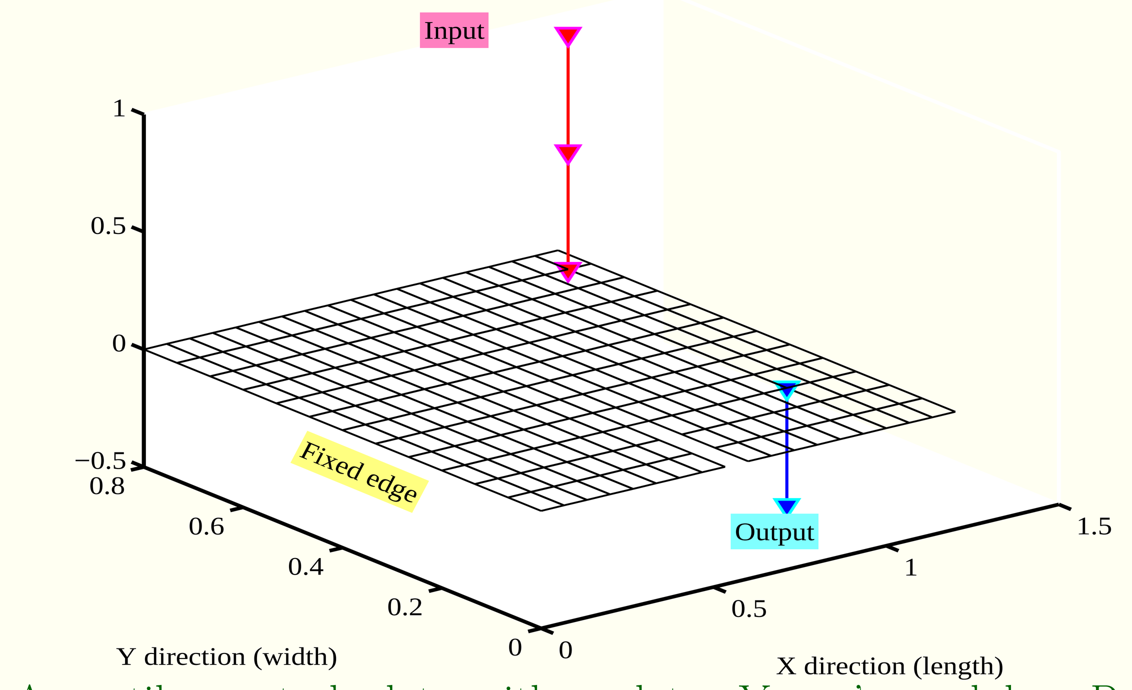
$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-nm/2}|\boldsymbol{\Sigma}|^{-m/2}\,\text{etr}\left\{-\frac{1}{2}\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \mathbf{M})(\mathbf{X} - \mathbf{M})^T\right\}. \quad (5)$$

Using this pdf in Eq. (4), it can be shown that the pdf of $\mathbf{G}$ can be expressed by non-central Wishart distribution given by Eq. (2). From this discussion, we have the following basic result:

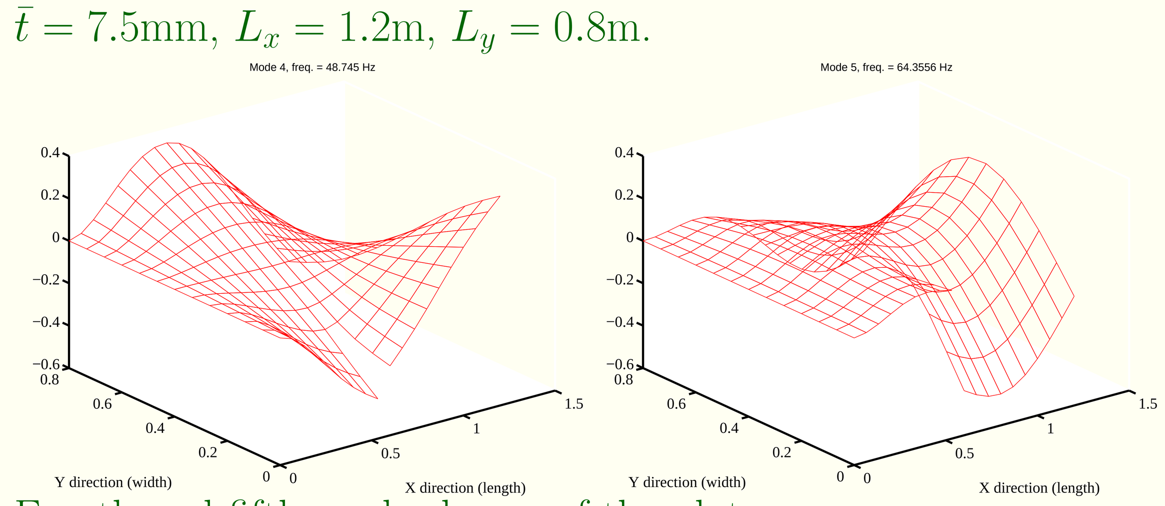
Theorem 1. *The unified parametric-nonparametric probability density function a random system matrix* $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ *follows the non-central Wishart distribution, that is,* $\mathbf{G} \sim W_n(m, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$.

Here the parameter $\boldsymbol{\Theta} = \boldsymbol{\Sigma}^{-1}\mathbf{M}\mathbf{M}^T \in \mathbb{R}_n^+$. The unknown parameters of the unified distributions are $m \in \mathbb{R}$, $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$ and $\boldsymbol{\Theta} \in \mathbb{R}_n^+$. There are in total $n + n(n+1)$ number of unknowns in this distribution. A least-square error minimization approach based on the expressions of the mean and covariance matrix of the noncentral Wishart distribution have been developed.

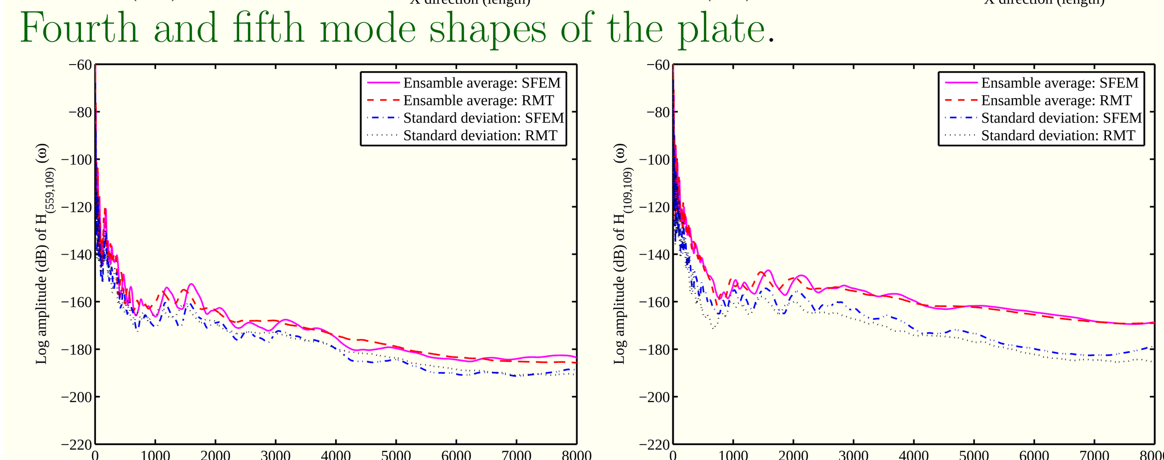
Example: A Plate With Random Properties



A cantilever steel plate with a slot. Young's modulus, Poissons ratio, mass density and thickness are random fields of the form $E(\mathbf{x}) = \bar{E}\,(1 + \epsilon_E f_1(\mathbf{x}))$, $\mu(\mathbf{x}) = \bar{\mu}\,(1 + \epsilon_\mu f_2(\mathbf{x}))$, $\rho(\mathbf{x}) = \bar{\rho}\,(1 + \epsilon_\rho f_3(\mathbf{x}))$ and $t(\mathbf{x}) = \bar{t}\,(1 + \epsilon_t f_4(\mathbf{x}))$. The deterministic properties are: $\bar{E} = 200 \times 10^9 \text{N/m}^2$, $\bar{\mu} = 0.3$, $\bar{\rho} = 7860 \text{kg/m}^3$, $\bar{t} = 7.5$mm, $L_x = 1.2$m, $L_y = 0.8$m.



Fourth and fifth mode shapes of the plate.



Comparison of the mean and standard deviation of the amplitude of the frequency response function obtained using the direct stochastic finite element simulation and proposed unified distribution.

Conclusions

- A new unified parametric-nonparametric uncertainty quantification method for linear dynamical systems has been proposed.
- The main outcome of this study is that the unified parametric-nonparametric probability density function a random system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ follows the non-central Wishart distribution, that is, $\mathbf{G} \sim W_n(m, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$.
- The discovery of the non-central Wishart distribution in this context is significant. On the one hand it eliminates some of the drawbacks of the nonparametric distribution, and in the other hand it can incorporate some parametric features while keeping the nonparametric features unchanged.

Application of this modeling technology to complex aerospace systems are currently being investigated.

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Acknowledgements: The author acknowledges the support of the Engineering and Physical Sciences Research Council (EPSRC) through the award of an advanced research fellowship, grant number GR/T03369/01.