Uncertainty Propagation and Quantification in Structural Dynamics

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Description of real-life engineering structural systems is inevitably associated with some amount of uncertainty in specifying material properties, geometric parameters, boundary conditions and applied loads. In order to come up with high-fidelity numerical models, it is required to consider these uncertainties explicitly. There are various ways to model uncertainties, for example, probabilistic methods, interval algebra, fuzzy sets and convex sets to mention only a few. In this work probabilistic models of uncertainties are considered and two different types of problems are studied. In the uncertainty propagation problem, given the probabilistic description of the system parameters we develop tools to obtain probabilistic descriptions of the natural frequencies, mode shapes and dynamic response. The second problem, uncertainty quantification is the inverse problem. Here measurements obtained from a number of nominally identical structures are used to estimate the probability density function of the unknown parameters.

Uncertainty Propagation

Suppose that the system parameters, for example, material properties, geometric parameters and boundary conditions can be casted in a m-dimensional vector $\mathbf{x}$. Statistical properties of the system are completely described by the joint probability density function $p_{\mathbf{X}}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}$. Once this information is given, the objective is to obtain the probabilistic description of the natural frequencies, mode shapes and dynamic response. Recent works have considered the following problems of wide interest:

- *Random Eigenvalue Problems*
- *Probabilistic Reliability Analysis*
- *Reliability Based Optimal Design*

Random Eigenvalue Problems

Dynamic characteristics of linear structural systems are governed by the natural frequencies and the mode-shapes. The determination of natural frequency and mode shapes require the solution of an eigenvalue problem. When we take account of uncertainties in the system, it is necessary to consider random eigenvalue problems. The random eigenvalue problem of undamped or proportionally damped systems can be expressed by

$$\mathbf{K}(\mathbf{x})\boldsymbol{\phi}_j = \lambda_j \mathbf{M}(\mathbf{x})\boldsymbol{\phi}_j \qquad (1)$$

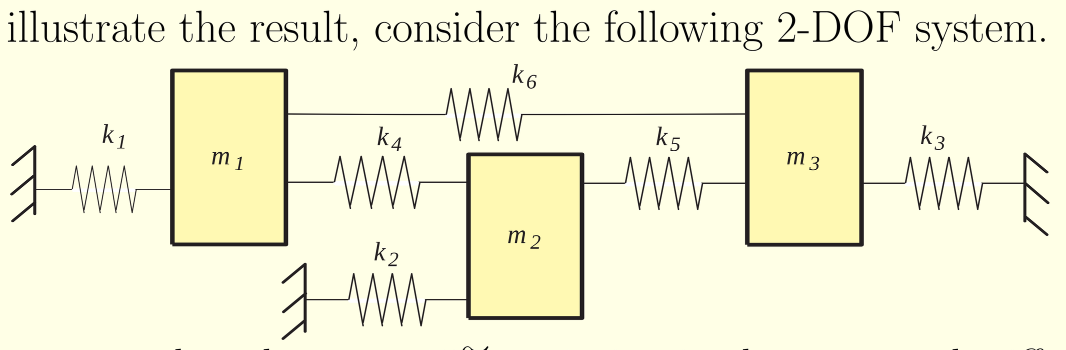
Here λ_j and $\boldsymbol{\phi}_j$ are the eigenvalues and the eigenvectors of the dynamic system. $\mathbf{M}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$ and $\mathbf{K}(\mathbf{x}) : \mathbb{R}^m \mapsto \mathbb{R}^{N\times N}$, the mass and stiffness matrices. Current methods to solve such problems are dominated by mean-centered perturbation based methods. In this work new tools are being developed [1, 2, 3] to obtain moments, cumulants and probability density functions of the eigenvalues. It is shown that an arbitrary rth order moment of the eigenvalues can be obtained from

$$\mu_j^{(r)} = \mathrm{E}\left[\lambda_j^r(\mathbf{x})\right] = \int_{\mathbb{R}^m} \lambda_j^r(\mathbf{x}) p_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \approx (2\pi)^{m/2} \lambda_j^r(\boldsymbol{\theta}) \exp\left\{-L(\boldsymbol{\theta})\right\} \left\| \mathbf{D}_L(\boldsymbol{\theta}) + \frac{1}{r}\mathbf{d}_L(\boldsymbol{\theta})\mathbf{d}_L(\boldsymbol{\theta})^T - \frac{r}{\lambda_j(\boldsymbol{\theta})}\mathbf{D}_{\lambda_j}(\boldsymbol{\theta}) \right\|^{-1/2}$$

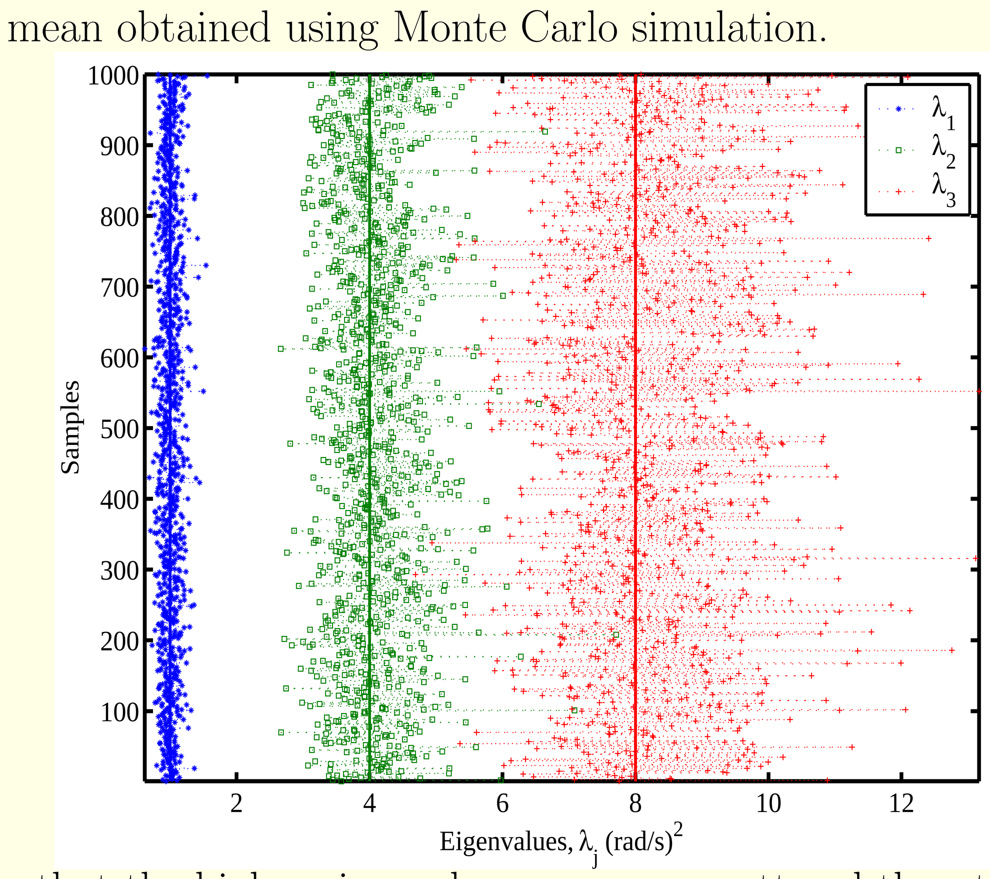
Here $L(\mathbf{x}) = -\ln p_{\mathbf{X}}(\mathbf{x})$, $\mathbf{d}_{(\bullet)} \in \mathbb{R}^m$ is the gradient vector and $\mathbf{H}_{(\bullet)} \in \mathbb{R}^{m\times m}$ is the Hessian matrix. The optimal point $\boldsymbol{\theta}$ can be obtained from

$$\mathbf{d}_{\lambda_j}(\boldsymbol{\theta}) r = \lambda_j(\boldsymbol{\theta}) \mathbf{d}_L(\boldsymbol{\theta}) \qquad (2)$$

To illustrate the result, consider the following 2-DOF system.

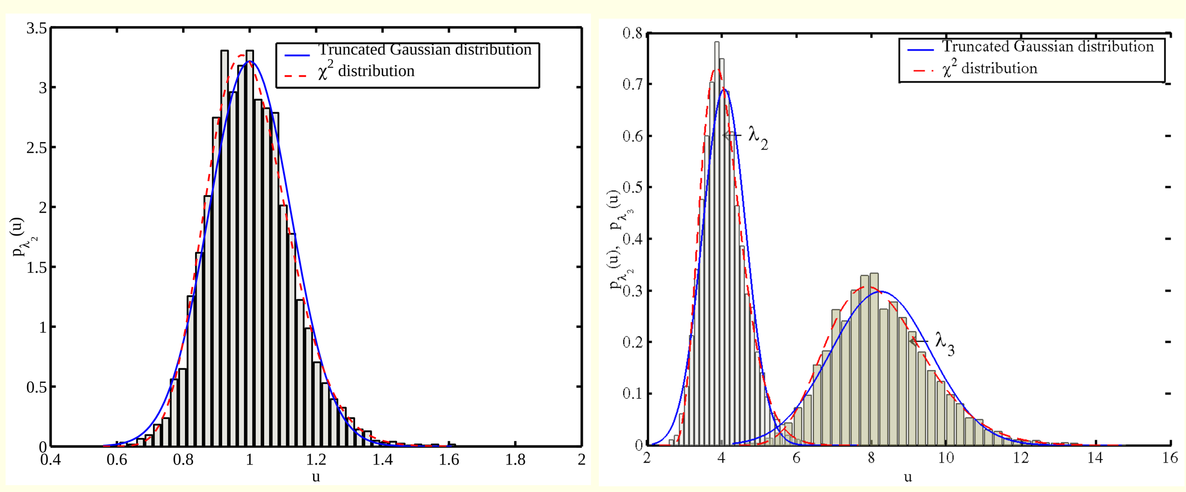


We assume that there are 15% variation in the mass and stiffness values. Following figure show the scatter in the eigenvalues about the mean obtained using Monte Carlo simulation.



Note that the higher eigenvalues are more scattered than the lower ones.

Probability density function of the eigenvalues obtained from the proposed analytical methods are compared with Monte Carlo simulation in the following figures.



It is clear that the proposed method matches well with the simulation results.

Probabilistic Reliability Analysis of Large Systems

The aim of this study is to obtain probability of failure of a structure whose properties are specified probabilistically. If $\mathbf{x} \in \mathbb{R}^n$ is the random parameter vector, then the condition of the structure for every $\mathbf{x}$ can be described by a safety margin $g(\mathbf{x})$ such the structure has failed if $g(\mathbf{x}) \leq 0$ and is safe if $g(\mathbf{x}) > 0$. Thus, the probability of failure is given by

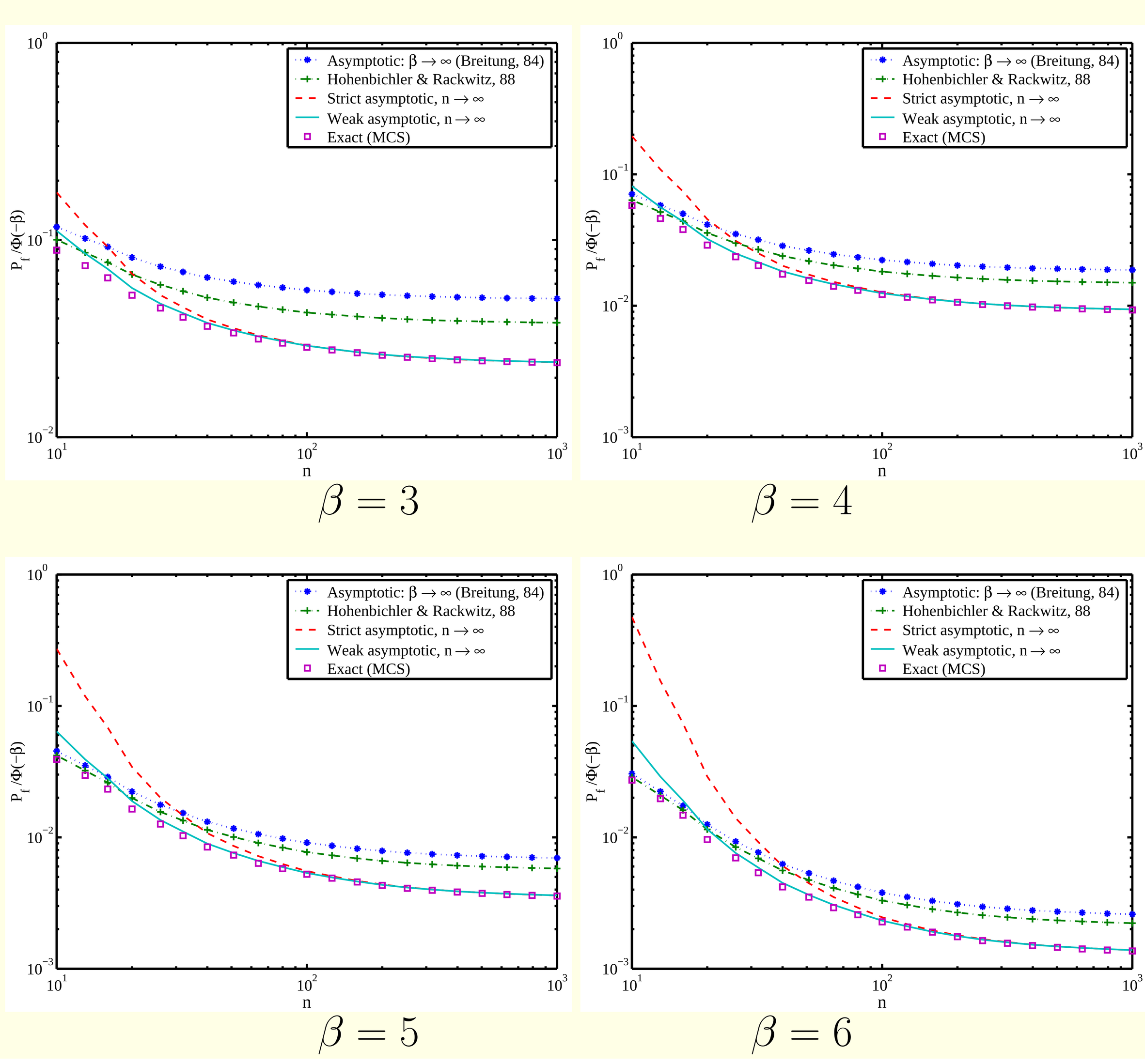
$$P_f = \int_{g(\mathbf{x})\leq 0} p(\mathbf{x}) d\mathbf{x}. \qquad (3)$$

Over the past three decades there has been extensive research to develop approximate numerical methods for the efficient calculation of the reliability integral. They can be broadly grouped into (a) first-order reliability method (FORM), and (b) second-order reliability method (SORM).

If the number of random variables n is very large, the computation of P_f using any available methods will be difficult. We address the following questions of fundamental interest:

1. Suppose we have followed the 'usual route' and did all the necessary calculations. Can we still expect the same level of accuracy from the classical FORM/SORM formula in high dimensions as we do in low dimensions? If not, what are the exact reasons behind it?
2. From the point of view of classical FORM/SORM, what do we mean by 'high dimension'? Is it a problem dependent quantity, or is it simply our perception based on available computational tools so that what we regard as a high dimension today may not be considered as a high dimension in the future when more powerful computational tools will be available?

These questions are answered using a new asymptotic theory [4]. It is shown that minor modifications to the classical FORM and SORM formula can improve their accuracy in high dimensions.



$\beta = 3$ $\beta = 4$ $\beta = 5$ $\beta = 6$

Uncertainty Quantification

The approach adopted for uncertainty quantification is to parameterize the probability density function (PDF) of the parameters and to estimate these parameters from multiple measured data sets using a maximum likelihood estimator. For example, if the parameters were assumed to be taken from a Gaussian distribution then the mean vector and covariance matrix is all that is required. This is essentially a form of regularization, since a smaller number of parameters are estimated for a given measured data set. The efficient evaluation of the response PDF through the perturbation and Monte Carlo approaches is vital. The probability that the measurements, occur given the PDF of the parameters, is tehn obtained. An optimization procedure, such as the simplex algorithm, may then be used to determine the optimum parameters for the PDFs that maximize the likelihood of the measurements [5].

Two methods of uncertainty propagation, namely the perturbation and Monte-Carlo methods, are used. In the perturbation approach, the response outputs are expanded as a Taylor series in the physical parameters about their nominal values. If the physical parameters are derived from a normal distribution then the linearised response is also normal, and the mean and covariance are easily computed from the Jacobian. The log likelihood function is then obtained from the standard PDF for the normal distribution. The Monte Carlo approach samples physical parameters based on the assumed PDF. These samples are used to calculate the response of the structure and the PDF of the response may be approximated using a kernel density estimator. The efficiency of the method may be improved significantly by retaining the results for the samples we have already computed. When the parameter PDF changes the contribution of these samples to the response PDF are simply re-weighted.

A laboratory structure, namely a free-free beams with a moving mass, will demonstrate the approach. Two parameters were estimated, namely the mass position and the beam stiffness, and their mean and variance estimated using measured natural frequencies from 50 tests, where a known PDF was used for the position. The table below shows the results. The beam properties were constant and so, as expected, the estimated variance of the beam stiffness was very small. The mean and variance of the mass position are estimated accurately.

Method	Mean	Variance	Mean	Variance
Real	0.8	0.03	–	0
Effective	0.7921	0.02993	–	0
Perturbation	0.7724	0.03469	0.9786	0.0102
Monte Carlo	0.7938	0.02905	0.9710	0.0003

Relevant Publications

[1] Adhikari, S. and Langley, R. S., "Distribution of eigenvalues of linear stochastic systems," *Ninth International Conference on Applications of Statistics and Probability in Civil Engineering, San Fransisco*, 2003.

[2] Adhikari, S., "Complex Modes in Stochastic Systems," *Advances in Vibration Engineering*, Vol. 3, No. 1, 2004, pp. 1–11.

[3] Adhikari, S. and Friswell, M. I., "Random eigenvalue problems in structural dynamics," *45th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics & Materials Conference*, Palm Springs, California, 2004.

[4] Adhikari, S., "Asymptotic distribution method for structural reliability analysis in high dimensions," *Proceedings of the Royal Society of London, Series- A*, 2005, Under review.

[5] Fonseca, J. R., Friswell, M. I., Mottershead, J. E. & Lees, A. W., "Uncertainty identification by the maximum likelihood method," *Journal of Sound and Vibration*, 2005, in press.

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