On the dynamics of a Duffing oscillator with an exponential non-viscous damping model

D J Wagg and S Adhikari

Department of Aerospace Engineering, University of Bristol, Bristol, U.K.

Email: S.Adhikari@bristol.ac.uk

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- Review of linear systems
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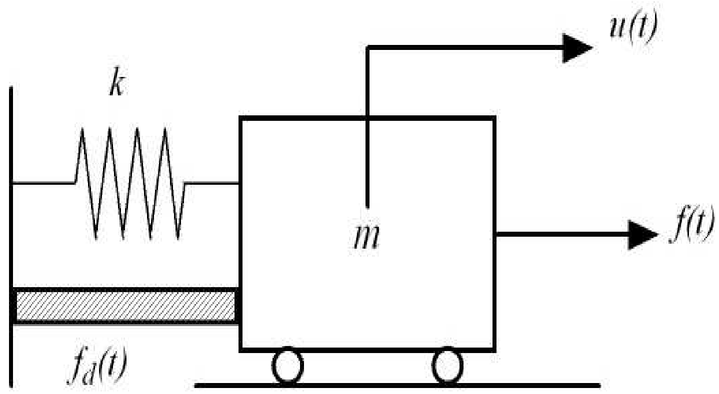
Linear non-viscous system



The equation of motion:

$$m\,\ddot{u}(t) + \int_0^t c\,\mu e^{-\mu(t-\tau)}\dot{u}(\tau)\,\mathrm{d}\tau + k\,u(t) = f(t) \qquad (1)$$

Frequency domain representation

$$\overline{d}(s)\,\overline{u}(s) = \overline{p}(s) \tag{2}$$

where

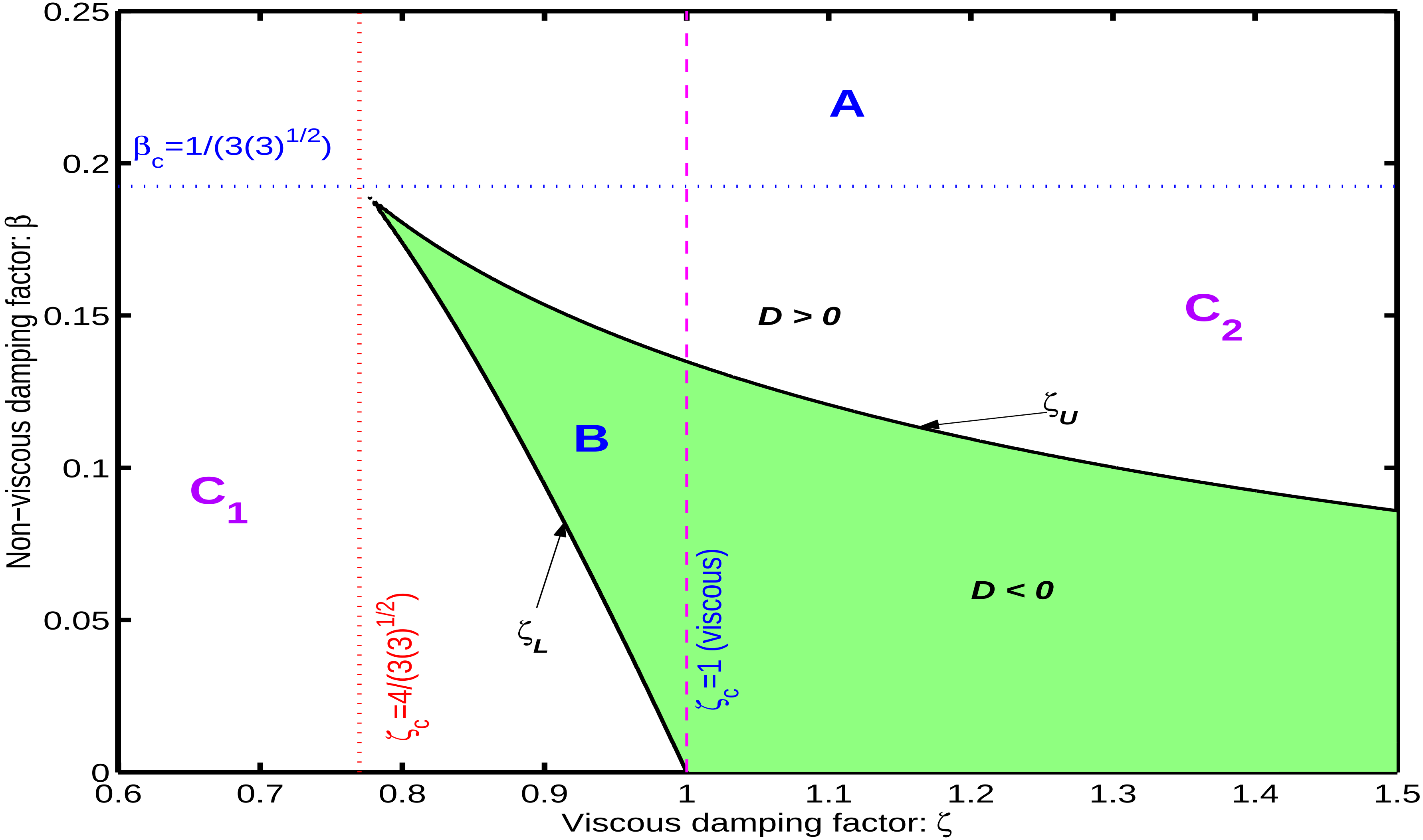
$$\overline{d}(s) = s^2 + s\,2\zeta\omega_n\left(\frac{\omega_n}{s\beta+\omega_n}\right) + \omega_n^2 \tag{3}$$

$\overline{p}(s)$ is the equivalent forcing function and

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2\sqrt{k\,m}}, \quad \text{and} \quad \beta = \frac{\omega_n}{\mu}. \tag{4}$$

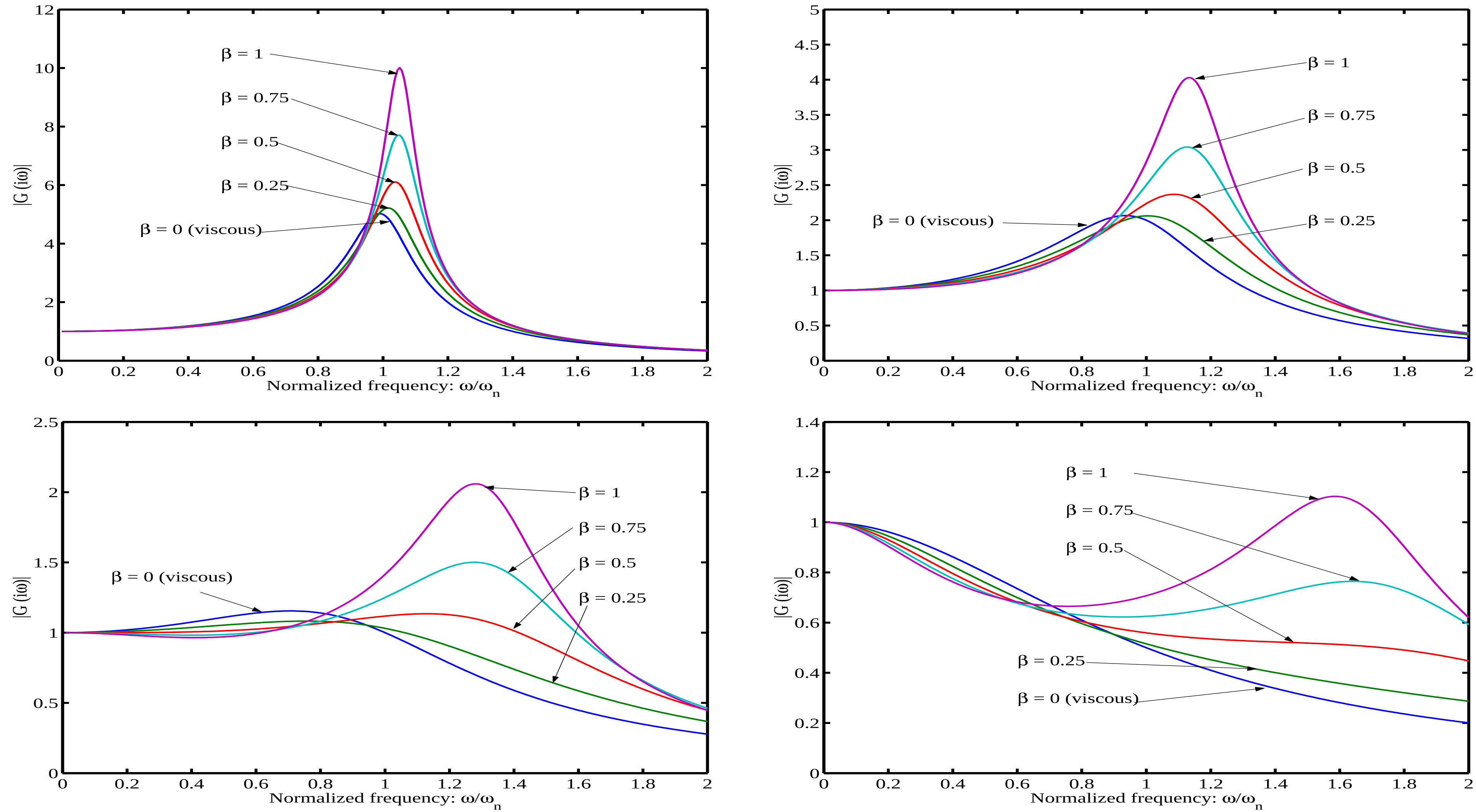
ω_n: undamped natural frequency, ζ: viscous damping factor and β: non-viscous damping factor.

Conditions for oscillatory motion



Critical values of ζ and β for oscillatory (periodic) motion.

Frequency response function



(a) $\zeta = 0.1$ (b) $\zeta = 0.25$ (c) $\zeta = 0.5$ (d) $\zeta = 1.0$

Partial summary of new results

- A non-viscously damped oscillator will have oscillatory motion if $\zeta < \frac{4}{3\sqrt{3}}$ or $\beta > \frac{1}{3\sqrt{3}}$.
- If $\beta < \frac{1}{3\sqrt{3}}$, the oscillatory motion is possible if and only if $\zeta \notin [\zeta_L, \zeta_U]$. ζ_L and ζ_U are the lower and upper critical damping factors.
- If $\beta > 1/4$, the natural frequency of a non-viscously damped oscillator will be more than that of an equivalent undamped oscillator.
- The amplitude of the frequency response function of a non-viscously damped oscillator can reach a maximum value if $\zeta < \frac{1}{2}\sqrt{\sqrt{5}-1}$ or $\beta > \frac{1}{2}\sqrt{3\sqrt{3}-4}$.

Some References

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Equation of motion

The governing equation is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}\hat{t}^2} + c\int_{\hat{\tau}=0}^{\hat{\tau}=\hat{t}} \mu \mathrm{e}^{-\mu(\hat{t}-\hat{\tau})}\frac{\mathrm{d}x}{\mathrm{d}\hat{\tau}}\mathrm{d}\hat{\tau} + \alpha_1 kx + \alpha_2 kx^3 = A\cos(\Omega\hat{t}),$$

- x: the displacement of mass m; k: linear spring stiffness
- α_1, α_2: strength of linear and nonlinear spring stiffness
- c: viscous damping coefficient
- The non-viscous damping effect is represented by the parameter μ via the convolution integral. $\mu \to \infty$ implies viscous damping, i.e., classical Duffing oscillator

The nondimensional equation

The nondimensional governing equation is

$$\ddot{x} + 2\zeta \int_0^t \frac{\mathrm{e}^{-\frac{1}{\beta}(t-\tau)}}{\beta} \dot{x} d\tau + \alpha_1 x + \alpha_2 x^3 = x_0 \cos(\omega t),$$

We now define the integral term as

$$y = \int_0^t \frac{\mathrm{e}^{-\frac{1}{\beta}(t-\tau)}}{\beta} \dot{x} d\tau$$

Then by using the Leibniz rule for differentiation of an integral we can write

$$\dot{y} = \frac{1}{\beta}\dot{x} - \frac{1}{\beta}y$$

The first-order form

We can then write a set of three first order ordinary differential equations

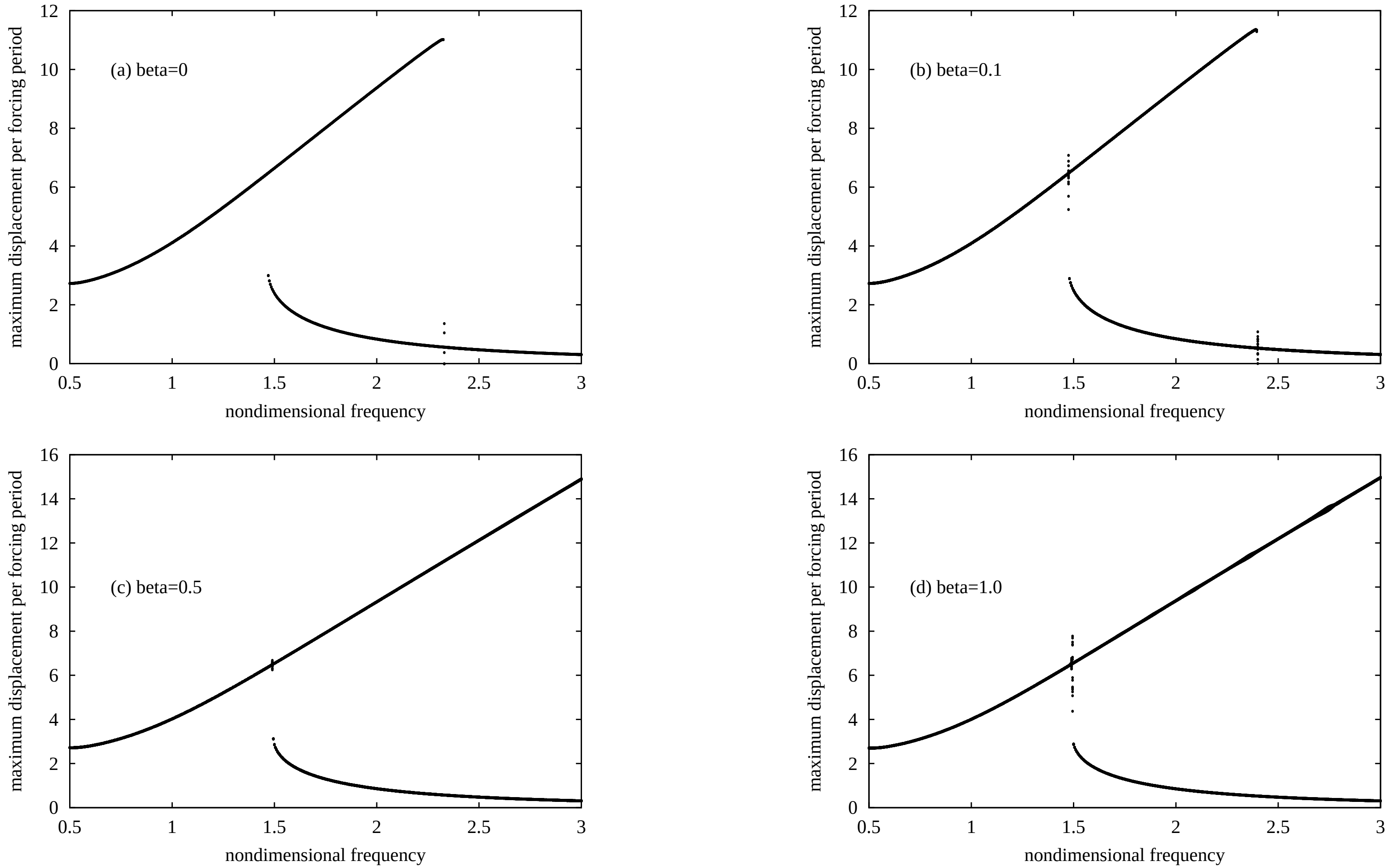
$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -2\zeta y - \alpha_1 x_1 - \alpha_2 x_1^3 + x_0 \cos(\omega t), \\ \dot{y} &= \frac{1}{\beta} x_2 - \frac{1}{\beta} y, \end{aligned}$$

Note that if we multiply through the last line by β, then as $\beta \to 0$, $y \to x_2$ and the viscous damping case is obtained.

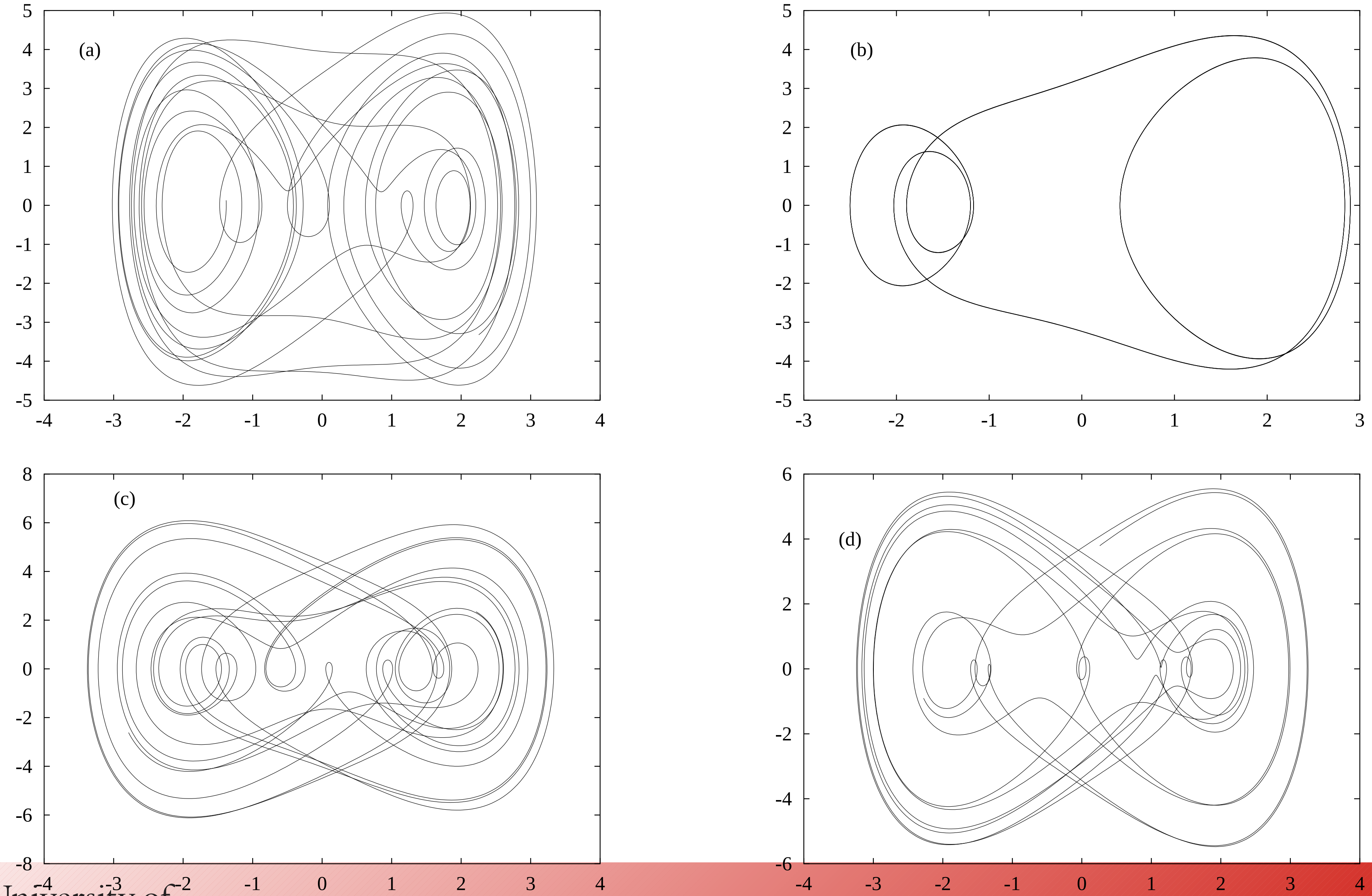
Computing solutions

- 4th order Runge-Kutta integration algorithm
- Start at the lowest ω value
- Compute transient periods (typically 100–200)
- Max displacement recorded for 20–50 steady state periods
- Increase ω and repeat
- At max ω, reverse increment and the process continued to ω_{min}

Weak coupling: $\alpha_1 = 1.0$ and $\alpha_2 = 0.05$



Strong coupling: $\alpha_1 = 0$ and $\alpha_2 = 1$



(a) Viscous case: $\beta = 0$ (b) $\beta = 0.1$ (c) $\beta = 0.3$ (d) $\beta = 0.8$

Conclusions

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- Many new features cannot be predicted (or institutively guessed) by 'simple extension' of the classical results known for viscously damped systems
- More new dynamical features are yet to be discovered in the future ... this is far from over!