Homogenisation and dynamics of randomly irregular metamaterials

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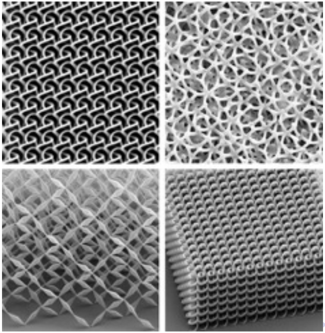
Outline

1. **Introduction**
   - Regular lattices
   - Irregular lattices
2. **Formulation for the viscoelastic analysis**
3. **Equivalent elastic properties of randomly irregular lattices**
4. **Effective properties of irregular lattices: uncorrelated uncertainty**
   - General results - closed-form expressions
   - Special case 1: Only spatial variation of the material properties
   - Special case 2: Only geometric irregularities
   - Special case 3: Regular hexagonal lattices
5. **Effective properties of irregular lattices: correlated uncertainty**
6. **Results and discussions**
   - Spatially correlated irregular elastic lattices
   - Viscoelastic properties of regular lattices
   - Spatially correlated irregular viscoelastic lattices
7. **Conclusions**

Lattice based metamaterials

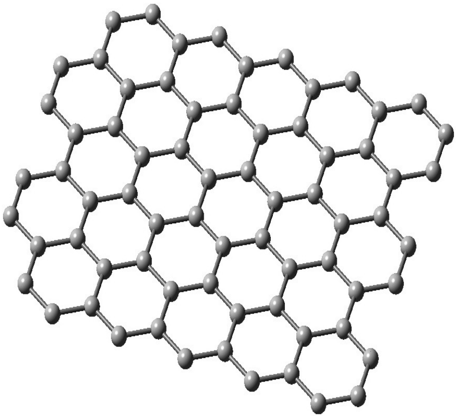
- Metamaterials are artificial materials designed to outperform naturally occurring materials in various fronts. These include, but are not limited to, electromagnetics, acoustics, optics, terahertz, infrared, dynamics and mechanical properties.
- Lattice based metamaterials are abundant in man-made and natural systems at various length scales
- Lattice based metamaterials are made of periodic identical/near-identical geometric units

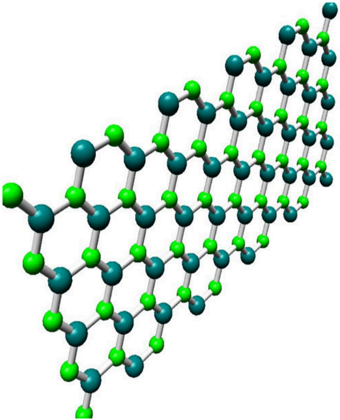


Hexagonal lattices in 2D

- Among various lattice geometries (triangle, square, rectangle, pentagon, hexagon), hexagonal lattice is most common (note that hexagon is the highest "space filling" pattern in 2D).
- This talk is about in-plane elastic properties of 2D hexagonal lattice structures - commonly known as "honeycombs"

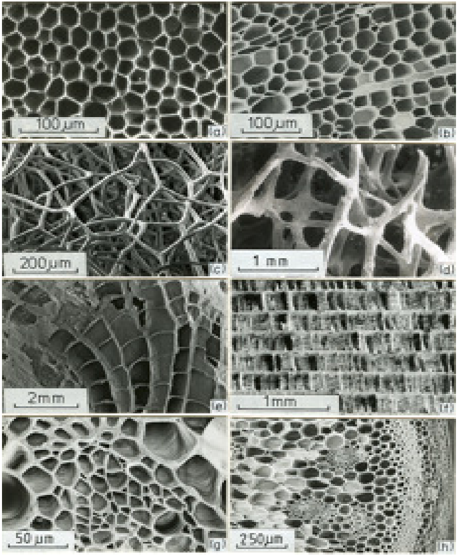
Lattice structures - nano scale

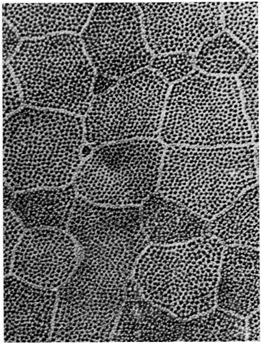




Single layer graphene sheet and born nitride nano sheet

Lattice structures - nature





Top left: cork, top right: balsa, next down left: sponge, next down right: trabecular bone, next down left: coral, next down right: cuttlefish bone, bottom left: leaf tissue, bottom right: plant stem, third column - epidermal cells (from web.mit.edu)

Some questions of general interest

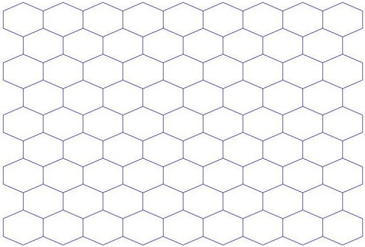
- Shall we consider lattices as “structures” or “materials” from a mechanics point of view?
- At what relative length-scale a lattice *structure* can be considered as a *material* with equivalent elastic properties?
- In what ways structural irregularities “mess up” equivalent elastic / viscoelastic properties? Can we evaluate it in a quantitative as well as in a qualitative manner?
- What is the consequence of *random* structural irregularities on the homogenisation approach in general? Can we obtain statistical measures?
- How can we efficiently *compute* equivalent elastic / viscoelastic properties of random lattice structures?

Regular lattice structures

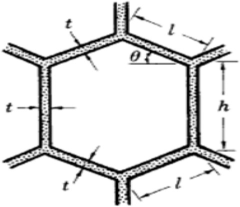
- Honeycombs have been modelled as a continuous solid with an equivalent elastic moduli throughout its domain.
- This approach eliminates the need of detail finite element modelling of honeycombs in complex structural systems like sandwich structures.
- Extensive amount of research has been carried out to predict the equivalent elastic / viscoelastic properties of regular honeycombs consisting of perfectly periodic hexagonal cells (El-Sayed et al., 1979; Gibson and Ashby, 1999; Goswami, 2006; Masters and Evans, 1996; Zhang and Ashby, 1992).
- Analysis of two dimensional honeycombs dealing with in-plane elastic properties are commonly based on an unit cell approach, which is applicable only for perfectly periodic cellular structures.
- For the dynamic analysis of perfectly periodic structures, Floquet-Bloch theorem is normally employed to characterise wave propagation.

Equivalent elastic properties of regular honeycombs

- Unit cell approach - Gibson and Ashby (1999)



(a) Regular hexagon ($\theta = 30^\circ$)



(b) Unit cell

- We are interested in homogenised equivalent in-plane elastic properties
- This way, we can avoid a detailed structural analysis considering all the beams and treat it as a material

Equivalent elastic properties of regular honeycombs

- The cell walls are treated as beams of thickness t, depth b and Young's modulus E_s. l and h are the lengths of inclined cell walls having inclination angle θ and the vertical cell walls respectively.
- The equivalent elastic properties are:

$$E_1 = E_s \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l} + \sin\theta)\sin^2\theta} \tag{1}$$

$$E_2 = E_s \left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l} + \sin\theta)}{\cos^3\theta} \tag{2}$$

$$\nu_{12} = \frac{\cos^2\theta}{(\frac{h}{l} + \sin\theta)\sin\theta} \tag{3}$$

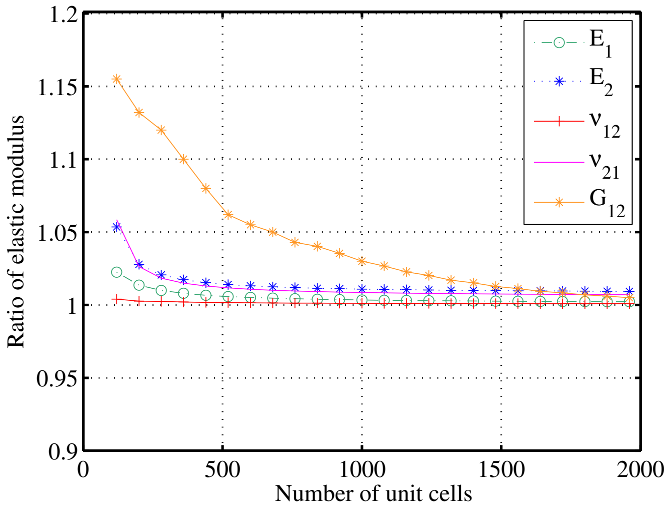
$$\nu_{21} = \frac{(\frac{h}{l} + \sin\theta)\sin\theta}{\cos^2\theta} \tag{4}$$

$$G_{12} = E_s \left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l} + \sin\theta)}{\left(\frac{h}{l}\right)^2 (1 + 2\frac{h}{l})\cos\theta} \tag{5}$$

Finite element modelling and verification

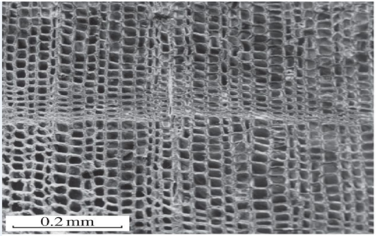
- A finite element code has been developed to obtain the in-plane elastic moduli numerically for honeycombs.
- Each cell wall has been modelled as an Euler-Bernoulli beam element having three degrees of freedom at each node.
- For E_1 and ν_{12}: two opposite edges parallel to direction-2 of the entire honeycomb structure are considered. Along one of these two edges, uniform stress parallel to direction-1 is applied while the opposite edge is restrained against translation in direction-1. Remaining two edges (parallel to direction-1) are kept free.
- For E_2 and ν_{21}: two opposite edges parallel to direction-1 of the entire honeycomb structure are considered. Along one of these two edges, uniform stress parallel to direction-2 is applied while the opposite edge is restrained against translation in direction-2. Remaining two edges (parallel to direction-2) are kept free.
- For G_{12}: uniform shear stress is applied along one edge keeping the opposite edge restrained against translation in direction-1 and 2, while the remaining two edges are kept free.

Finite element modelling and verification



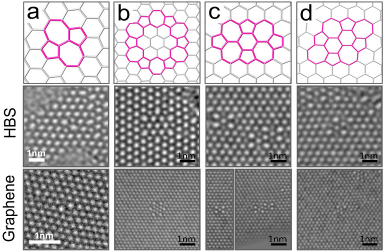
$\theta = 30^\circ$, $h/l = 1.5$. FE results converge to analytical predictions after 1681 cells.

Irregular lattice structures

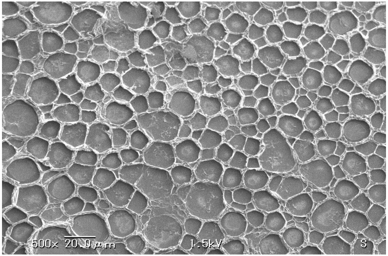


(c) Cedar wood

(d) Manufactured honeycomb core



(e) Graphene image



(f) Fabricated CNT surface

Irregular lattice structures

- A significant limitation of the aforementioned unit cell approach is that it cannot account for the spatial irregularity, which is practically inevitable.
- Spatial irregularity in honeycomb may occur due to manufacturing uncertainty, structural defects, variation in temperature, pre-stressing and micro-structural variability in honeycombs.
- To include the effect of irregularity, voronoi honeycombs have been considered in several studies (Li et al., 2005; Zhu et al., 2001, 2006).
- The effect of different forms of irregularity on elastic properties and structural responses of honeycombs are generally based on direct finite element (FE) simulation.
- In the FE approach, a small change in geometry of a single cell may require completely new geometry and meshing of the entire structure. In general this makes the entire process time-consuming and tedious.
- The problem becomes even worse for uncertainty quantification of the responses associated with irregular honeycombs, where the expensive finite element model is needed to be simulated for a large number of samples while using a Monte Carlo based approach.

Fundamental equation for the viscoelastic behaviour

- When a linear viscoelastic model is employed, the stress at some point of a structure can be expressed as a convolution integral over a kernel function as

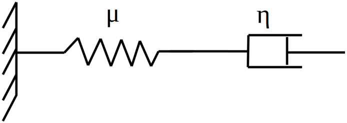
$$\sigma(t) = \int_{-\infty}^{t} g(t-\tau)\frac{\partial \epsilon(\tau)}{\partial \tau}\, \tau \tag{6}$$

- $t \in \mathbb{R}^+$ is the time, $\sigma(t)$ is stress and $\epsilon(t)$ is strain.
- The kernel function $g(t)$ also known as 'hereditary function', 'relaxation function' or 'after-effect function' in the context of different subjects.
- In practice, the kernel function is often defined in the frequency domain (or Laplace domain). Taking the Laplace transform of Equation (6), we have

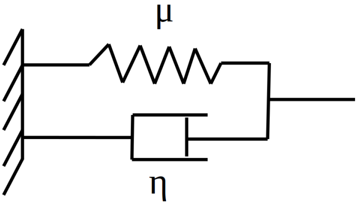
$$\bar{\sigma}(s) = s\bar{G}(s)\bar{\epsilon}(s) \tag{7}$$

  Here $\bar{\sigma}(s)$, $\bar{\epsilon}(s)$ and $\bar{G}(s)$ are Laplace transforms of $\sigma(t)$, $\epsilon(t)$ and $g(t)$ respectively and $s \in \mathbb{C}$ is the (complex) Laplace domain parameter.

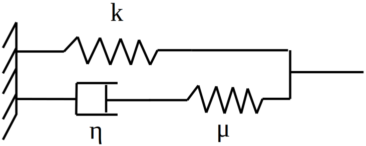
Viscoelastic models



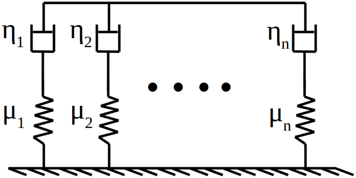
(g) Maxwell model



(h) Voigt model



(i) Standard linear model



(j) Generalised Maxwell model

Figure: Springs and dashpots based models viscoelastic materials.

Viscoelastic models

The viscoelastic kernel function can be expressed for the four models as

- *Maxwell model:*
$$g(t) = \mu e^{-(\mu/\eta)t}\mathcal{U}(t) \tag{8}$$
- *Voigt model:*
$$g(t) = \eta\delta(t) + \mu\mathcal{U}(t) \tag{9}$$
- *Standard linear model:*
$$g(t) = E_R\left[1 - (1 - \frac{\mathcal{T}_\sigma}{\mathcal{T}_\epsilon})e^{-t/\mathcal{T}_\epsilon}\right]\mathcal{U}(t) \tag{10}$$
- *Generalised Maxwell model:*
$$g(t) = \left[\sum_{j=1}^{n} \mu_j e^{-(\mu_j/\eta_j)t}\right]\mathcal{U}(t) \tag{11}$$

Models similar to this is also known as the Pony series model.

Viscoelastic models

Viscoelastic model	Complex modules
Biot model	$G(\omega) = G_0 + \sum_{k=1}^{n} \frac{a_k \mathrm{i}\omega}{\mathrm{i}\omega + b_k}$
Fractional derivative	$G(\omega) = \frac{G_0 + G_\infty (\mathrm{i}\omega\tau)^\beta}{1 + (\mathrm{i}\omega\tau)^\beta}$
GHM	$G(\omega) = G_0 \left[1 + \sum_k \alpha_k \frac{-\omega^2 + 2\mathrm{i}\xi_k\omega_k\omega}{-\omega^2 + 2\mathrm{i}\xi_k\omega_k\omega + \omega_k^2}\right]$
ADF	$G(\omega) = G_0 \left[1 + \sum_{k=1}^{n} \Delta_k \frac{\omega^2 + \mathrm{i}\omega\Omega_k}{\omega^2 + \Omega_k^2}\right]$
Step-function	$G(\omega) = G_0 \left[1 + \eta \frac{1 - e^{-st_0}}{st_0}\right]$
Half cosine model	$G(\omega) = G_0 \left[1 + \eta \frac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}\right]$
Gaussian model	$G(\omega) = G_0 \left[1 + \eta\, e^{\omega^2/4\mu} \left\{1 - \mathrm{erf}\left(\frac{\mathrm{i}\omega}{2\sqrt{\mu}}\right)\right\}\right]$

Complex modulus for some viscoelastic models in the frequency domain

The Biot Model

- We consider that each constitutive element of a hexagonal unit with the honeycomb structure is modelled using viscoelastic properties. For simplicity, we use Biot model with only one term. Frequency dependent complex elastic modulus for an element is expresses as

$$E(\omega) = E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \tag{12}$$

  where μ and ϵ are the relaxation parameter and a constant defining the 'strength' of viscosity, respectively. E_s is the intrinsic Young's modulus.
- The amplitude of this complex elastic modulus is given by

$$|E(\omega)| = E_S \sqrt{\frac{\mu^2 + \omega^2 \left(1+\epsilon\right)^2}{\mu^2 + \omega^2}} \tag{13}$$

- The phase (ϕ) of this complex elastic modulus is given by

$$\phi(E(\omega)) = \tan^{-1}\left(\frac{\epsilon\mu\omega}{\mu^2 + \omega^2(1+\epsilon)}\right) \tag{14}$$

Mathematical idealisation of irregularity in lattice structures

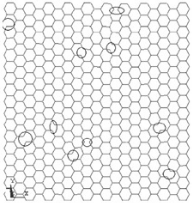


Fig. Randomly missing cell wall

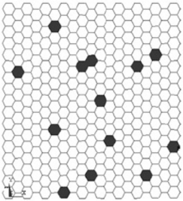


Fig. Random filled cell

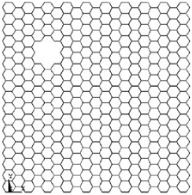
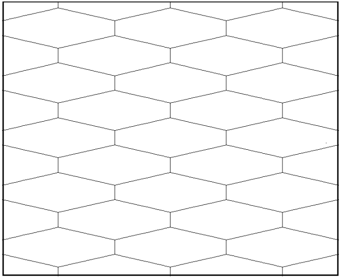
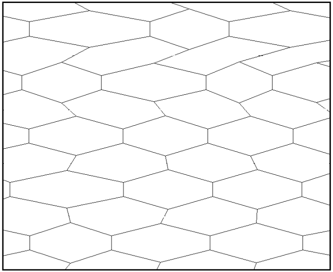


Fig. Missing cell cluster

Irregular honeycomb



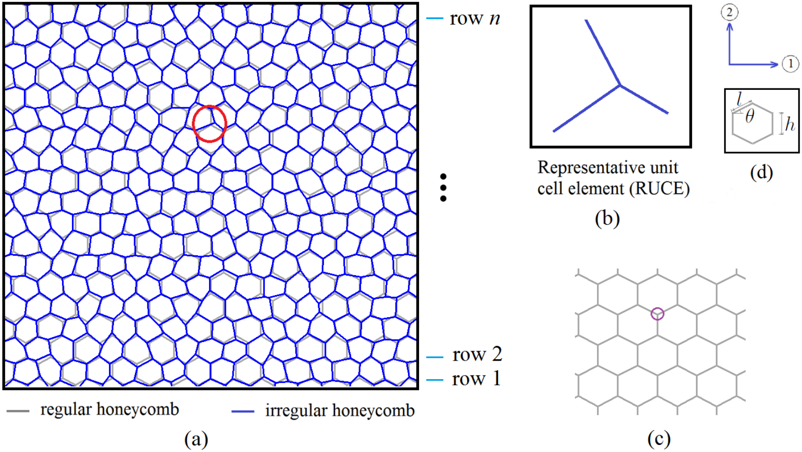


- Random spatial irregularity in cell angle is considered in this study.

Irregular lattice structures

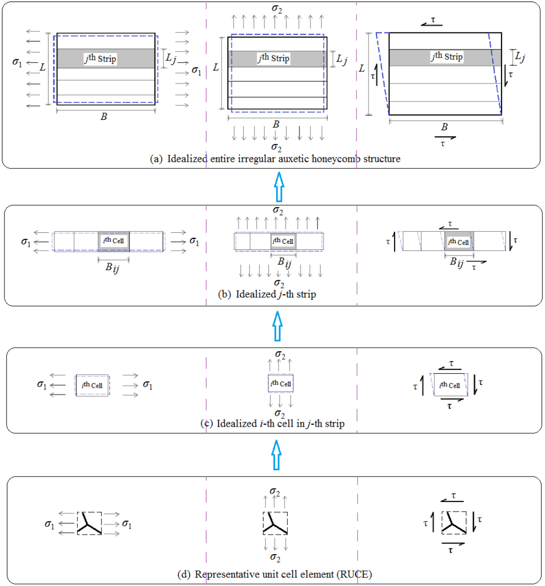
- Direct numerical simulation to deal with irregularity in honeycombs may not necessarily provide proper understanding of the underlying physics of the system. An analytical approach could be a simple, insightful, yet an efficient way to obtain effective elastic properties of honeycombs.
- This work develops a structural mechanics based analytical framework for predicting equivalent in-plane elastic properties of irregular honeycomb having spatially random variations in cell angles.
- Closed-form analytical expressions will be derived for equivalent in-plane elastic properties.
- An approach based on the frequency-domain representation of the viscoelastic property of the constituent elements in the cells is used.

The philosophy of the analytical approach for irregular honeycombs

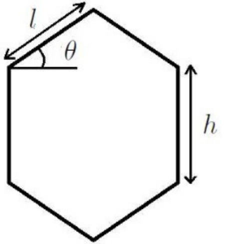


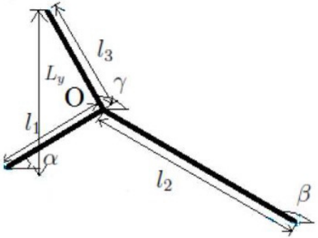
(a) Typical representation of an irregular honeycomb (b) Representative unit cell element (RUCE) (c) Illustration to define degree of irregularity (d) Unit cell considered for regular hexagonal lattice by Gibson and Ashby (1999).

The idealisation of RUCE and the bottom-up homogenisation approach



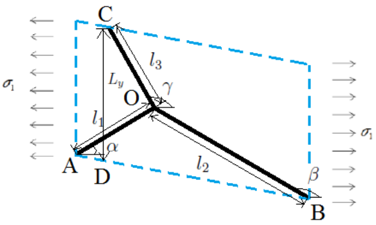
Unit cell geometry



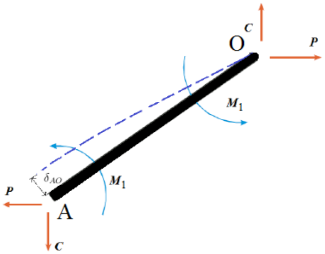


(a) Classical unit cell for regular lattices (b) Representative unit cell element (RUCE) geometry for irregular lattices

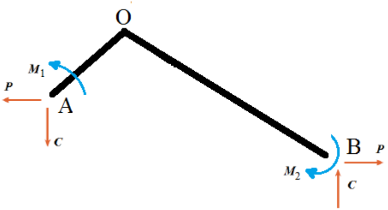
RUCE and free-body diagram for the derivation of E_1



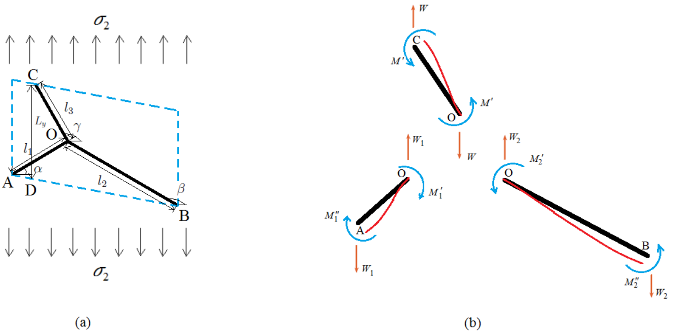
(a)

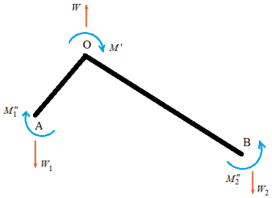


(b)

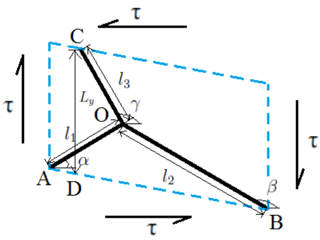


RUCE and free-body diagram for the derivation of E_2

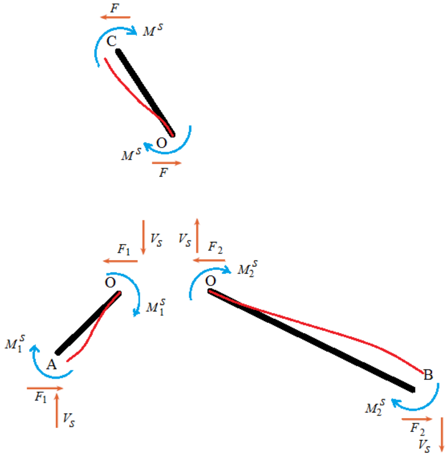




RUCE and free-body diagram for the derivation of G_{12}



(a)



(b)

Equivalent E_1, E_2

Equivalent E_1

$$E_{1v}(\omega) = \frac{t^3}{L}\sum_{j=1}^{n} \frac{\sum\limits_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum\limits_{i=1}^{m} \frac{l_{1ij}^2 l_{2ij}^2\left(l_{1ij}+l_{2ij}\right)\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)^2}{E_{sij}\left(1+\epsilon_{ij}\frac{\mathrm{i}\omega}{\mu_{ij}+\mathrm{i}\omega}\right)\left(\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)^2\right)}} \tag{15}$$

Equivalent Young's moduli E_2

$$E_{2v}(\omega) = \frac{Lt^3}{\sum\limits_{j=1}^{n} \frac{\sum\limits_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum\limits_{i=1}^{m} E_{sij}\left(1+\epsilon_{ij}\frac{\mathrm{i}\omega}{\mu_{ij}+\mathrm{i}\omega}\right)\left(l_{3ij}^2\cos^2\gamma_{ij}\left(l_{3ij}+\frac{l_{1ij}l_{2ij}}{l_{1ij}+l_{2ij}}\right)+\frac{l_{1ij}^2 l_{2ij}^2\left(l_{1ij}+l_{2ij}\right)\cos^2\alpha_{ij}\cos^2\beta_{ij}}{\left(l_{1ij}\cos\alpha_{ij}-l_{2ij}\cos\beta_{ij}\right)^2}\right)^{-1}}} \tag{16}$$

Equivalent shear Modulus G_{12}

Equivalent G_{12}

$$G_{12v}(\omega) = \frac{Lt^3}{\displaystyle\sum_{j=1}^{n} \frac{\displaystyle\sum_{i=1}^{m} \left(l_{1ij} \cos\alpha_{ij} - l_{2ij} \cos\beta_{ij} \right)}{\displaystyle\sum_{i=1}^{m} E_{sij} \left(1 + \epsilon_{ij} \frac{\mathrm{i}\omega}{\mu_{ij} + \mathrm{i}\omega} \right) \left(l_{3ij}^2 \sin^2\gamma_{ij} \left(l_{3ij} + \frac{l_{1ij} l_{2ij}}{l_{1ij} + l_{2ij}} \right) \right)^{-1}}} \quad (17)$$

Poisson's ratios ν_{12}, ν_{21}

Equivalent ν_{12}

$$\nu_{12eq} = -\frac{1}{L}\sum_{j=1}^{n} \frac{\sum_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum_{i=1}^{m}\frac{\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)}{\cos\alpha_{ij}\cos\beta_{ij}}} \tag{18}$$

Equivalent ν_{21}

$$\nu_{21eq} = -\frac{L}{\sum_{j=1}^{n}\frac{\sum_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum_{i=1}^{m}\frac{l_{1ij}^{2}l_{2ij}^{2}\left(l_{1ij}+l_{2ij}\right)\cos\alpha_{ij}\cos\beta_{ij}\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)}{\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)^{2}\left(l_{3ij}^{2}\cos^{2}\gamma_{ij}\left(l_{3ij}+\frac{l_{1ij}l_{2ij}}{l_{1ij}+l_{2ij}}\right)+\frac{l_{1ij}^{2}l_{2ij}^{2}\left(l_{1ij}+l_{2ij}\right)\cos^{2}\alpha_{ij}\cos^{2}\beta_{ij}}{\left(l_{1ij}\cos\alpha_{ij}-l_{2ij}\cos\beta_{ij}\right)^{2}}\right)}}} \tag{19}$$

Only spatial variation of the material properties

- According to the notations used for a regular honeycomb by Gibson and Ashby (1999), the notations for honeycombs without any structural irregularity can be expressed as: $L = n(h + l\sin\theta)$; $l_{1ij} = l_{2ij} = l_{3ij} = l$; $\alpha_{ij} = \theta$; $\beta_{ij} = 180^\circ - \theta$; $\gamma_{ij} = 90^\circ$, for all i and j.
- Using these transformations in case of the spatial variation of only material properties, the closed-form formulae for compound variation of material and geometric properties (equations 15–17) can be reduced to:

$$E_{1v} = \kappa_1 \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l} + \sin\theta)\sin^2\theta} \tag{20}$$

$$E_{2v} = \kappa_2 \left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l} + \sin\theta)}{\cos^3\theta} \tag{21}$$

and

$$G_{12v} = \kappa_2 \left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l} + \sin\theta\right)}{\left(\frac{h}{l}\right)^2 (1 + 2\frac{h}{l})\cos\theta} \tag{22}$$

Only spatial variation of the material properties

- The multiplication factors κ_1 and κ_2 arising due to the consideration of spatially random variation of intrinsic material properties can be expressed as

$$\kappa_1 = \frac{m}{n}\sum_{j=1}^{n} \frac{1}{\sum\limits_{i=1}^{m} \frac{1}{E_{sij}\left(1+\epsilon_{ij}\frac{\mathrm{i}\omega}{\mu_{ij}+\mathrm{i}\omega}\right)}} \tag{23}$$

and

$$\kappa_2 = \frac{n}{m} \frac{1}{\sum\limits_{j=1}^{n} \frac{1}{\sum\limits_{i=1}^{m} E_{sij}\left(1+\epsilon_{ij}\frac{\mathrm{i}\omega}{\mu_{ij}+\mathrm{i}\omega}\right)}} \tag{24}$$

- In the special case when $\omega \rightarrow 0$ and there is no spatial variabilities in the material properties of the lattice, all viscoelastic material properties become identical (i.e. $E_{sij} = E_s$, $\mu_{ij} = \mu$ and $\epsilon_{ij} = \epsilon$ for $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., n$) and subsequently the amplitude of κ_1 and κ_2 becomes exactly 1. This confirms that the expressions in 23 and 24 give the necessary generalisations of the classical expressions of Gibson and Ashby (1999) through 20–22.

Only geometric irregularities

- In case of only spatially random variation of structural geometry but constant viscoelastic material properties (i.e. $E_{sij} = E_s$, $\mu_{ij} = \mu$ and $\epsilon_{ij} = \epsilon$ for $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., n$) the 15–17 lead to

$$E_{1v} = E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \zeta_1 \tag{25}$$

$$E_{2v} = E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \zeta_2 \tag{26}$$

$$G_{12v} = E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \zeta_3 \tag{27}$$

Only geometric irregularities

- The random coefficients $\zeta_i \quad (i = 1, 2, 3)$ are

$$\zeta_1 = \frac{t^3}{L} \sum_{j=1}^{n} \frac{\sum_{i=1}^{m} (l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij})}{\sum_{i=1}^{m} \frac{l_{1ij}^2 l_{2ij}^2 (l_{1ij} + l_{2ij}) (\cos \alpha_{ij} \sin \beta_{ij} - \sin \alpha_{ij} \cos \beta_{ij})^2}{(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij})^2}} \tag{28}$$

$$\zeta_2 = \frac{Lt^3}{\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} (l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij})}{\sum_{i=1}^{m} \left(l_{3ij}^2 \cos^2 \gamma_{ij} \left(l_{3ij} + \frac{l_{1ij} l_{2ij}}{l_{1ij} + l_{2ij}} \right) + \frac{l_{1ij}^2 l_{2ij}^2 (l_{1ij} + l_{2ij}) \cos^2 \alpha_{ij} \cos^2 \beta_{ij}}{(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij})^2} \right)^{-1}}} \tag{29}$$

$$\zeta_3 = \frac{Lt^3}{\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} (l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij})}{\sum_{i=1}^{m} \left(l_{3ij}^2 \sin^2 \gamma_{ij} \left(l_{3ij} + \frac{l_{1ij} l_{2ij}}{l_{1ij} + l_{2ij}} \right) \right)^{-1}}} \tag{30}$$

Regular hexagonal lattices

- The geometric notations for regular lattices can be expressed as: $L = n(h + l\sin\theta)$; $l_{1ij} = l_{2ij} = l_{3ij} = l$; $\alpha_{ij} = \theta$; $\beta_{ij} = 180^\circ - \theta$; $\gamma_{ij} = 90^\circ$, for all i and j. Using these transformations, the expressions of in-plane elastic moduli for regular hexagonal lattices (without the viscoelastic effect) can be obtained.
- The in-plane Young's moduli and shear modulus (viscosity dependent in-plane elastic properties) can be expressed as

$$E_{1v} = E_s\left(1 + \epsilon\frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right)\left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l} + \sin\theta)\sin^2\theta} \tag{31}$$

$$E_{2v} = E_s\left(1 + \epsilon\frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right)\left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l} + \sin\theta)}{\cos^3\theta} \tag{32}$$

$$G_{12v} = E_s\left(1 + \epsilon\frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right)\left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l} + \sin\theta\right)}{\left(\frac{h}{l}\right)^2(1 + 2\frac{h}{l})\cos\theta} \tag{33}$$

- The amplitude of the elastic moduli obtained based on the above expressions converge to the closed-form equation provided by Gibson and Ashby (1999) in the limiting case of $\omega \to 0$.

Regular uniform hexagonal lattices

- In the case of regular uniform honeycombs with $\theta = 30°$, we have

$$E_{1v} = E_{2v} = 2.3E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \left(\frac{t}{l}\right)^3 \tag{34}$$

- Similarly, in the case of shear modulus for regular uniform honeycombs ($\theta = 30°$)

$$G_{12v} = 0.57E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \left(\frac{t}{l}\right)^3 \tag{35}$$

- Regular viscoelastic honeycombs satisfy the reciprocal theorem

$$E_{2v}\nu_{12v} = E_{1v}\nu_{21v} = E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \left(\frac{t}{l}\right)^3 \frac{1}{\sin\theta\cos\theta} \tag{36}$$

Random field model for material and geometric properties

- Correlated structural and material attributes can be modelled random fields $\mathcal{H}(\mathbf{x}, \theta)$.
- The traditional way of dealing with random field is to discretise the random field into finite number of random variables. The available schemes for discretising random fields can be broadly divided into three groups: (1) point discretisation (e.g., midpoint method, shape function method, integration point method, optimal linear estimate method); (2) average discretisation method (e.g., spatial average, weighted integral method), and (3) series expansion method (e.g., orthogonal series expansion).
- An advantageous alternative for discretising $\mathcal{H}(\mathbf{x}, \theta)$ is to represent it in a generalised Fourier type of series as, often termed as Karhunen-Loève (KL) expansion.

Karhunen-Loève (KL) expansion

- Suppose, $\mathcal{H}(\mathbf{x}, \theta)$ is a random field with covariance function $\Gamma_{\mathcal{H}}(\mathbf{x}_1, \mathbf{x}_2)$ defined in the probability space $(\Theta, \mathcal{F}, \mathcal{P})$. The KL expansion for $\mathcal{H}(\mathbf{x}, \theta)$ takes the following form

$$\mathcal{H}(\mathbf{x}, \theta) = \bar{\mathcal{H}}(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\theta)\psi_i(\mathbf{x}) \tag{37}$$

  where $\{\xi_i(\theta)\}$ is a set of uncorrelated random variables.
- $\{\lambda_i\}$ and $\{\psi_i(\mathbf{x})\}$ are the eigenvalues and eigenfunctions of the covariance kernel $\Gamma_{\mathcal{H}}(\mathbf{x}_1, \mathbf{x}_2)$, satisfying the integral equation

$$\int_{\Re^N} \Gamma_{\mathcal{H}}(\mathbf{x}_1, \mathbf{x}_2)\psi_i(\mathbf{x}_1)\, d\mathbf{x}_1 = \lambda_i\psi_i(\mathbf{x}_2) \tag{38}$$

- In practise, the infinite series of 37 must be truncated, yielding a truncated KL approximation

$$\tilde{\mathcal{H}}(\mathbf{x}, \theta) \cong \bar{\mathcal{H}}(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i}\xi_i(\theta)\psi_i(\mathbf{x}) \tag{39}$$

Karhunen-Loève (KL) expansion

- Gaussian and lognormal random fields have been considered. The covariance function is represented as:

$$\Gamma_{\alpha z} = \sigma_{\alpha z}^2 e^{(-|y_1 - y_2|/b_y) + (-|z_1 - z_2|/b_z)} \tag{40}$$

where b_y and b_z are the correlation parameters at y and z directions (that corresponds to direction - 1 and direction - 2 respectively). These quantities control the rate at which the covariance decays.

- In a two dimensional physical space the eigensolutions of the covariance function are obtained by solving the integral equation analytically

$$\lambda_i \psi_i(y_2, z_2) = \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} \Gamma(y_1, z_1; y_2, z_2) \psi_i(y_1, z_1) dy_1 dz_1 \tag{41}$$

where $-a_1 \leqslant y \leqslant a_1$ and $-a_2 \leqslant z \leqslant a_2$.

- Assume the eigen-solutions are separable in y and z directions, i.e.

$$\psi_i(y_2, z_2) = \psi_i^{(y)}(y_2) \psi_i^{(z)}(z_2) \tag{42}$$

$$\lambda_i(y_2, z_2) = \lambda_i^{(y)}(y_2) \lambda_i^{(z)}(z_2) \tag{43}$$

Karhunen-Loève (KL) expansion

- The solution of the integral equation reduces to the product of the solutions of two equations of the form

$$\lambda_i^{(y)}\psi_i^{(y)}(y_1) = \int_{-a_1}^{a_1} e^{(-|y_1-y_2|/b_y)}\psi_i^{(y)}(y_2)dy_2 \tag{44}$$

- The solution of this equation, which is the eigensolution (eigenvalues and eigenfunctions) of an exponential covariance kernel for a one-dimensional random field is obtained as

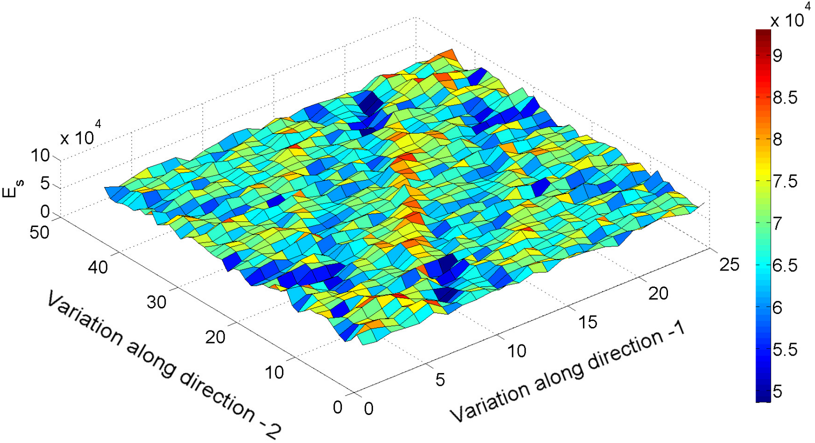
$$\begin{cases} \psi_i(\zeta) = \dfrac{\cos(\omega_i\zeta)}{\sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}} & \lambda_i = \dfrac{2\sigma_{\alpha_z}^2 b}{\omega_i^2 + b^2} \quad \text{for } i \text{ odd} \\ \psi_i(\zeta) = \dfrac{\sin(\omega_i^*\zeta)}{\sqrt{a - \frac{\sin(2\omega_i^* a)}{2\omega_i*}}} & \lambda_i^* = \dfrac{2\sigma_{\alpha_z}^2 b}{\omega_i^{*2} + b^2} \quad \text{for } i \text{ even} \end{cases} \tag{45}$$

where $b = 1/b_y$ or $1/b_z$ and $a = a_1$ or a_2. ζ can be either y or z and ω_i presents the period of the random field.

- The final eigenfunctions are given by

$$\psi_k(y,z) = \psi_i^{(y)}(y)\psi_i^{(z)}(z) \tag{46}$$

Samples of the random fields



Spatial variability of the intrinsic elastic modulus (E_s) with $\Delta_m = 0.002$

The degree of geometric irregularity

- To define the degree of irregularity, it is assumed that each connecting node of the lattice moves randomly within a certain radius (r_d) around the respective node corresponding to the regular deterministic configuration. For physically realistic variabilities, it is considered that a given node do not cross a neighbouring node, that is

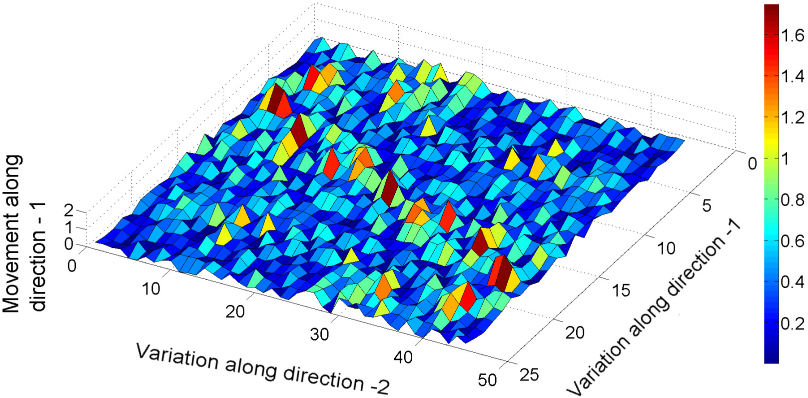
$$r_d < \min\left(\frac{h}{2}, \frac{l}{2}, l\cos\theta\right) \tag{47}$$

- In each realization of the Monte Carlo simulation, all the nodes of the lattice move simultaneously to new random locations within the specified circular bounds. Thus, the degree of irregularity (r) is defined as a non-dimensional ratio of the area of the circle and the area of one regular hexagonal unit as

$$r = \frac{\pi r_d^2 \times 100}{2l\cos\theta(h + l\sin\theta)} \tag{48}$$

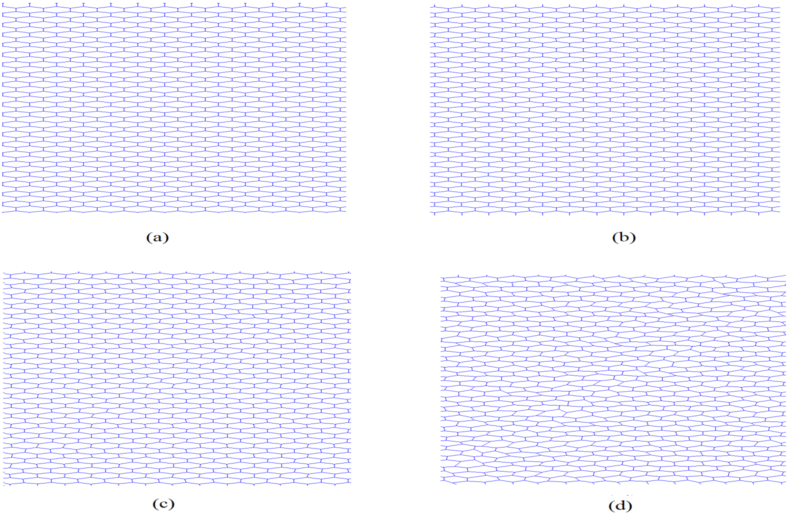
- The degree of irregularity (r) has been expressed as percentage values for presenting the results.

Samples of the random fields



Movement of the top vertices of a tessellating hexagonal unit cell with respect to the corresponding deterministic locations ($r = 6$)

Random geometric configurations



Structural configurations for a single random realisation of an irregular hexagonal lattice considering deterministic cell angle $\theta = 30^{\circ}$ and $h/l = 1$: (a) $r = 0$ (b) $r = 2$ (c) $r = 4$ (d) $r = 6$

Samples of random geometric configurations

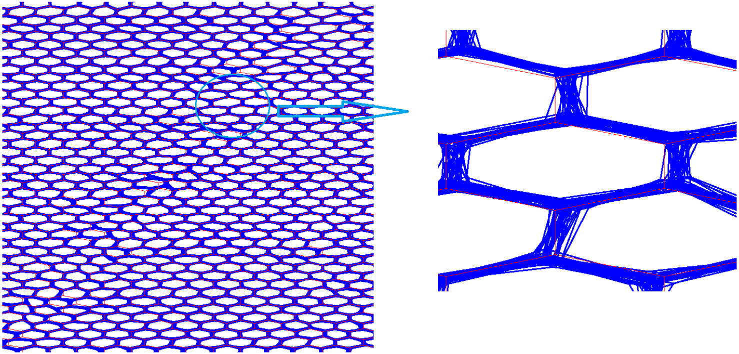


Figure: Simulation bound of the structural configuration of an irregular hexagonal lattice for multiple random realisations considering $\theta = 30^\circ$, $h/l = 1$ and $r = 6$. The regular configuration is presented using red colour.

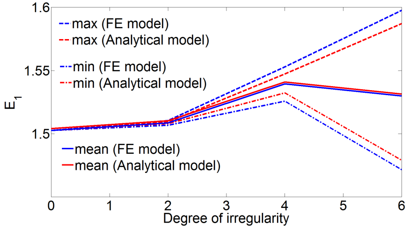
Samples of random geometric configurations

- In randomly inhomogeneous correlated system, spatial variability of the stochastic structural attributes are accounted, wherein each sample of the Monte Carlo simulation includes the spatially random distribution of structural and materials attributes with a rule of correlation.
- The spatial variability in structural and material properties (E_s, μ and ϵ) are physically attributed by degree of structural irregularity (r) and degree of material property variation (Δ_m) respectively.
- As the two Young's moduli and shear modulus for low density lattices are proportional to $E_s\rho^3$ (Zhu et al., 2001), the non-dimensional results for in-plane elastic moduli E_1, E_2, and G_{12}, unless otherwise mentioned, are presented as:

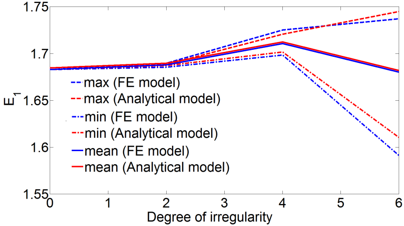
  $\bar{E_1} = \frac{E_{1eq}}{E_s\rho^3}$, $\bar{E_2} = \frac{E_{2eq}}{E_s\rho^3}$

  $\bar{G_{12}} = \frac{G_{12eq}}{E_s\rho^3}$
- ρ is the relative density of the lattice (defined as a ratio of the planar area of solid to the total planar area of the lattice).

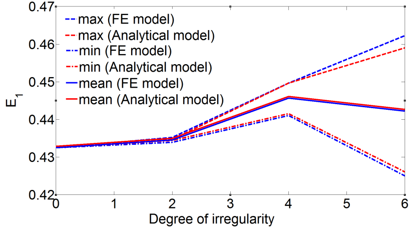
Spatially correlated irregular elastic lattices: E_1



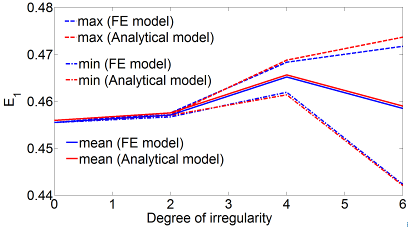
(a) $\theta = 30°; \frac{h}{l} = 1$



(b) $\theta = 30°; \frac{h}{l} = 1.5$

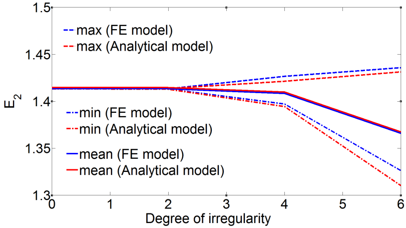


(c) $\theta = 45°; \frac{h}{l} = 1$

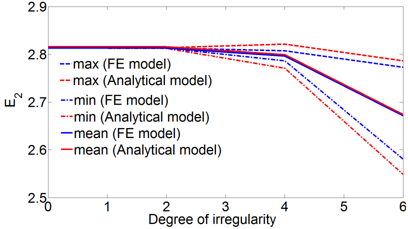


(d) $\theta = 45°; \frac{h}{l} = 1.5$

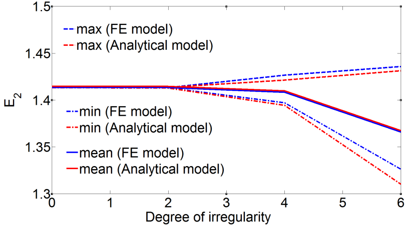
Spatially correlated irregular elastic lattices: E_2



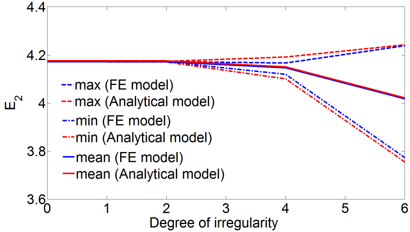
(a) $\theta = 30^\circ$; $\frac{h}{l} = 1$



(b) $\theta = 30^\circ$; $\frac{h}{l} = 1.5$

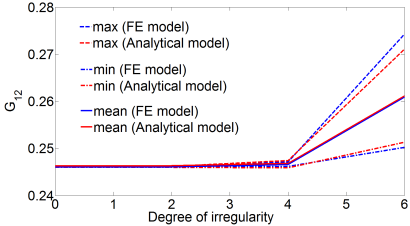


(c) $\theta = 45^\circ$; $\frac{h}{l} = 1$

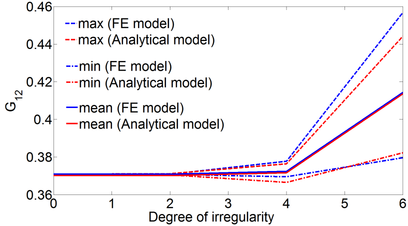


(d) $\theta = 45^\circ$; $\frac{h}{l} = 1.5$

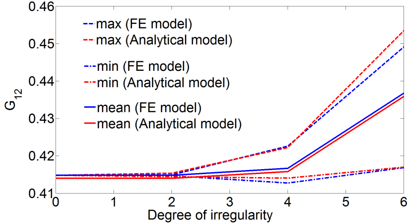
Spatially correlated irregular elastic lattices: G_{12}



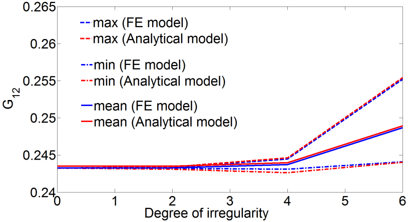
(a) $\theta = 30°; \frac{h}{l} = 1$



(b) $\theta = 30°; \frac{h}{l} = 1.5$

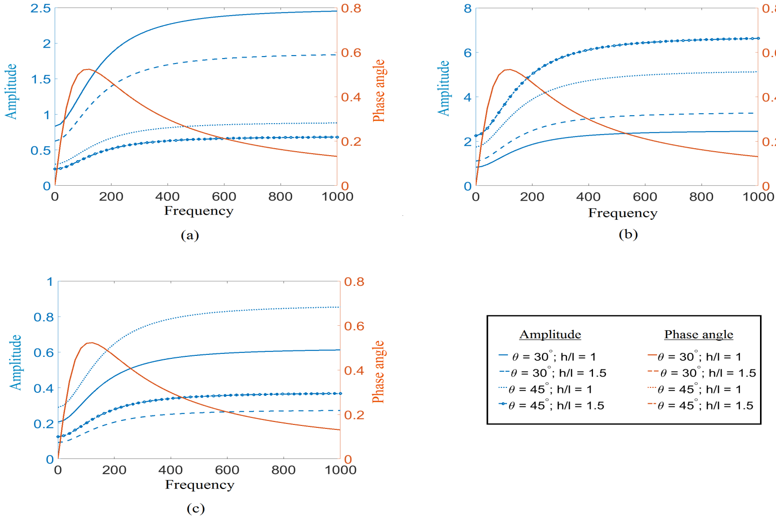


(c) $\theta = 45°; \frac{h}{l} = 1$



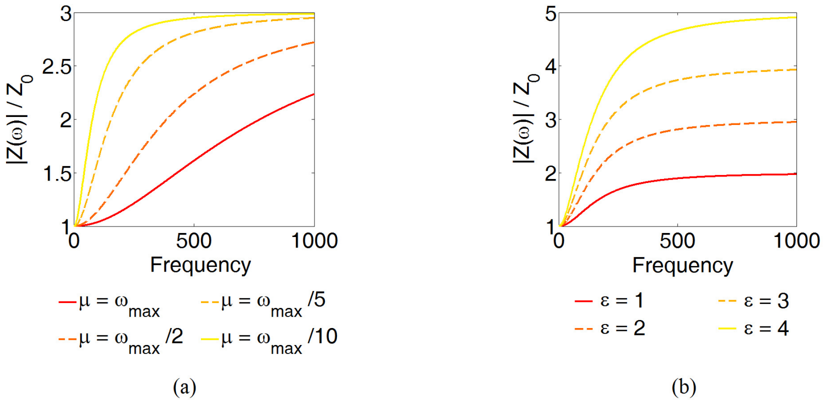
(d) $\theta = 45°; \frac{h}{l} = 1.5$

Viscoelastic properties of regular lattices: E_1, E_2, G_{12}



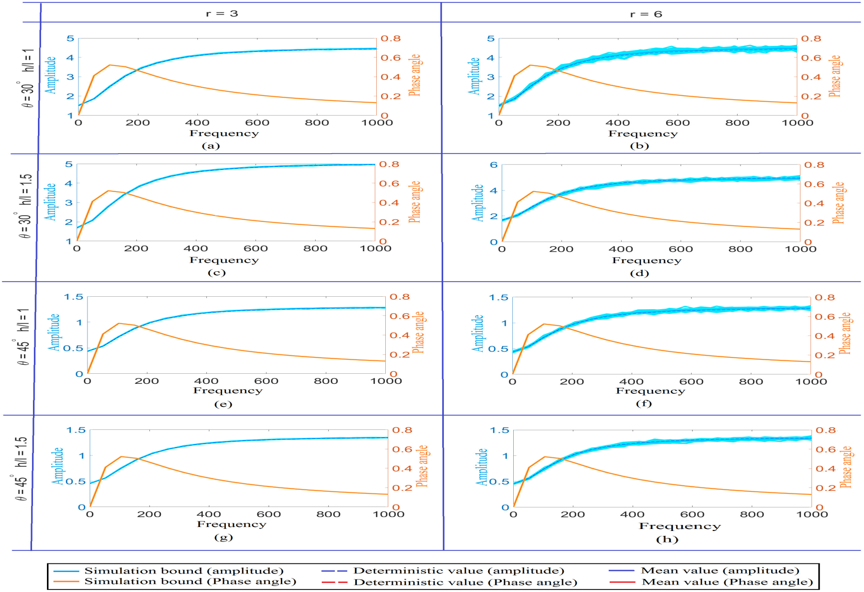
(a) Effect of viscoelasticity on the magnitude and phase angle of E_1 for regular hexagonal lattices (b) Effect of viscoelasticity on the magnitude and phase angle of E_2 for regular hexagonal lattices (c) Effect of viscoelasticity on the magnitude and phase angle of G_{12} for regular hexagonal lattices

Viscoelastic properties of regular lattices

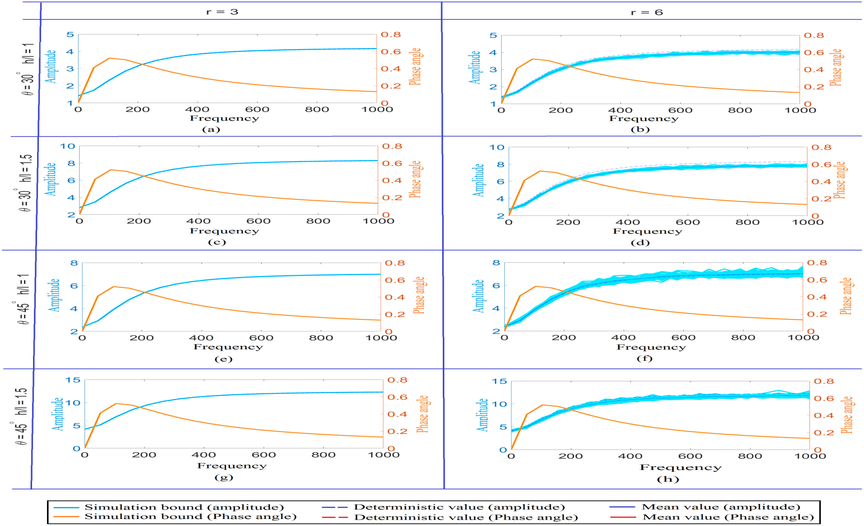


(a) Effect of variation of μ on the viscoelastic modulus of regular hexagonal lattices (considering a constant value of $\epsilon = 2$) (b) Effect of variation of ϵ on the viscoelastic modulus of regular hexagonal lattices (considering a constant value of $\mu = \omega_{max}/5$). Here Z represents the viscoelastic moduli (i.e. E_1, E_2 and G_{12}) and Z_0 is the corresponding elastic modulus value for $\omega = 0$.

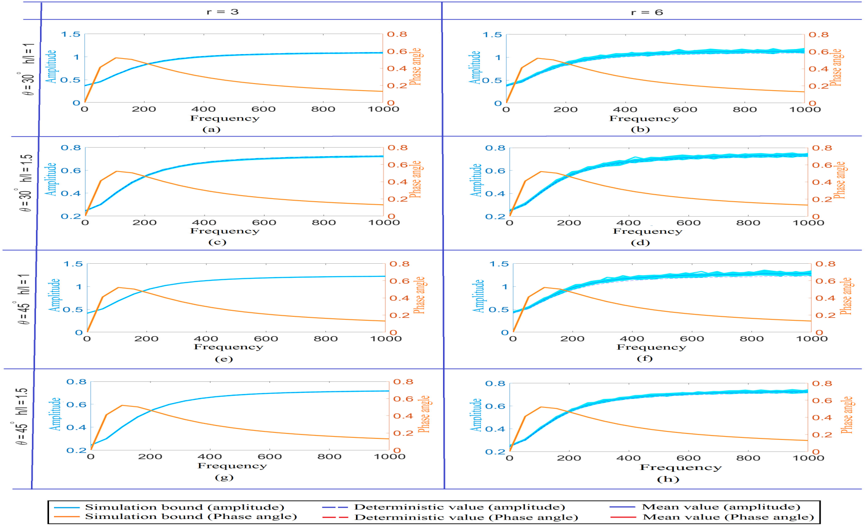
Spatially correlated irregular viscoelastic lattices: E_1



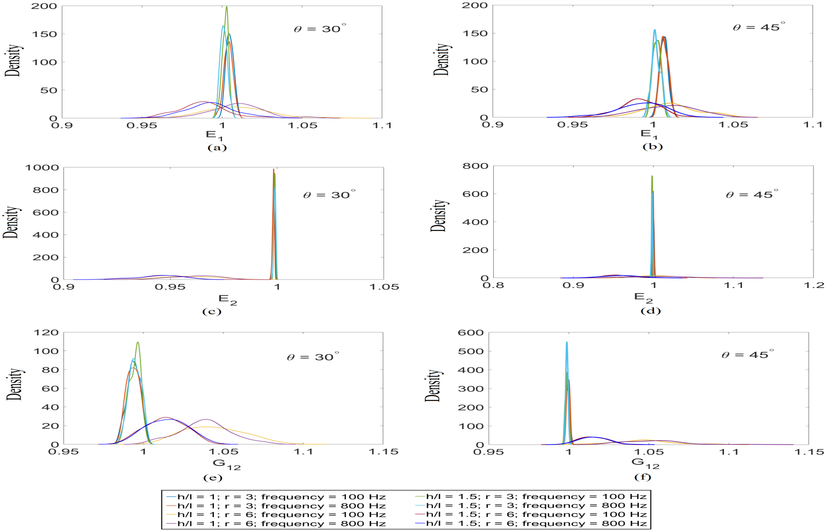
Spatially correlated irregular viscoelastic lattices: E_2



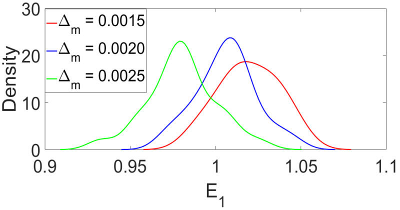
Spatially correlated irregular elastic lattices: G_{12}



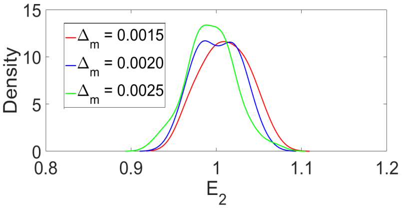
Probability density function: random geometry



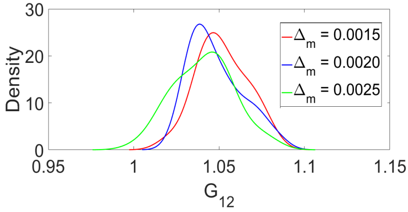
Probability density function: random material property



(a)



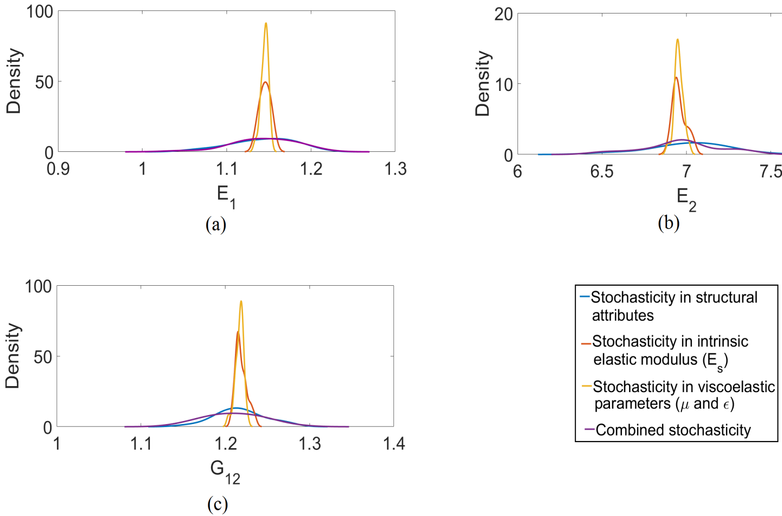
(b)



(c)

Probability density function plots for the amplitude of the elastic moduli considering randomly inhomogeneous form of stochasticity for different values of Δ_m (i.e. coefficient of variation for spatially random correlated material properties, such as E_s, μ and ϵ). Results are presented as a ratio of the values corresponding to irregular configurations and respective deterministic values (for a frequency of 800 Hz).

Combined material and geometric uncertianty



Probabilistic descriptions for the amplitudes of three effective viscoelastic properties corresponding to a frequency of 800 Hz considering individual and compound effect of stochasticity in material and structural attributes with $\Delta_{COV} = 0.006$

Conclusions

- The effect of viscoelasticity on irregular hexagonal lattices is investigated in frequency domain considering two different forms of irregularity in structural and material parameters (spatially uncorrelated and correlated).
- Spatially correlated structural and material attributes are considered to account for the effect of randomly inhomogeneous form of irregularity based on Karhunen-Loève expansion.
- The two Young's moduli and shear modulus are dependent on the viscoelastic parameters. Two in-plane Poisson's ratios depend only on structural geometry of the lattice structure.
- The classical closed-form expressions for equivalent in-plane and out of plane elastic properties of regular hexagonal lattice structures have been generalised to consider geometric and material irregularity and viscoelasticity.
- Using the principle of basic structural mechanics on a newly defined unit cell with a homogenisation technique, closed-form expressions have been obtained for E_1, E_2, ν_{12}, ν_{21} and G_{12}.
- The new results reduce to classical formulae of Gibson and Ashby for the special case of no irregularities and no viscoelastic effect.
- Future research will consider dynamic analysis (inertia effect).

Closed-form expressions: Elastic Moduli

$$E_{1V}(\omega) = \frac{t^3}{L} \sum_{j=1}^{n} \frac{\sum_{i=1}^{m} \left(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij} \right)}{\sum_{i=1}^{m} \frac{l_{1ij}^2 l_{2ij}^2 \left(l_{1ij} + l_{2ij} \right) \left(\cos \alpha_{ij} \sin \beta_{ij} - \sin \alpha_{ij} \cos \beta_{ij} \right)^2}{E_{sij} \left(1 + \epsilon_{ij} \frac{\mathrm{i}\omega}{\mu_{ij} + \mathrm{i}\omega} \right) \left(\left(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij} \right)^2 \right)}} \tag{49}$$

$$E_{2V}(\omega) = \frac{L t^3}{\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} \left(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij} \right)}{\sum_{i=1}^{m} E_{sij} \left(1 + \epsilon_{ij} \frac{\mathrm{i}\omega}{\mu_{ij} + \mathrm{i}\omega} \right) \left(l_{3ij}^2 \cos^2 \gamma_{ij} \left(l_{3ij} + \frac{l_{1ij} l_{2ij}}{l_{1ij}+l_{2ij}} \right) + \frac{l_{1ij}^2 l_{2ij}^2 \left(l_{1ij}+l_{2ij} \right) \cos^2 \alpha_{ij} \cos^2 \beta_{ij}}{\left(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij} \right)^2} \right)^{-1}}} \tag{50}$$

$$G_{12V}(\omega) = \frac{L t^3}{\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} \left(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij} \right)}{\sum_{i=1}^{m} E_{sij} \left(1 + \epsilon_{ij} \frac{\mathrm{i}\omega}{\mu_{ij} + \mathrm{i}\omega} \right) \left(l_{3ij}^2 \sin^2 \gamma_{ij} \left(l_{3ij} + \frac{l_{1ij} l_{2ij}}{l_{1ij}+l_{2ij}} \right) \right)^{-1}}} \tag{51}$$

Closed-form expressions: Poisson's ratios

$$\nu_{12eq} = -\frac{1}{L}\sum_{j=1}^{n} \frac{\sum_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum_{i=1}^{m}\frac{\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)}{\cos\alpha_{ij}\cos\beta_{ij}}} \quad (52)$$

$$\nu_{21eq} = -\frac{L}{\sum_{j=1}^{n}\frac{\sum_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum_{i=1}^{m}\frac{l_{1ij}^2 l_{2ij}^2\left(l_{1ij}+l_{2ij}\right)\cos\alpha_{ij}\cos\beta_{ij}\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)}{\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)^2\left(l_{3ij}^2\cos^2\gamma_{ij}\left(l_{3ij} + \frac{l_{1ij}l_{2ij}}{l_{1ij}+l_{2ij}}\right) + \frac{l_{1ij}^2 l_{2ij}^2\left(l_{1ij}+l_{2ij}\right)\cos^2\alpha_{ij}\cos^2\beta_{ij}}{\left(l_{1ij}\cos\alpha_{ij}-l_{2ij}\cos\beta_{ij}\right)^2}\right)}}} \quad (53)$$

Some of our papers on this topic

1. Mukhopadhyay, T. and Adhikari, S., "Effective in-plane elastic properties of quasi-random spatially irregular hexagonal lattices", International Journal of Engineering Science, (revised version submitted).
2. Mukhopadhyay, T., Mahata, A., Asle Zaeem, M. and Adhikari, S., "Effective elastic properties of two dimensional multiplanar hexagonal nano-structures", 2D Materials, 4[2] (2017), pp. 025006:1-15.
3. Mukhopadhyay, T. and Adhikari, S., "Stochastic mechanics of metamaterials", Composite Structures, 162[2] (2017), pp. 85-97.
4. Mukhopadhyay, T. and Adhikari, S., "Free vibration of sandwich panels with randomly irregular honeycomb core", ASCE Journal of Engineering Mechanics,141[6] (2016), pp. 06016008:1-5..
5. Mukhopadhyay, T. and Adhikari, S., "Equivalent in-plane elastic properties of irregular honeycombs: An analytical approach", International Journal of Solids and Structures, 91[8] (2016), pp. 169-184.
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