Spectral methods for fuzzy structural dynamics: modal vs direct approach

S Adhikari

Zienkiewicz Centre for Computational Engineering, College of Engineering, Swansea University, Wales, UK

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Outline of the talk

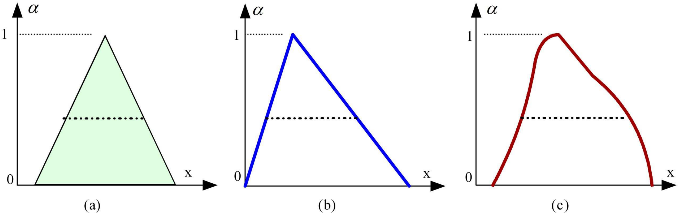
Fuzzy variables

- The consideration of uncertainties in numerical models to obtain the variability of response is becoming more common for finite element models arising in practical problems.
- When substantial statistical information exists, the theory of probability and stochastic processes offer a rich mathematical framework to represent such uncertainties.
- In a probabilistic setting, uncertainty associated with the system parameters, such as the geometric properties and constitutive relations (i.e. Young's modulus, mass density, Poisson's ratio, damping coefficients), can be modeled as random variables or stochastic processes.
- These uncertainties can be quantified and propagated, for example, using the stochastic finite element method

Fuzzy variables

- The reliable application of probabilistic approaches requires information to construct the probability density functions of uncertain parameters. This information may not be easily available for many complex practical problems.
- In such situations, non-probabilistic approaches such as interval algebra, convex modelsand fuzzy set based methods can be used.
- Fuzzy finite element analysis aims to combine the power of finite element method and uncertainty modelling capability of fuzzy variables.
- In the context of computational mechanics, the aim of a fuzzy finite element analysis is to obtain the range of certain output quantities (such as displacement, acceleration and stress) given the membership of data in the set of input variables.
- This problem, known as the uncertainty propagation problem, has taken the centre stage in recent research activities in the field.

Membership functions of a fuzzy variable



Membership functions of a fuzzy variable; (a) symmetric triangular membership function; (b) asymmetric triangular membership function; (c) general membership function. The value corresponding to $\alpha = 1$ is the crisp (or deterministic) value. The range corresponding to $\alpha = 0$ is the widest interval. Any intermediate value of $0 < \alpha < 1$, yields an interval variable with a finite lower and upper bound.

Uncertainty propagation

- We are interested in the propagation of a *m*-dimensional vector of Fuzzy variables, which can be expressed as

$$\mathbf{y} = \widehat{\mathbf{f}}(\mathbf{x}) \in \mathbb{R}^n \tag{1}$$

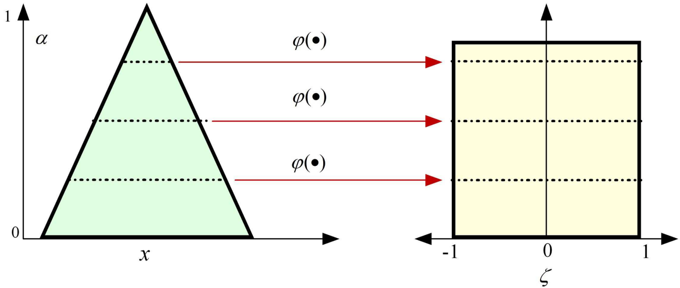
where $\widehat{\mathbf{f}}(\bullet) : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a smooth nonlinear function of the input fuzzy vector $\mathbf{x}$.

- The fuzzy uncertainty propagation problem can be formally defined as the solution of the following set of optimisation problems

$$\left.\begin{aligned} y_{jk_{\min}} &= \min\left(\widehat{f}_j\left(\mathbf{x}_{\alpha_k}\right)\right) \\ y_{jk_{\max}} &= \max\left(\widehat{f}_j\left(\mathbf{x}_{\alpha_k}\right)\right) \end{aligned}\right\} \forall \; j = 1, 2, \ldots, n;\, k = 1, 2, \ldots, r \tag{2}$$

- Two ways the solution can be made efficient: (1) evaluate $\widehat{\mathbf{f}}(\mathbf{x})$ in Eq (1) fast, and/or (2) do a better job with the optimisation in Eq (2).
- The main idea proposed here is based on efficient and accurate construction of a (hopefully cheaper!) 'response surface' for every α-cut followed by an optimization approach.

Transformation of a fuzzy variable



Transformation of a Fuzzy variable x to $\zeta \in [-1, 1]$ for different α-cuts. The transformation $\varphi(\bullet)$ maps the interval $\left[x_l^{(\alpha)}, x_h^{(\alpha)}\right] \to [-1, 1]$.

Transformation of a fuzzy variable

- Suppose the transformation $\varphi(\bullet)$ maps the interval $\left[x_{i_l}^{(\alpha)}, x_{i_h}^{(\alpha)}\right] \to [-1, 1]$. Here $x_{i_l}^{(\alpha)}$ and $x_{i_h}^{(\alpha)}$ denotes the lower and upper bound of the Fuzzy variable x_i for a given α-cut.
- Denote ζ_i as the variable bounded between $[-1, 1]$.
- Considering the variable $x_i^{(\alpha)}$ lies between $\left[x_{i_l}^{(\alpha)}, x_{i_h}^{(\alpha)}\right]$, the linear transformation $\varphi(\bullet)$ can be identified as

$$\zeta_i = \varphi(x_i^{(\alpha)}) = 2\left\{\frac{\left(x_i^{(\alpha)} - x_{i_l}^{(\alpha)}\right)}{\left(x_{i_h}^{(\alpha)} - x_{i_l}^{(\alpha)}\right)} - \frac{1}{2}\right\} \tag{3}$$

- The inverse transformation of (3) can be obtained as

$$x_i^{(\alpha)} = \left(\frac{x_{i_h}^{(\alpha)} - x_{i_l}^{(\alpha)}}{2}\right)\zeta_i + \frac{\left(x_{i_h}^{(\alpha)} + x_{i_l}^{(\alpha)}\right)}{2} \tag{4}$$

- Substituting $x_i^{(\alpha)}$ in Eq. (1) for all α and i, we can can formally express the uncertainty propagation problem in terms of the vector valued variable $\zeta = \{\zeta_i\}_{\forall i=1,\cdots,m} \in [-1,1]^m$ as

$$\mathbf{y}^{(\alpha)} = \mathbf{f}^{(\alpha)}(\zeta) \in \mathbb{R}^n \tag{5}$$

- Now we utilise the orthogonal property of Legendre polynomials in $[-1,1]$ for this uncertainty propagation problem[1].

[1] Adhikari, S. and Khodaparast, H. H., "A spectral approach for fuzzy uncertainty propagation in finite element analysis", Fuzzy Sets and Systems, 243[1] (2014), pp. 1-24.

Univariate Legendre polynomials

- In $[-1, 1]$, Legendre polynomials are orthogonal with respect to the $\mathcal{L}_2$ inner product norm

$$\int_{-1}^{1} L_j(\zeta) L_k(\zeta) d\zeta = \frac{2}{2k+1}\delta_{jk} \tag{6}$$

  δ_{jk} denotes the Kronecker delta (equal to 1 if $j = k$ and to 0 otherwise) and $L_j(\zeta)$ is the jth order Legendre polynomial.

- Each Legendre polynomial $L_j(\zeta)$ is an kth-degree polynomial and can be expressed using Rodrigues' formula as

$$L_k(\zeta) = \frac{1}{2^k k!}\frac{d^k}{d\zeta^k}\left\{(1-\zeta^2)^k\right\}; \quad k = 0, 1, 2, \cdots \tag{7}$$

Table : One dimensional Legendre Polynomials upto order 10

k	Legendre Polynomials of order k: $L_k(\zeta)$; $\zeta \in [-1, 1]$
0	1
1	ζ
2	$\frac{1}{2}\left(3\zeta^2 - 1\right)$
3	$\frac{1}{2}\left(5\zeta^3 - 3\zeta\right)$
4	$\frac{1}{8}\left(35\zeta^4 - 15\zeta^2 + 3\right)$
5	$\frac{1}{8}\left(63\zeta^5 - 70\zeta^3 + 15\zeta\right)$
6	$\frac{1}{16}\left(231\zeta^6 - 315\zeta^4 + 105\zeta^2 - 5\right)$
7	$\frac{1}{16}\left(429\zeta^7 - 693\zeta^5 + 325\zeta^3 - 35\zeta\right)$
8	$\frac{1}{128}\left(6435\zeta^8 - 12012\zeta^6 + 6930\zeta^4 - 1260\zeta^2 + 35\right)$
9	$\frac{1}{128}\left(12155\zeta^9 - 25740\zeta^7 + 18018\zeta^5 - 4620\zeta^3 + 315\zeta\right)$
10	$\frac{1}{256}\left(46189\zeta^{10} - 109395\zeta^8 + 90090\zeta^6 - 30030\zeta^4 + 3465\zeta^2 - 63\right)$

Multivariate Legendre polynomials

- Multivariate Legendre polynomials can be defined as products of univariate Legendre polynomials, similar to that of Hermite polynomials. This can be constructed by considering products of different orders. We consider following sets of integers

$$\mathbf{i} = \{i_1, i_2, \cdots i_p\}, \quad i_k \geq 1 \tag{8}$$
$$\boldsymbol{\beta} = \{\beta_1, \beta_2, \cdots \beta_p\}, \quad \beta_k \geq 0 \tag{9}$$

- The multivariate Legendre polynomial associated with sequence $(\mathbf{i}, \boldsymbol{\beta})$ as the products of univariate Legendre polynomials

$$\mathfrak{L}_{\mathbf{i},\boldsymbol{\beta}}(\zeta) = \prod_{k=1}^{p} L_{\beta_k}(\zeta_k) \tag{10}$$

The multivariate Legendre polynomials also satisfy the orthogonality condition with respect to the $\mathcal{L}_2$ inner product norm in the respective dimension.

Projection approach

- We follow the idea of homogeneous chaos proposed by Wienner involving Hermite polynomials.
- If a function $f(\zeta)$ is square integrable, it can be expanded in Homogeneous Chaos as

$$
\begin{aligned}
f(\zeta) = \hat{y}_{i_0} h_0 + \sum_{i_1=1}^{\infty} \hat{y}_{i_1} \Gamma_1(\zeta_{i_1}) \\
+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \hat{y}_{i_1, i_2} \Gamma_2(\zeta_{i_1}, \zeta_{i_2}) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \hat{y}_{i_1 i_2 i_3} \Gamma_3(\zeta_{i_1}, \zeta_{i_2}, \zeta_{i_3}) \\
+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \sum_{i_4=1}^{i_3} \hat{y}_{i_1 i_2 i_3 i_4} \Gamma_4(\zeta_{i_1}, \zeta_{i_2}, \zeta_{i_3}, \zeta_{i_4}) + \dots,
\end{aligned}
\tag{11}
$$

- $\Gamma_p(\zeta_{i_1}, \zeta_{i_2}, \cdots \zeta_{i_m})$ is m-dimensional homogeneous chaos of order p (obtained by products of one-dimensional Legendre polynomials).

- We can concisely write

$$f(\zeta) = \sum_{j=0}^{P-1} y_j \Psi_j(\zeta) \tag{12}$$

where the constant y_j and functions $\Psi_j(\bullet)$ are effectively constants $\hat{y}_k$ and functions $\Gamma_k(\bullet)$ for corresponding indices.

- Equation (12) can be viewed as the projection in the basis functions $\Psi_j(\zeta)$ with corresponding 'coordinates' y_j. The number of terms P in Eq. (12) depends on the number of variables m and maximum order of polynomials p as

$$P = \sum_{j=0}^{p} \frac{(m+j-1)!}{j!(m-1)!} = \binom{m+p}{p} \tag{13}$$

Table : Two dimensional Legendre polynomial based homogeneous chaos basis up to 4*th* order

j	p	Construction of Ψ_j	Ψ_j
0	$p=0$	L_0	1
1	$p=1$	$L_1(\zeta_1)$	ζ_1
2		$L_1(\zeta_2)$	ζ_2
3		$L_2(\zeta_1)$	$3/2\,{\zeta_1}^2 - 1/2$
4	$p=2$	$L_1(\zeta_1)L_1(\zeta_2)$	$\zeta_1\zeta_2$
5		$L_2(\zeta_2)$	$3/2\,{\zeta_2}^2 - 1/2$
6		$L_3(\zeta_1)$	$5/2\,{\zeta_1}^3 - 3/2\,\zeta_1$
7	$p=3$	$L_2(\zeta_1)L_1(\zeta_2)$	$\left(3/2\,{\zeta_1}^2 - 1/2\right)\zeta_2$
8		$L_1(\zeta_1)L_2(\zeta_2)$	$\zeta_1\left(3/2\,{\zeta_2}^2 - 1/2\right)$
9		$L_3(\zeta_2)$	$5/2\,{\zeta_2}^3 - 3/2\,\zeta_2$
10		$L_4(\zeta_1)$	$\frac{35}{8}\,{\zeta_1}^4 - \frac{15}{4}\,{\zeta_1}^2 + 3/8$
11		$L_3(\zeta_1)L_1(\zeta_2)$	$\left(5/2\,{\zeta_1}^3 - 3/2\,\zeta_1\right)\zeta_2$
12	$p=4$	$L_2(\zeta_1)L_2(\zeta_2)$	$\left(3/2\,{\zeta_1}^2 - 1/2\right)\left(3/2\,{\zeta_2}^2 - 1/2\right)$
13		$L_1(\zeta_1)L_3(\zeta_2)$	$\zeta_1\left(5/2\,{\zeta_2}^3 - 3/2\,\zeta_2\right)$
14		$L_4(\zeta_2)$	$\frac{35}{8}\,{\zeta_2}^4 - \frac{15}{4}\,{\zeta_2}^2 + 3/8$

- The unknown coefficients are obtains using a least-square method with respect to the inner product norm in $[-1,1]^m$ as

$$< \bullet, \bullet >= \frac{1}{V_m} \underbrace{\int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1}}_{m-\text{fold}} (\bullet)(\bullet) d\zeta_1 d\zeta_2 \cdots d\zeta_m \tag{14}$$

  Here the volume

$$V_m = 2^m \tag{15}$$

- Using this we have

$$\widehat{f}(\zeta) = \sum_{j=0}^{P-1} \left[\frac{\langle \Psi_j(\zeta), f(\zeta) \rangle}{\langle \Psi_j(\zeta), \Psi_j(\zeta) \rangle} \right] \Psi_j(\zeta) \tag{16}$$

  Here $\widehat{f}(\zeta)$ is an approximation to the original function $f(\zeta)$ for polynomial order upto p.

An example with two variables

- We consider the 'Camelback' function ($-3 \leq x_1 \leq 3; -2 \leq x_2 \leq 2$)

$$f_1(\mathbf{x}) = (4 - 2.1x_1^2 + x_1^4/3)x_1^2 + x_1 x_2 + (4x_2^2 - 4)x_2^2 \tag{17}$$

- Transform the variables in $[-1, 1]$: Omitting the notation α for convenience, we have

$$x_1 = 3\zeta_1 \quad \text{and} \quad x_2 = 2\zeta_2 \tag{18}$$

- The function in the transformed variables

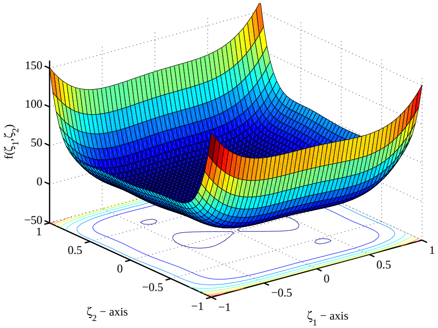
$$f_1(\zeta) = 9\left(4 - \frac{189}{10}{\zeta_1}^2 + 27{\zeta_1}^4\right){\zeta_1}^2 + 6\zeta_1\zeta_2 + 4\left(-4 + 16{\zeta_2}^2\right){\zeta_2}^2 \tag{19}$$

- The values of y_j can be obtained as

$$y_1 = \frac{21169}{1050}, y_4 = \frac{1488}{35}, y_5 = 6, y_6 = \frac{544}{21}, y_{11} = \frac{70956}{1925}, y_{15} = \frac{512}{35} \tag{20}$$

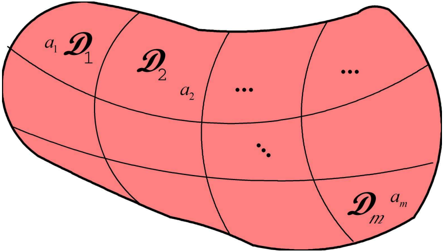
with all other values 0.

Original and fitted function

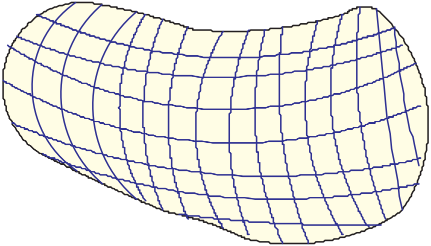


(a) The exact function

(b) Fitted function using Legendre polynomials



(c) Subdomains $\mathcal{D}_i$ with Fuzzy parameter a_i for $i = 1, 2, \cdots, M$



(d) Finite element mesh

We define the discretised fuzzy variables a_i for all the subdomains as

$$a_i = a(\mathbf{r}); \mathbf{r} \in \mathcal{D}_i, \forall i = 1, 2, \cdots, M \tag{21}$$

- A particular subdomain $\mathcal{D}_i$ is expected to contain several finite elements.
- The element stiffness matrix can be obtained following the finite element approach as

$$\mathbf{K}_e = \int_{\mathcal{D}_e} a(\mathbf{r}) \mathbf{B}^{(e)^T}(\mathbf{r}) \mathbf{B}^{(e)}(\mathbf{r}) \mathrm{d}\mathbf{r} = a_i \int_{\mathcal{D}_e} \mathbf{B}^{(e)^T}(\mathbf{r}) \mathbf{B}^{(e)}(\mathbf{r}) \mathrm{d}\mathbf{r}; \quad \mathbf{r} \in \mathcal{D}_i, \tag{22}$$

  where $\mathbf{B}^{(e)}(\mathbf{r})$ is a deterministic matrix related to the shape function used to interpolate the solution within the element e.
- Suppose $a_{i_l}^{(\alpha)}$ and $a_{i_h}^{(\alpha)}$ denotes the lower and upper bound of the Fuzzy variable a_i for a given α-cut. We transform a_i into the standard variable $\zeta_i \in [-1, 1]$ for every α-cut as

$$a_i^{(\alpha)} = \left(\frac{a_{i_h}^{(\alpha)} - x_{i_l}^{(\alpha)}}{2} \right) \zeta_i + \frac{\left(a_{i_h}^{(\alpha)} + a_{i_l}^{(\alpha)} \right)}{2} \tag{23}$$

- Substituting this into Eq. (22) we have

$$\mathbf{K}_e^{(\alpha)} = \mathbf{K}_{e_0}^{(\alpha)} + \zeta_i \mathbf{K}_{e_i}^{(\alpha)} \tag{24}$$

  where the crisp and fuzzy parts of the element stiffness matrix is given by

$$\mathbf{K}_{e_0}^{(\alpha)} = \frac{\left(a_{i_h}^{(\alpha)} + a_{i_l}^{(\alpha)}\right)}{2} \int_{\mathcal{D}_e} \mathbf{B}^{(e)^T}(\mathbf{r})\mathbf{B}^{(e)}(\mathbf{r})\mathrm{d}\mathbf{r} \tag{25}$$

$$\text{and} \quad \mathbf{K}_{e_i}^{(\alpha)} = \frac{\left(a_{i_h}^{(\alpha)} - a_{i_l}^{(\alpha)}\right)}{2} \int_{\mathcal{D}_e} \mathbf{B}^{(e)^T}(\mathbf{r})\mathbf{B}^{(e)}(\mathbf{r})\mathrm{d}\mathbf{r};\ \mathbf{r} \in \mathcal{D}_i, \tag{26}$$

- The global stiffness matrix can be obtained by the usual finite element assembly procedure and taking account of the domains of the discretised fuzzy variables. Similar approach as be used for the mass and damping matrices also.

The finite element equation for a given α-cut can be expressed as

$$\mathbf{M}^{(\alpha)}\ddot{\mathbf{U}}^{(\alpha)}(t) + \mathbf{C}^{(\alpha)}\dot{\mathbf{U}}^{(\alpha)}(t) + \mathbf{K}^{(\alpha)}\mathbf{U}^{(\alpha)}(t) = \mathbf{F}(t) \tag{27}$$

In the preceding equation $\mathbf{F} \in \mathbb{R}^n$ is the global forcing vector and $\mathbf{U}^{(\alpha)}(t)$ is the dynamic response for a given α-cut.
The global stiffness, mass and damping matrices can be expressed as

$$\mathbf{K}^{(\alpha)} = \mathbf{K}_0^{(\alpha)} + \sum_{i=1}^{m_K} +\zeta_i \mathbf{K}_i^{(\alpha)},\ \mathbf{M}^{(\alpha)} = \mathbf{M}_0^{(\alpha)} + \sum_{i=1}^{m_M} +\zeta_i \mathbf{M}_i^{(\alpha)},\ \mathbf{C}^{(\alpha)} = \mathbf{C}_0^{(\alpha)} + \sum_{i=1}^{m_C} +\zeta_i \mathbf{C}_i^{(\alpha)} \tag{28}$$

Considering the stiffness matrix for example, the crisp part can be obtained as

$$\mathbf{K}_0^{(\alpha)} = \bigoplus_e \mathbf{K}_{e_0}^{(\alpha)} \tag{29}$$

where $\bigoplus_e$ denotes the summation over all the elements.

Fuzzy FE formulation

- The fuzzy parts of the stiffness matrix related to the variable ζ_j can be given by

$$\mathbf{K}_i^{(\alpha)} = \bigoplus_{e:\mathbf{r}\in\mathcal{D}_i} \mathbf{K}_{e_i}^{(\alpha)}; \quad i = 1, 2, \cdots, m_K \tag{30}$$

  where $\bigoplus_{e:\mathbf{r}\in\mathcal{D}_i}$ denotes the summation over those elements for which the domain belongs to $\mathcal{D}_i$.

- An expression of the stiffness matrix similar to Eq. (28) can alternatively obtained directly for complex systems where different subsystems may contain different fuzzy variables. In that case $\mathbf{K}_i^{(\alpha)}$ would be block matrices, influenced by only the transformed fuzzy variable ζ_i within a particular subsystem.

Frequency domain analysis

- Considering that the initial conditions are zero for all the coordinates, taking the Fourier transformation of Eq. (27) we have

$$\left[-\omega^2\mathbf{M}^{(\alpha)} + \mathrm{i}\omega\mathbf{C}^{(\alpha)} + \mathbf{K}^{(\alpha)}\right]\mathbf{u}^{(\alpha)}(\omega) = \mathbf{f}(\omega) \tag{31}$$

  Here ω is the frequency parameter and $\mathbf{f}(\omega)$ and $\mathbf{u}^{(\alpha)}(\omega)$ are the Fourier transforms of $\mathbf{F}(t)$ and $\mathbf{U}^{(\alpha)}(t)$ respectively.
- Equation (31) can be conveniently rewritten as

$$\mathbf{D}(\omega)^{(\alpha)}\mathbf{u}^{(\alpha)}(\omega,\zeta) = \mathbf{f}(\omega) \tag{32}$$

  where the dynamic stiffness matrix

$$\mathbf{D}^{(\alpha)} = \mathbf{D}_0^{(\alpha)}(\omega) + \sum_{i=1}^{M} +\zeta_i\mathbf{D}_i^{(\alpha)}(\omega) \tag{33}$$

- Note that $\mathbf{u}^{(\alpha)}(\omega,\zeta)$ is effectivly a function of the frequency ω and well as the variables ζ.

- The crisp part of the dynamic stiffness matrix is given by

$$\mathbf{D}_0^{(\alpha)} = -\omega^2 \mathbf{M}_0 + \mathrm{i}\omega \mathbf{C}_0 + \mathbf{K}_0 \tag{34}$$

  and M is the total number of variables so that

$$M = m_K + m_M + m_C \tag{35}$$

  The matrices $\mathbf{D}_i^{(\alpha)}$ becomes the deviatoric part of the system matrices for the corresponding values of the index i.
- The elements of the solution vector $\mathbf{u}^{(\alpha)}$ in the fuzzy finite element equation (32) can be viewed as nonlinear functions of the variables $\zeta_j \in [-1, 1]$ for each value of the frequency ω.

- We project the solution vector $\mathbf{u}^{(\alpha)}(\omega, \zeta)$ in the basis of Legendre polynomials as

$$\mathbf{u}^{(\alpha)}(\omega, \zeta) = \sum_{j=0}^{P-1} \mathbf{u}_j^{(\alpha)}(\omega)\Psi_j(\zeta) \tag{36}$$

  The aim to obtain the coefficient vectors $\mathbf{u}_j \in \mathbb{C}^{N \times N}$ using a Galerkin type of error minimisation approach.
- Substituting expansion of $\mathbf{u}^{(\alpha)}(\omega, \zeta)$ in the governing equation (27), the error vector can be obtained as

$$\varepsilon^{(\alpha)}(\omega) = \left(\sum_{i=0}^{M} \mathbf{D}_i^{(\alpha)}(\omega)\zeta_i\right)\left(\sum_{j=1}^{P-1} \mathbf{u}_j^{(\alpha)}(\omega)\Psi_j(\zeta)\right) - \mathbf{f}(\omega) \tag{37}$$

  where $\zeta_0 = 1$ is used to simplify the first summation expression.

- The coefficients can be obtained by solving

$$\begin{bmatrix} \mathbf{D}_{0,0}^{(\alpha)}(\omega) & \mathbf{D}_{0,1}^{(\alpha)}(\omega) & \cdots & \mathbf{D}_{0,P-1}^{(\alpha)}(\omega) \\ \mathbf{D}_{1,0}^{(\alpha)}(\omega) & \mathbf{D}_{1,1}^{(\alpha)}(\omega) & \cdots & \mathbf{D}_{1,P-1}^{(\alpha)}(\omega) \\ \vdots & \vdots & \vdots & \\ \mathbf{D}_{P-1,0}^{(\alpha)}(\omega) & \mathbf{D}_{P-1,1}^{(\alpha)}(\omega) & \cdots & \mathbf{D}_{P-1,P-1}^{(\alpha)}(\omega) \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0^{(\alpha)}(\omega) \\ \mathbf{u}_1^{(\alpha)}(\omega) \\ \vdots \\ \mathbf{u}_{P-1}^{(\alpha)}(\omega) \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_0(\omega) \\ \mathbf{f}_1(\omega) \\ \vdots \\ \mathbf{f}_{P-1}(\omega) \end{Bmatrix} \tag{38}$$

  or in a compact notation

$$\boldsymbol{\mathcal{D}}^{(\alpha)}(\omega)\boldsymbol{\mathcal{U}}^{(\alpha)}(\omega) = \boldsymbol{\mathcal{F}}(\omega) \tag{39}$$

  where $\boldsymbol{\mathcal{D}}^{(\alpha)}(\omega) \in \mathbb{C}^{nP\times nP}$, $\boldsymbol{\mathcal{U}}^{(\alpha)}(\omega), \boldsymbol{\mathcal{F}}(\omega) \in \mathbb{C}^{nP}$.

- This equation needs to be solved for every α-cut. Once all $\mathbf{u}_j^{(\alpha)}(\omega)$ for $j = 0, 1, \ldots, P-1$ are obtained, the solution vector can be obtained from (36) for every α-cut.

Problem description

- The generalised eigenvalue problem for a given α-cut can be expressed as
$$\mathbf{K}^{(\alpha)}\mathbf{y}_j = \lambda_j^{(\alpha)}\mathbf{M}^{(\alpha)}\mathbf{y}_j^{(\alpha)}, \quad j = 1, 2, 3, \cdots \tag{40}$$
where $\lambda_j^{(\alpha)}$ and $\mathbf{y}_j^{(\alpha)}$ denote the eigenvalues and eigenvectors of the system.
- The eigenvalues and eigenvectors can be expressed by spectral decomposition as[2]

$$\lambda^{(j)}(\xi_1, \ldots \xi_M) = \sum_{k=1}^{P} \lambda_{jk}\Psi_k(\xi_1, \ldots \xi_M) \tag{41}$$
$$\mathbf{y}^{(j)}(\xi_1, \ldots \xi_M) = \sum_{k=1}^{P} \mathbf{y}_k^{(j)}\Psi_k(\xi_1, \ldots \xi_M) \tag{42}$$

where λ_{jk} and $\mathbf{y}_k^{(j)}$ are unknowns and the basis functions considered here are multivariate Legendre's polynomials.

[2] Pascual, B., Adhikari, S., "Hybrid perturbation-polynomial chaos approaches to the random algebraic eigenvalue problem", Computer Methods in Applied Mechanics and Engineering, 217-220[1] (2012), pp. 153-167.

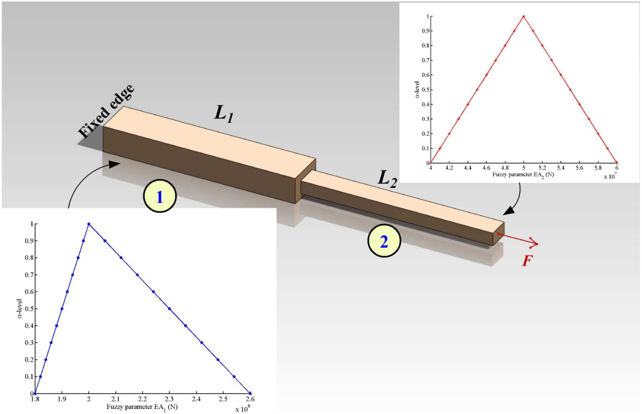


Figure : A cantilever rod subjected to an axial force modelled using two fuzzy variables. $L_1 = 1$m, $L_2 = 1$m, crisp values $\overline{EA}_1 = 20 \times 10^7$N, $\overline{EA}_2 = 5 \times 10^7$N, $F = 1$kN. The linear (triangular) membership functions for the two variables are shown in the figure.

Fuzzy variables

- At $\alpha = 0$ we have the maximum variability

$$18 \times 10^7 \leq EA_1 \leq 26 \times 10^7 \text{ and } 4 \times 10^7 \leq EA_2 \leq 6 \times 10^7 \tag{43}$$

This implies a variability between -10% and 30% for EA_1 and $\pm 20\%$ for EA_2. The maximum and minimum values of the interval for a given α-cut can be described for the two fuzzy variables as

$$EA_{1_{\min}} = \overline{EA}_1(0.9 + 0.1\alpha) \text{ and } EA_{1_{\max}} = \overline{EA}_1(1.3 - 0.3\alpha) \tag{44}$$

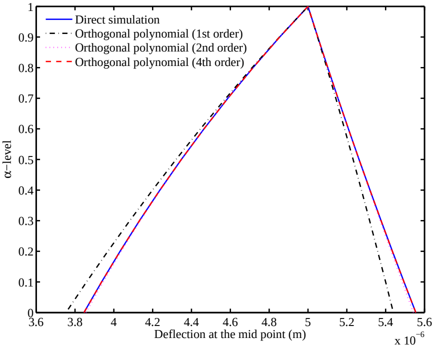
$$EA_{2_{\min}} = \overline{EA}_2(0.8 + 0.2\alpha) \text{ and } EA_{2_{\max}} = \overline{EA}_2(1.2 - 0.2\alpha) \tag{45}$$

- The stiffness matrix:

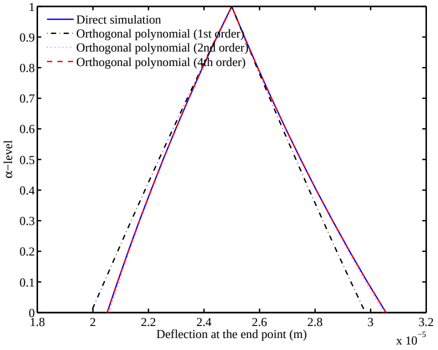
$$\mathbf{K} = \mathbf{K}_0 + \zeta_1 \mathbf{K}_1^{(\alpha)} + \zeta_2 \mathbf{K}_2^{(\alpha)} \tag{46}$$

Here $\mathbf{K}_0 \in \mathbb{R}^{100 \times 100}$ is the stiffness matrix corresponding to the crisp values, $\mathbf{K}_1^{(\alpha)}$ and $\mathbf{K}_2^{(\alpha)}$ are the coefficient matrices corresponding to a given α-cut.

Static Results



(a) Response at the mid point



(b) Response at the end point

Fuzzy description of the response of the rod at two points computed using different order of Legendre polynomial based homogeneous chaos and direct simulation.

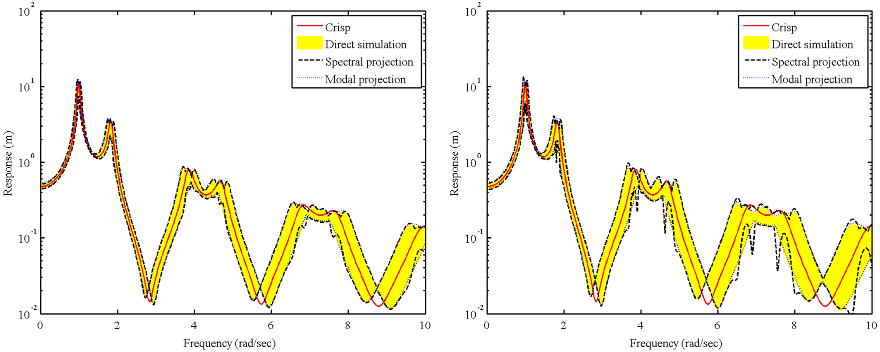
Percentage error in the mid point: Static analysis

	1st order		2nd order		4th order	
α-cut	min	max	min	max	min	max
0.0	2.4136	2.0837	0.2669	0.2280	0.0026	0.0026
0.1	1.9713	1.7254	0.1976	0.1713	0.0016	0.0016
0.2	1.5711	1.3944	0.1410	0.1241	0.0009	0.0009
0.3	1.2138	1.0925	0.0961	0.0858	0.0005	0.0005
0.4	0.9001	0.8219	0.0615	0.0558	0.0002	0.0002
0.5	0.6312	0.5848	0.0362	0.0334	0.0001	0.0001
0.6	0.4081	0.3837	0.0189	0.0177	0.0000	0.0000
0.7	0.2320	0.2214	0.0081	0.0077	0.0000	0.0000
0.8	0.1042	0.1010	0.0025	0.0024	0.0000	0.0000
0.9	0.0264	0.0259	0.0003	0.0003	0.0000	0.0000
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Percentage error at the end point: Static analysis

	1st order		2nd order		4th order	
α-cut	min	max	min	max	min	max
0.0	2.8522	2.4338	0.3426	0.2891	0.0039	0.0040
0.1	2.2891	1.9846	0.2471	0.2121	0.0023	0.0023
0.2	1.7931	1.5795	0.1719	0.1500	0.0013	0.0013
0.3	1.3618	1.2187	0.1141	0.1013	0.0007	0.0007
0.4	0.9930	0.9028	0.0713	0.0644	0.0003	0.0003
0.5	0.6848	0.6325	0.0410	0.0376	0.0001	0.0001
0.6	0.4354	0.4086	0.0208	0.0195	0.0000	0.0000
0.7	0.2435	0.2321	0.0087	0.0083	0.0000	0.0000
0.8	0.1076	0.1043	0.0026	0.0025	0.0000	0.0000
0.9	0.0268	0.0264	0.0003	0.0003	0.0000	0.0000
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Frequency response function: mid point

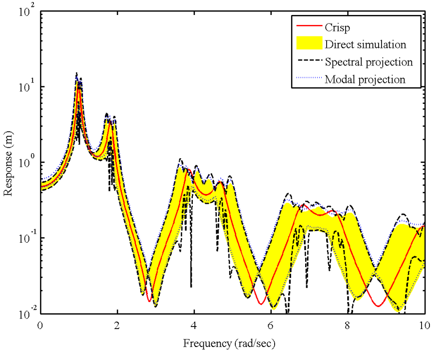


(c) $\alpha = 0.75$

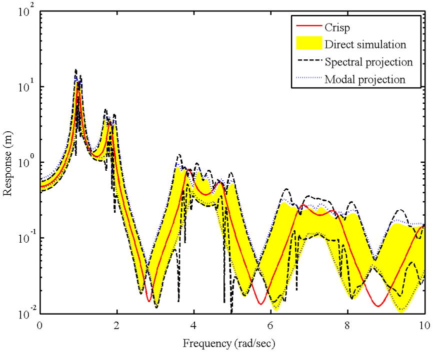
(d) $\alpha = 0.5$

Variability of the frequency response of the rod at the mid-point computed using Legendre polynomial based homogeneous chaos (4th order) and direct simulation.

Frequency response function: mid point



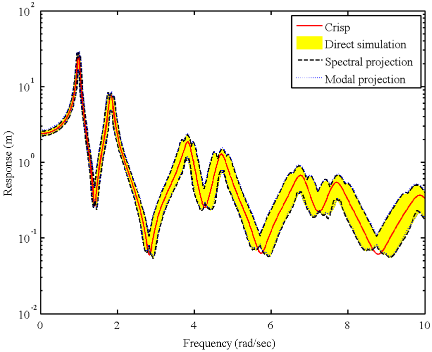
(e) $\alpha = 0.25$



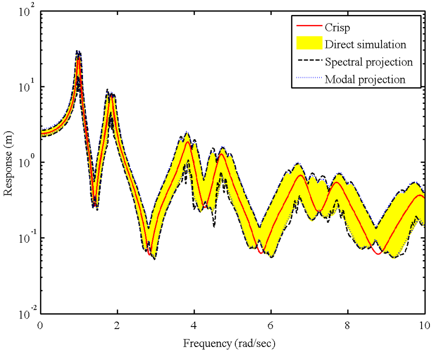
(f) $\alpha = 0$

Variability of the frequency response of the rod at the mid-point computed using Legendre polynomial based homogeneous chaos (4th order) and direct simulation.

Frequency response function: end point



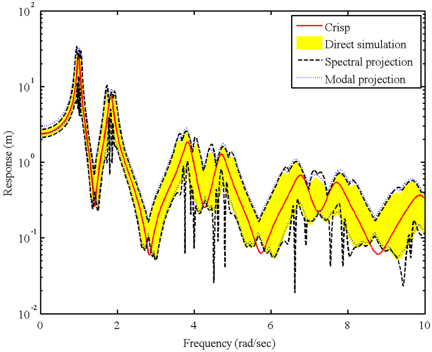
(g) $\alpha = 0.75$



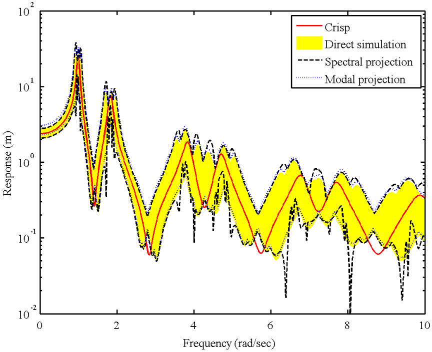
(h) $\alpha = 0.5$

Variability of the frequency response of the rod at the end-point computed using Legendre polynomial based homogeneous chaos (4th order) and direct simulation.

Frequency response function: end point



(i) $\alpha = 0.25$



(j) $\alpha = 0$

Variability of the frequency response of the rod at the end-point computed using Legendre polynomial based homogeneous chaos (4th order) and direct simulation.

Summary and conclusions

- Uncertainty propagation in complex systems with Fuzzy variables is considered. An orthogonal function expansion approach in conjunction with Galerkin type error minimisation is proposed.
- The method proposed has three major steps: (a) transformation of a fuzzy variable into a set of interval variables for different α-cuts via the membership function, (b) transformation of interval variables into a standard interval variable between $[-1, 1]$ for each α-cut, and (c) projection of the response function in the basis of multivariate orthogonal Legendre polynomials in terms of the transformed standard interval variables.
- The coefficients associated with the basis functions are obtained using a Galerkin type of error minimisation. Depending on the number of basis retained in the series expansion, it is shown that various order of approximation can be obtained.

Summary and conclusions

- A computational method is proposed for fuzzy finite element problems in structural dynamics, where the technique is generalised to vector valued functions with multiple fuzzy variables in the frequency domain.
- For a given α-cut, the complex dynamic stiffness matrix is expressed as a series involving standard interval variables and coefficient matrices. This representation significantly simplify the problem of response prediction via the proposed multivariate orthogonal Legendre polynomials expansion technique.
- The coefficient vectors in the polynomial expansion are calculated from the solution of an extended set of linear algebraic equations.
- A numerical example of axial deformation of a rod with fuzzy axial stiffness is considered to illustrate the proposed methods. The results are compared with direct numerical simulation results.

Summary and conclusions

- The spectral method proposed in this paper enables to propagate fuzzy uncertainty in a mathematically rigorous and general manner similar to what is available for propagation of probabilistic uncertainty.
- The main computational cost involves the solution of an extended set of linear algebraic equations necessary to obtain the coefficients associated with the polynomial basis functions. Future research is necessary to develop computationally efficient methods to deal with this problem arising in systems with large number of fuzzy variables.