A domain decomposition approach for hybrid stochastic problems in structural dynamics

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25th Biennial Numerical Analysis Conference, Glasgow, UK

Outline of the talk

1. **Introduction**
   - Mid-frequency vibration problem
   - Stochastic dynamical systems
2. **Domain decomposition for multi-frequency scale problems**
   - Domain decomposition for two domains
3. **Computational approach for uncertainty propagation**
   - Stochastic interface problem
   - Stochastic interior problems
4. **Numerical example**
5. **Summary and conclusion**

Linear structural dynamics

- The consideration of uncertainty becomes increasingly necessary for complex engineering structures.
- For structural dynamic systems, the approach to model uncertainty often depends on the frequency of excitation. For linear systems, this in turn is related to the wave-length scale of vibration.
- The discretised equation of motion of a linear dynamical system can be expressed as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) \tag{1}$$

M: mass matrix, **C**: damping matrix, **K**: stiffness matrix, **u**: response vector, **f**: forcing vector

Frequency domain representation

- Taking the Fourier transformation, the equation of motion of a linear dynamic system in the frequency domain can be given by

$$\mathbf{A}(\omega)\mathbf{u} = \mathbf{f} \tag{2}$$

where $\omega \in \mathbb{R}$ is the frequency of excitation and the dynamic stiffness matrix $\mathbf{A}(\omega)$ is given by

$$\mathbf{A}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K} \tag{3}$$

- Uncertainty modelling of engineering structures depends on the frequency of excitation ω.

Role of vibration frequency on uncertainty modelling

- For low- frequency vibration problems (longer wavelength), parametric uncertainty model is normally used.
- Random field or random variables can be used to model uncertain parameters and stochastic finite element method can be used to propagate uncertainty.
- For high-frequency vibration problems (shorter wavelength), nonparametric uncertainty model is normally used.
- Random matrix model, such as those based on Wishart random matrices, can be used for this purpose.
- In majority of practical engineering problems, one expects a mixture of wavelengths.

Multifrequency dynamics

- Complex dynamic structures such as aircrafts, helicopters contain several substructures.
- For a given frequency of excitation, the wavelength of vibration in different substructures can be significantly different.
- For example, in the context of an aircraft fuselage, the ring girders will have significantly longer wavelength of vibration compared to the thin panel for a given frequency of excitation.

Multifrequency dynamics

(a) Aircraft fusulage

(b) Car body

Possible sources of uncertainty

(a) parametric uncertainty - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved.
(b) model uncertainty - arising from the lack of scientific knowledge about the model which is a-priori unknown.
(c) experimental error - uncertain and unknown error percolate into the model when they are calibrated against experimental results.
(d) computational uncertainty - e.g, machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis

Parametric uncertainty: low-frequency vibration problem

- Fist few vibration modes (typically few tens) are participating in the dynamical response of interest
- Uncertainty models aim to characterise parametric uncertainty (type 'a')
- Random variable or random field models are used to represent uncertain parameters
- Well established methods such as stochastic finite element method (polynomial chaos, perturbation methods, spectral method) exist in literature
- A system matrix can be expressed as

$$\mathbf{A}(\theta_1) = \mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta_1)\mathbf{A}_i$$

$\mathbf{A}_0$: baseline model, $\xi_i(\theta_1)$: random variables

Non-parametric uncertainty: high-frequency vibration problem

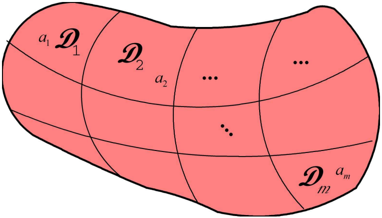
- Many vibration modes are (in hundreds) participating in the dynamical response of interest
- Uncertainty models aim to characterise non-parametric uncertainties (type 'b-d')
- Random matrix models can be used to represent uncertain system matrices
- A system matrix can be expressed as

$$\mathbf{A} = W_n(\delta_A, \mathbf{A}_0)$$

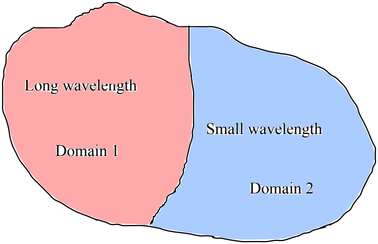
$\mathbf{A}_0$: baseline model, δ_A: dispersion parameter, W_n: Wishart random matrix of dimension n.

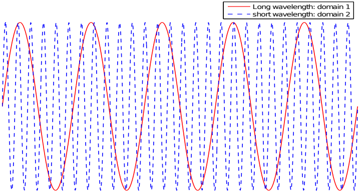
Domain decomposition method

- Developed to solve a boundary value problem by splitting it into smaller boundary value problems on subdomains
- The problems on the subdomains are independent, which makes domain decomposition methods suitable for parallel computing
- Originally developed for numerical solution of partial differential equations (not explicitly for uncertainty quantification)
- Excellent and powerful computational tools are available



Domain decomposition method





Domain 1: $\mathbf{A}(\theta_1) = \mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta_1)\mathbf{A}_i$ (dimension n_1) - parametric uncertainty

Domain 2: $\mathbf{A}(\theta_2) = W_{n_2}(\delta_A, \mathbf{A}_0)$ (dimension n_2) - nonparametric uncertainty

Two subdomains

The equation of motion of a linear dynamic system in the frequency domain is

$$\mathbf{A}(\omega)\mathbf{u} = \mathbf{f} \tag{4}$$

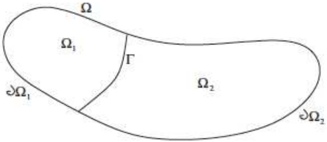
where the dynamic stiffness matrix over the whole domain Ω, $\mathbf{A}(\omega)$ is given by

$$\mathbf{A}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K} \tag{5}$$

We aim to subdivide the domain Ω into two nonoverlapping domains.

Two subdomains

The region Ω partitioned into two nonoverlapping subdomains Ω_1 and Ω_2 as below



The equilibrium equation of the system can be partitioned as

$$\begin{bmatrix} [\mathbf{A}_{II}^1]_{m_1\times m_1} & 0 & [\mathbf{A}_{I\Gamma}^1]_{m_1\times m_\Gamma} \\ 0 & [\mathbf{A}_{II}^2]_{m_2\times m_2} & [\mathbf{A}_{I\Gamma}^2]_{m2\times m_\Gamma} \\ [\mathbf{A}_{\Gamma I}^1]_{m_\Gamma\times m_1} & [\mathbf{A}_{\Gamma I}^2]_{m_\Gamma\times m_2} & [\mathbf{A}_{\Gamma\Gamma}^1+\mathbf{A}_{\Gamma\Gamma}^2]_{m_2\times m_2} \end{bmatrix} \times \begin{Bmatrix} \mathbf{u}_I^1 \\ \mathbf{u}_I^2 \\ \mathbf{u}_\Gamma \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_I^1 \\ \mathbf{f}_I^2 \\ \mathbf{f}_\Gamma^1+\mathbf{f}_\Gamma^2 \end{Bmatrix} \quad (6)$$

Equilibrium equations

The above equilibrium equation can be rearranged into following explicit forms (interior and interface problems):

$$[\mathbf{A}^1_{II}]\{\mathbf{u}^1_I\} = \{\mathbf{f}^1_I\} - [\mathbf{A}^1_{I\Gamma}]\{\mathbf{u}_\Gamma\} \tag{7}$$

$$[\mathbf{A}^2_{II}]\{\mathbf{u}^2_I\} = \{\mathbf{f}^2_I\} - [\mathbf{A}^2_{I\Gamma}]\{\mathbf{u}_\Gamma\} \tag{8}$$

$$[\underbrace{[\mathbf{A}^1_{\Gamma\Gamma}] - [\mathbf{A}^1_{\Gamma I}][\mathbf{A}^1_{II}]^{-1}[\mathbf{A}^1_{I\Gamma}]}_{\mathbf{S}_1} + \underbrace{[\mathbf{A}^2_{\Gamma\Gamma}] - [\mathbf{A}^2_{\Gamma I}][\mathbf{A}^2_{II}]^{-1}[\mathbf{A}^2_{I\Gamma}]}_{\mathbf{S}_2}]\{\mathbf{u}_\Gamma\} \tag{9}$$
$$= \underbrace{[\{\mathbf{f}^1_\Gamma\} - [\mathbf{A}^1_{\Gamma I}][\mathbf{A}^1_{II}]^{-1}]\{\mathbf{f}^1_I\}]}_{\mathbf{F}_1} + \underbrace{[\{\mathbf{f}^2_\Gamma\} - [\mathbf{A}^2_{\Gamma I}][\mathbf{A}^2_{II}]^{-1}]\{\mathbf{f}^2_I\}]}_{\mathbf{F}_2}$$

The coefficient matrix $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ is known as the *Schur complement* matrix.

Stochastic domain decomposition

We have two system matrices. For the domain with parametric uncertainty (long wavelength scale):

$$[\mathbf{A}^1(\theta_1)]_{n_1\times n_1} = \begin{bmatrix} \mathbf{A}^1_{II}(\theta_1) & \mathbf{A}^1_{I\Gamma}(\theta_1) \\ \mathbf{A}^1_{\Gamma I}(\theta_1) & \mathbf{A}^1_{\Gamma\Gamma}(\theta_1) \end{bmatrix} = \mathbf{A}^1_0 + \sum_{i=1}^{M} \xi_i(\theta_1)\mathbf{A}^1_i \tag{10}$$

with $n_1 = m_1 + m_\Gamma$.
For the domain with nonparametric uncertainty (short wavelength scale):

$$\mathbf{A}^2(\theta_2)_{n_2\times n_2} = \begin{bmatrix} \mathbf{A}^2_{II}(\theta_2) & \mathbf{A}^2_{I\Gamma}(\theta_2) \\ \mathbf{A}^2_{\Gamma I}(\theta_2) & \mathbf{A}^2_{\Gamma\Gamma}(\theta_2) \end{bmatrix} = W_{n_2}(\delta_{A_2}, \mathbf{A}^2_0) \tag{11}$$

with $n_2 = m_2 + m_\Gamma$.

Stochastic interface problem

For the stochastic interface problem we have a system of (densely) coupled m_Γ complex stochastic equations

$$[\mathbf{S}_1(\theta_1) + \mathbf{S}_2(\theta_2)]\mathbf{u}_\Gamma(\theta_1, \theta_2) = \mathbf{F}_1(\theta_1) + \mathbf{F}_2(\theta_2) \tag{12}$$

where

$$\mathbf{S}_1(\theta_1) = \mathbf{A}^1_{\Gamma\Gamma}(\theta_1) - \mathbf{A}^1_{\Gamma I}(\theta_1)[\mathbf{A}^1_{II}(\theta_1)]^{-1}\mathbf{A}^1_{I\Gamma}(\theta_1) \tag{13}$$

$$\mathbf{F}_1(\theta_1) = \mathbf{f}^1_\Gamma - \mathbf{A}^1_{\Gamma I}(\theta_1)[\mathbf{A}^1_{II}(\theta_1)]^{-1}\mathbf{f}^1_I \tag{14}$$

and

$$\mathbf{S}_2(\theta_2) = \mathbf{A}^2_{\Gamma\Gamma}(\theta_2) - \mathbf{A}^2_{\Gamma I}(\theta_2)[\mathbf{A}^2_{II}(\theta_2)]^{-1}\mathbf{A}^2_{I\Gamma}(\theta_2) \tag{15}$$

$$\mathbf{F}_2(\theta_2) = \mathbf{f}^2_\Gamma - \mathbf{A}^2_{\Gamma I}(\theta_2)[\mathbf{A}^2_{II}(\theta_2)]^{-1}\mathbf{f}^2_I \tag{16}$$

Stochastic interior problems

Solving the interface problem we have $\mathbf{u}_\Gamma(\theta_1, \theta_2)$. This can used to obtain the interior solutions as

$$\mathbf{u}_I^1(\theta_1, \theta_2) = [\mathbf{A}_{II}^1(\theta_1)]^{-1}[\mathbf{f}_I^1 - \mathbf{A}_{I\Gamma}^1(\theta_1)\mathbf{u}_\Gamma(\theta_1, \theta_2)] \quad (17)$$

$$\mathbf{u}_I^2(\theta_1, \theta_2) = [\mathbf{A}_{II}^2(\theta_1)]^{-1}[\mathbf{f}_I^2 - \mathbf{A}_{I\Gamma}^2(\theta_1)\mathbf{u}_\Gamma(\theta_1, \theta_2)] \quad (18)$$

The most computationally intensive parts of the solution process is obtaining $[\mathbf{A}_{II}^1(\theta_1)]^{-1}$ and $[\mathbf{A}_{II}^2(\theta_1)]^{-1}$ which involves the solution of m_1 and m_2 number of coupled complex stochastic equations.

Existing computational methods for uncertainty propagation can be used.

Stochastic interior problems

Recall that in the frequency domain

$$\mathbf{A}_{II}(\omega,\theta) = -\omega^2\mathbf{M}_{II}(\theta) + i\omega\mathbf{C}_{II}(\theta) + \mathbf{K}_{II}(\theta) \tag{19}$$

Assuming proportional damping model, we have

$$[\mathbf{A}_{II}(\omega,\theta)]^{-1} = \sum_{k=1}^{m} \frac{\phi_k(\theta)\phi_k^T(\theta)}{\omega_k^2(\theta) - \omega^2 + 2i\zeta_k\omega_k(\theta)} \tag{20}$$

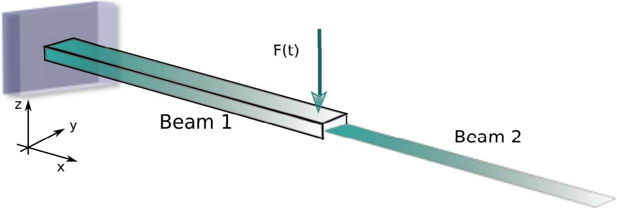
Here ζ_k are the modal damping factors and the eigenvalues are eigenvectors are obtained from

$$\mathbf{K}_{II}(\theta)\phi_k(\theta) = \omega_k^2\mathbf{M}_{II}(\theta)\phi_k(\theta), \quad k = 1, 2, \cdots \tag{21}$$

Any existing methods for random eigenvalue problem can be used (perturbation, polynomial chaos, Neumann series . . .).

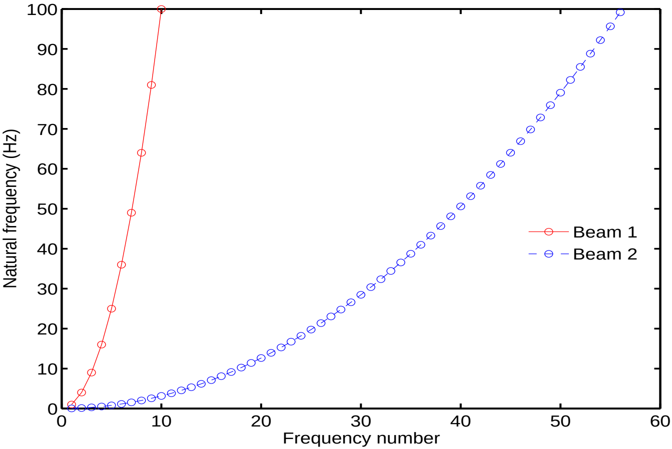
An Euler-Bernoulli beam example

- Two coupled Euler-Bernoulli beams with stochastic elasticity are considered



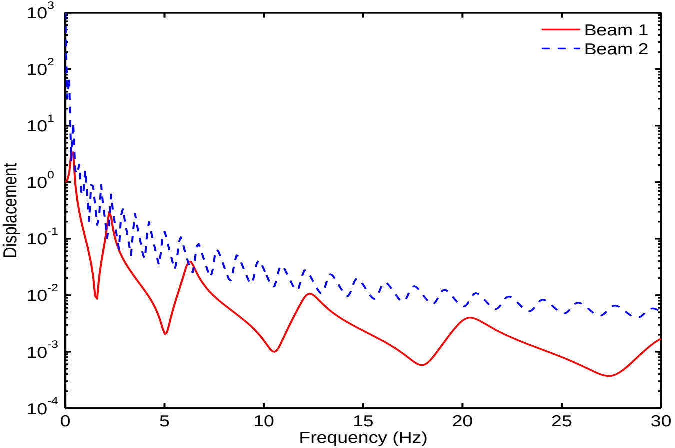
- $L_1 = 1$, $EI_{1_0} = 1/3$, $\rho A_1 = \pi^2/12$, $\zeta_1 = 0.04$
- $L_2 = L_1$, $EI_{2_0} = EI_{1_0}/10^3$, $\rho A_2 = \rho A_1$, $\zeta_2 = \zeta_1/2$
- We study the deflection of the beam under the action of a point harmonic load on the interior of beam 1.

Natural frequencies



Due to the difference in the stiffness values, beam 1 has less number of frequencies compared to beam 2 within a given frequency range.

Frequency response



Frequency response functions of the two beams in isolation (in cantilever configuration with a point load at the end).

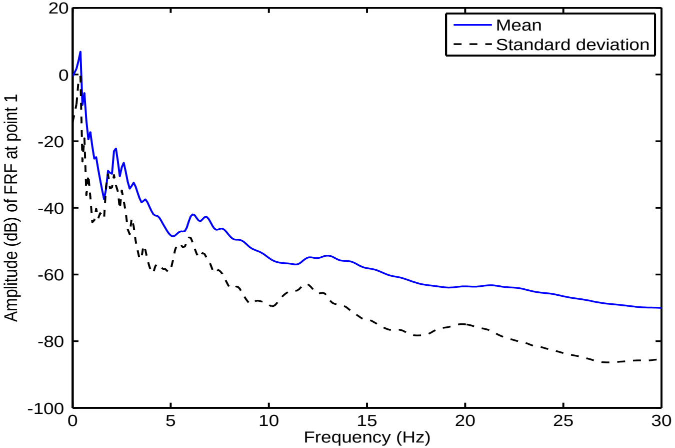
Stochastic models

- The bending modulus of the first beam is modelled by two Gaussian random variables (a discretised random field with standard deviation $\sigma_a = 0.2$). The stiffness matrix is of the form

$$\mathbf{K}^1(\theta_1) = \mathbf{K}_0 + \xi_1(\theta_1)\mathbf{K}_1^1 + \xi_2(\theta_1)\mathbf{K}_2^1$$

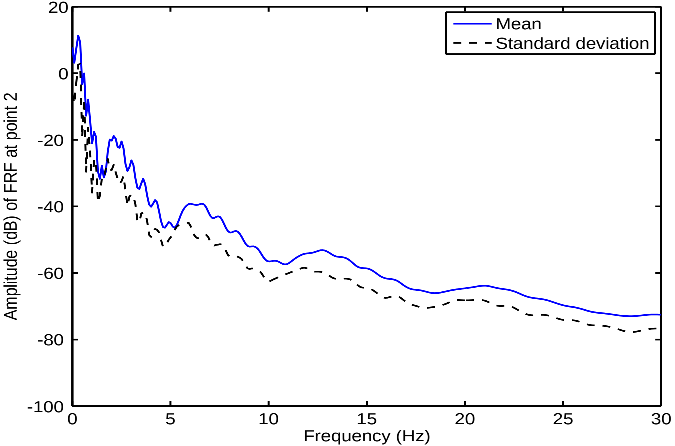
- For the second beam, an Wishart random matrix model with $\delta = 0.2$ is considered.
- The mass matrix and the damping factors are deterministic for both the beams.
- First-order perturbation is used for the interior random eigenvalue problems. 1000-sample Monte Carlo sample is used to for the interface problem.
- For the numerical calculation we used $n_1 = 60$, $n_2 = 328$. In the domain decomposition approach, $m_1 = 58$, $m_2 = 336$ and $m_\Gamma = 2$.

Stochastic response - driving point



Response statistics of the stochastic multiscale system at the driving point.

Stochastic response - tip point



Response statistics of the stochastic multiscale system at the tip.

Summary and conclusion

- The objective was to consider large and small wavelength-scale vibrations simultaneously in conjunction with relevant stochastic models.
- Parametric uncertainty model is considered for large wavelength-scale vibrations (low frequency). Random field/random variable models can be used for this purpose.
- Non-parametric uncertainty model is considered for small wavelength-scale vibrations (high frequency). Random matrix models can be used for this purpose.
- Domain decomposition method (originally proposed for parallel computation of deterministic boundary problems) is used to 'combine' two domains with two different uncertainty models.
- A simple numerical example with two wavelength-scale domains is used to illustrate the idea.