Uncertainty propagation in structural dynamics: Physics based methods

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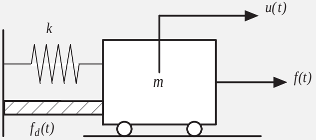
Stochastic dynamical systems across the length-scale

Outline of the talk

Few general questions

- How does system stochasticity impact the dynamic response? Does it matter?
- What is the underlying physics?
- How can we efficiently quantify uncertainty in the dynamic response for large dynamic systems?
- What about using 'black box' type response surface methods?
- Can we use modal analysis for stochastic systems?

Stochastic SDOF systems



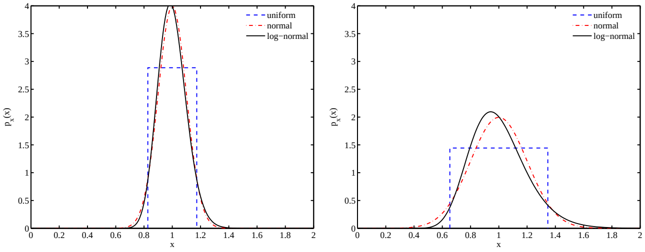
Consider a normalised single degrees of freedom system (SDOF):

$$\ddot{u}(t) + 2\zeta\omega_n\,\dot{u}(t) + \omega_n^2\,u(t) = f(t)/m \tag{1}$$

Here $\omega_n = \sqrt{k/m}$ is the natural frequency and $\xi = c/2\sqrt{km}$ is the damping ratio.

- We are interested in understanding the motion when the natural frequency of the system is perturbed in a stochastic manner.
- Stochastic perturbation can represent statistical scatter of measured values or a lack of knowledge regarding the natural frequency.

Frequency variability

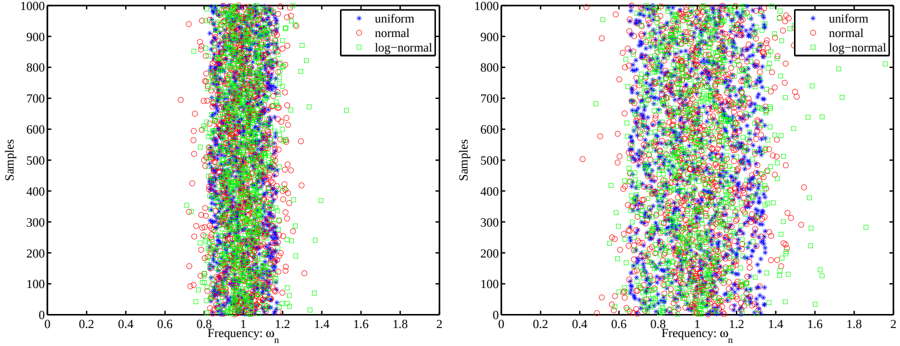


(a) Pdf: $\sigma_a = 0.1$ (b) Pdf: $\sigma_a = 0.2$

Figure : We assume that the mean of r is 1 and the standard deviation is σ_a.

- Suppose the natural frequency is expressed as $\omega_n^2 = \omega_{n_0}^2 r$, where ω_{n_0} is deterministic frequency and r is a random variable with a given probability distribution function.

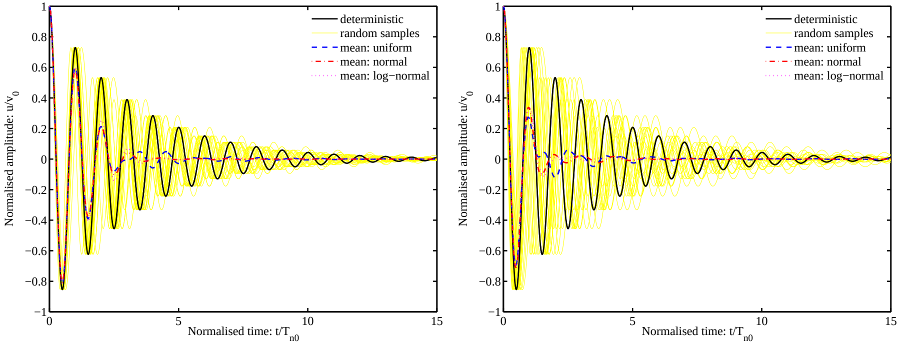
Frequency samples



(a) Frequencies: $\sigma_a = 0.1$ (b) Frequencies: $\sigma_a = 0.2$

Figure : 1000 sample realisations of the frequencies for the three distributions

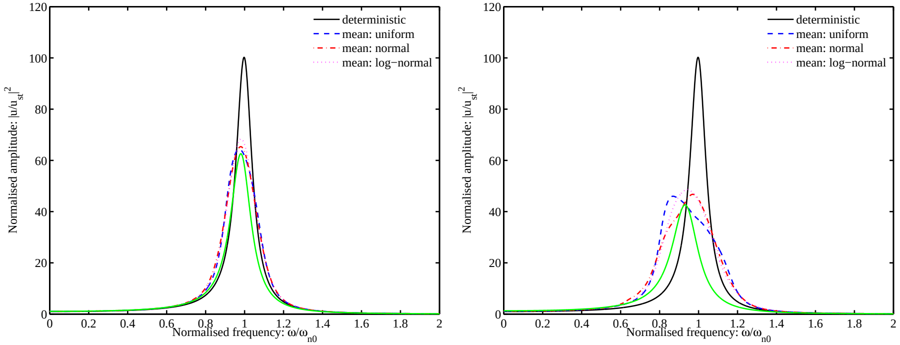
Response in the time domain



(a) Response: $\sigma_a = 0.1$ (b) Response: $\sigma_a = 0.2$

Figure : Response due to initial velocity v_0 with 5% damping

Frequency response function



(a) Response: $\sigma_a = 0.1$

(b) Response: $\sigma_a = 0.2$

Figure : Normalised frequency response function $|u/u_{st}|^2$, where $u_{st} = f/k$

Key observations

- The mean response response is more damped compared to deterministic response.
- The higher the randomness, the higher the "effective damping".
- The qualitative features are almost independent of the distribution the random natural frequency.
- We often use averaging to obtain more reliable experimental results - is it always true?

Assuming uniform random variable, we aim to explain some of these observations.

Equivalent damping

- Assume that the random natural frequencies are $\omega_n^2 = \omega_{n_0}^2(1 + \epsilon x)$, where x has zero mean and unit standard deviation.
- The normalised harmonic response in the frequency domain

$$\frac{u(\mathrm{i}\omega)}{f/k} = \frac{k/m}{[-\omega^2 + \omega_{n_0}^2(1 + \epsilon x)] + 2\mathrm{i}\xi\omega\omega_{n_0}\sqrt{1 + \epsilon x}} \tag{2}$$

- Considering $\omega_{n_0} = \sqrt{k/m}$ and frequency ratio $r = \omega/\omega_{n_0}$ we have

$$\frac{u}{f/k} = \frac{1}{[(1 + \epsilon x) - r^2] + 2\mathrm{i}\xi r\sqrt{1 + \epsilon x}} \tag{3}$$

Equivalent damping

- The squared-amplitude of the normalised dynamic response at $\omega = \omega_{n_0}$ (that is $r = 1$) can be obtained as

$$\hat{U} = \left(\frac{|u|}{f/k}\right)^2 = \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} \tag{4}$$

- Since x is zero mean unit standard deviation uniform random variable, its pdf is given by $p_x(x) = 1/2\sqrt{3}, -\sqrt{3} \leq x \leq \sqrt{3}$
- The mean is therefore

$$\begin{aligned}
\mathrm{E}\left[\hat{U}\right] &= \int \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} p_x(x)\mathrm{d}x \\
&= \frac{1}{4\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}} - \frac{\xi}{\sqrt{1-\xi^2}}\right) \\
&+ \frac{1}{4\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}} + \frac{\xi}{\sqrt{1-\xi^2}}\right)
\end{aligned} \tag{5}$$

Equivalent damping

- Note that

$$\frac{1}{2}\left\{\tan^{-1}(a+\delta)+\tan^{-1}(a-\delta)\right\}=\tan^{-1}(a)+O(\delta^2) \tag{6}$$

- Neglecting terms of the order $O(\xi^2)$ we have

$$\mathrm{E}\left[\hat{U}\right]\approx\frac{1}{2\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}}\tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}}\right)=\frac{\tan^{-1}(\sqrt{3}\epsilon/2\xi)}{2\sqrt{3}\epsilon\xi} \tag{7}$$

Equivalent damping

- For small damping, the maximum deterministic amplitude at $\omega = \omega_{n_0}$ is $1/4\xi_e^2$ where ξ_e is the equivalent damping for the mean response
- Therefore, the equivalent damping for the mean response is given by

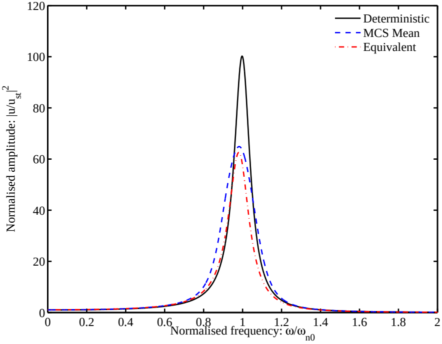
$$(2\xi_e)^2 = \frac{2\sqrt{3}\epsilon\xi}{\tan^{-1}(\sqrt{3}\epsilon/2\xi)} \tag{8}$$

- For small damping, taking the limit we can obtain

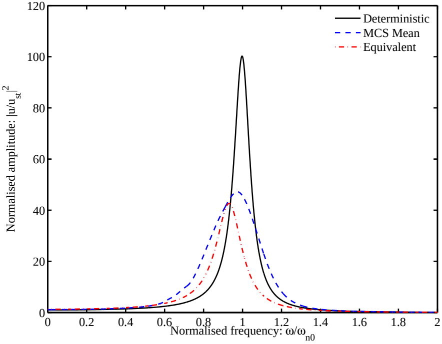
$$\xi_e \approx \frac{3^{1/4}\sqrt{\epsilon}}{\sqrt{\pi}}\sqrt{\xi} \tag{9}$$

- *The equivalent damping factor of the mean system is proportional to the square root of the damping factor of the underlying baseline system*

Equivalent frequency response function



(a) Response: $\sigma_a = 0.1$



(b) Response: $\sigma_a = 0.2$

Figure : Normalised frequency response function with equivalent damping ($\xi_e = 0.05$ in the ensembles). For the two cases $\xi_e = 0.0643$ and $\xi_e = 0.0819$ respectively.

Can we extend the ideas based on stochastic SDOF systems to stochastic MDOF systems?

Equation for motion

- The equation for motion for stochastic linear MDOF dynamic systems:

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}(\theta,t) + \mathbf{C}(\theta)\dot{\mathbf{u}}(\theta,t) + \mathbf{K}(\theta)\mathbf{u}(\theta,t) = \mathbf{f}(t) \tag{10}$$

- $\mathbf{M}(\theta) = \mathbf{M}_0 + \sum_{i=1}^{p} \mu_i(\theta_i)\mathbf{M}_i \in \mathbb{R}^{n\times n}$ is the random mass matrix, $\mathbf{K}(\theta) = \mathbf{K}_0 + \sum_{i=1}^{p} \nu_i(\theta_i)\mathbf{K}_i \in \mathbb{R}^{n\times n}$ is the random stiffness matrix, $\mathbf{C}(\theta) \in \mathbb{R}^{n\times n}$ as the random damping matrix and $\mathbf{f}(t)$ is the forcing vector
- The mass and stiffness matrices have been expressed in terms of their deterministic components ($\mathbf{M}_0$ and $\mathbf{K}_0$) and the corresponding random contributions ($\mathbf{M}_i$ and $\mathbf{K}_i$). These can be obtained from discretising stochastic fields with a finite number of random variables ($\mu_i(\theta_i)$ and $\nu_i(\theta_i)$) and their corresponding spatial basis functions.
- Proportional damping model is considered for which $\mathbf{C}(\theta) = \zeta_1\mathbf{M}(\theta) + \zeta_2\mathbf{K}(\theta)$, where ζ_1 and ζ_2 are scalars.

Stochastic modal analysis

Idea: *Extend conventional modal analysis to diagonalise the system and use the SDOF results presented earlier*

$$\mathbf{u}(\omega,\theta) = \sum_{j=1}^{n_r} \frac{\phi_j^T(\theta)\mathbf{f}}{-\omega^2 + 2i\omega\xi_j\omega_j(\theta) + \omega_j^2(\theta)} \phi_j(\theta) \tag{11}$$

Difficulty: Need to solve a random eigenvalue problem

$$\mathbf{K}(\theta)\phi_j(\theta) = \omega_j^2(\theta)\mathbf{M}(\theta)\phi_j(\theta), \quad j = 1, 2, \cdots \tag{12}$$

- Computationally very challenging, probably more challenging than the solution problem itself! A research field in its own right.
- Normalising the modes is an open problem
- We also have the conceptual problem of ‘statistical overlap’ of the modes.

Stochastic modal analysis

- Stochastic modal analysis to obtain the dynamic response in general is not a good idea
- Consider the following 3DOF example:

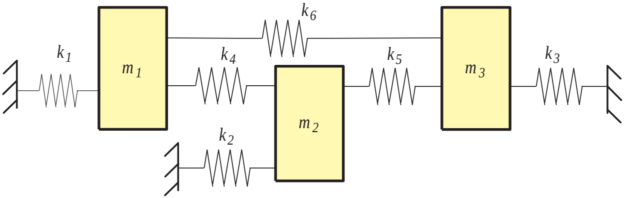
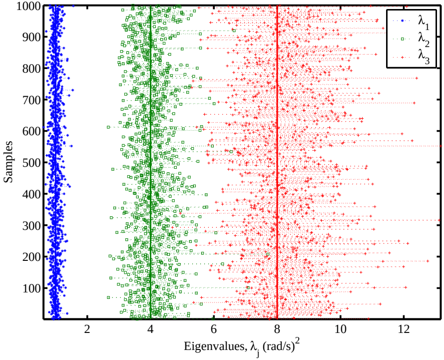
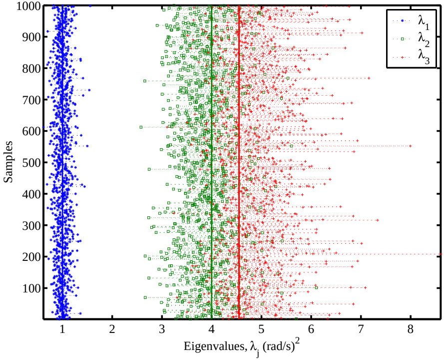


Figure : A 3DOF system with parametric uncertainty in m_i and k_i

Statistical overlap



(a) Eigenvalues are seperated



(b) Someeigenvalues are close

Figure : Scatter of the eigenvalues due to parametric uncertainties

Frequency domain representation

- For the harmonic analysis of the structural system, taking the Fourier transform

$$\left[-\omega^2\mathbf{M}(\theta) + i\omega\mathbf{C}(\theta) + \mathbf{K}(\theta)\right]\widetilde{\mathbf{u}}(\omega,\theta) = \widetilde{\mathbf{f}}(\omega) \tag{13}$$

  where $\widetilde{\mathbf{u}}(\omega,\theta)$ is the complex frequency domain system response amplitude, $\widetilde{\mathbf{f}}(\omega)$ is the amplitude of the harmonic force.
- For convenience we group the random variables associated with the mass and stiffness matrices as

$$\xi_i(\theta) = \mu_i(\theta) \quad \text{and} \quad \xi_{j+p_1}(\theta) = \nu_j(\theta) \quad \begin{array}{l} \text{for} \quad i = 1, 2, \ldots, p_1 \\ \text{and} \quad j = 1, 2, \ldots, p_2 \end{array}$$

Frequency domain representation

- Using $M = p_1 + p_2$ which we have

$$\left(\mathbf{A}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i(\omega)\right) \widetilde{\mathbf{u}}(\omega,\theta) = \widetilde{\mathbf{f}}(\omega) \tag{14}$$

  where $\mathbf{A}_0$ and $\mathbf{A}_i \in \mathbb{C}^{n\times n}$ represent the complex deterministic and stochastic parts respectively of the mass, the stiffness and the damping matrices ensemble.

- For the case of proportional damping the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written as

$$\mathbf{A}_0(\omega) = \left[-\omega^2 + i\omega\zeta_1\right]\mathbf{M}_0 + \left[i\omega\zeta_2 + 1\right]\mathbf{K}_0, \tag{15}$$

$$\mathbf{A}_i(\omega) = \left[-\omega^2 + i\omega\zeta_1\right]\mathbf{M}_i \quad \text{for} \quad i = 1,2,\ldots,p_1 \tag{16}$$

$$\text{and} \quad \mathbf{A}_{j+p_1}(\omega) = \left[i\omega\zeta_2 + 1\right]\mathbf{K}_j \quad \text{for} \quad j = 1,2,\ldots,p_2\,.$$

Time domain representation

If the time steps are fixed to Δt, then the equation of motion can be written as

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{C}(\theta)\dot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{K}(\theta)\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t}. \tag{17}$$

Following the Newmark method based on constant average acceleration scheme, the above equations can be represented as

$$[a_0\mathbf{M}(\theta) + a_1\mathbf{C}(\theta) + \mathbf{K}(\theta)]\,\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta) \tag{18}$$

$$\text{and,}\quad \mathbf{p}^{eqv}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t} + f(\mathbf{u}_t(\theta), \dot{\mathbf{u}}_t(\theta), \ddot{\mathbf{u}}_t(\theta), \mathbf{M}(\theta), \mathbf{C}(\theta)) \tag{19}$$

where $\mathbf{p}^{eqv}_{t+\Delta t}(\theta)$ is the equivalent force at time $t + \Delta t$ which consists of contributions of the system response at the previous time step.

Newmark's method

The expressions for the velocities $\dot{\mathbf{u}}_{t+\Delta t}(\theta)$ and accelerations $\ddot{\mathbf{u}}_{t+\Delta t}(\theta)$ at each time step is a linear combination of the values of the system response at previous time steps (Newmark method) as

$$\ddot{\mathbf{u}}_{t+\Delta t}(\theta) = a_0\left[\mathbf{u}_{t+\Delta t}(\theta) - \mathbf{u}_t(\theta)\right] - a_2\dot{\mathbf{u}}_t(\theta) - a_3\ddot{\mathbf{u}}_t(\theta) \tag{20}$$

$$\text{and,}\quad \dot{\mathbf{u}}_{t+\Delta t}(\theta) = \dot{\mathbf{u}}_t(\theta) + a_6\ddot{\mathbf{u}}_t(\theta) + a_7\ddot{\mathbf{u}}_{t+\Delta t}(\theta) \tag{21}$$

where the integration constants $a_i,\ i = 1, 2, \ldots, 7$ are independent of system properties and depends only on the chosen time step and some constants:

$$a_0 = \frac{1}{\alpha\Delta t^2}; \qquad a_1 = \frac{\delta}{\alpha\Delta t}; \qquad a_2 = \frac{1}{\alpha\Delta t}; \qquad a_3 = \frac{1}{2\alpha} - 1; \tag{22}$$

$$a_4 = \frac{\delta}{\alpha} - 1; \qquad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right); \qquad a_6 = \Delta t(1-\delta); \qquad a_7 = \delta\Delta t \tag{23}$$

Newmark's method

Following this development, the linear structural system in (18) can be expressed as

$$\underbrace{\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right]}_{\mathbf{A}(\theta)} \mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta). \tag{24}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. For the case of proportional damping, the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written similar to the case of frequency domain as

$$\mathbf{A}_0 = [a_0 + a_1\zeta_1]\,\mathbf{M}_0 + [a_1\zeta_2 + 1]\,\mathbf{K}_0 \tag{25}$$

$$\text{and,} \quad \mathbf{A}_i = [a_0 + a_1\zeta_1]\,\mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p_1 \tag{26}$$

$$= [a_1\zeta_2 + 1]\,\mathbf{K}_i \quad \text{for} \quad i = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\,.$$

General mathematical representation

- Whether time-domain or frequency domain methods were used, in general the main equation which need to be solved can be expressed as

$$\left(\mathbf{A}_0 + \sum_{i=1}^{M} \Gamma_i(\boldsymbol{\xi}(\theta))\mathbf{A}_i\right)\mathbf{u}(\theta) = \mathbf{f}(\theta) \tag{27}$$

  where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. These can be real or complex matrices.
- The functions $\Gamma_i(\boldsymbol{\xi}(\theta))$ can be used to introduce non-Gaussian random variables. In the special case $\Gamma_i(\boldsymbol{\xi}(\theta)) = \xi_i(\theta)$
- Generic response surface based methods have been used in literature

Polynomial Chaos expansion

After the finite truncation, the polynomial chaos expansion can be written as

$$\mathbf{u}(\theta) = \sum_{k=1}^{P} H_k(\xi(\theta))\mathbf{u}_k \tag{28}$$

where $H_k(\xi(\theta))$ are the polynomial chaoses. We need to solve a $nP \times nP$ linear equation to obtain all $\mathbf{u}_k \in \mathbb{R}^n$.

$$\begin{bmatrix} \mathbf{A}_{0,0} & \cdots & \mathbf{A}_{0,P-1} \\ \mathbf{A}_{1,0} & \cdots & \mathbf{A}_{1,P-1} \\ \vdots & \vdots & \vdots \\ \mathbf{A}_{P-1,0} & \cdots & \mathbf{A}_{P-1,P-1} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{P-1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{P-1} \end{Bmatrix} \tag{29}$$

The number of terms P increases exponentially with M:

M	2	3	5	10	20	50	100
2nd order PC	5	9	20	65	230	1325	5150
3rd order PC	9	19	55	285	1770	23425	176850

Some Observations

- The basis is a function of the pdf of the random variables only. For example, Hermite polynomials for Gaussian pdf, Legender's polynomials for uniform pdf.
- The physics of the underlying problem (static, dynamic, heat conduction, transients....) cannot be incorporated in the basis.
- For an n-dimensional output vector, the number of terms in the projection can be more than n (depends on the number of random variables). This implies that many of the vectors $\mathbf{u}_k$ are linearly dependent.
- The physical interpretation of the coefficient vectors $\mathbf{u}_k$ is not immediately obvious.
- The functional form of the response is a pure polynomial in random variables.

Possibilities of solution types

As an example, consider the frequency domain response vector of the stochastic system $\mathbf{u}(\omega, \theta)$ governed by $\left[-\omega^2 \mathbf{M}(\xi(\theta)) + i\omega \mathbf{C}(\xi(\theta)) + \mathbf{K}(\xi(\theta))\right] \mathbf{u}(\omega, \theta) = \mathbf{f}(\omega)$. Some possibilities are

$$
\begin{aligned}
\mathbf{u}(\omega, \theta) &= \sum_{k=1}^{P_1} H_k(\xi(\theta)) \mathbf{u}_k(\omega) \\
\text{or} \quad &= \sum_{k=1}^{P_2} \Gamma_k(\omega, \xi(\theta)) \phi_k \\
\text{or} \quad &= \sum_{k=1}^{P_3} a_k(\omega) H_k(\xi(\theta)) \phi_k \\
\text{or} \quad &= \sum_{k=1}^{P_4} a_k(\omega) H_k(\xi(\theta)) \mathbf{U}_k(\xi(\theta)) \quad \ldots \text{etc.}
\end{aligned}
\tag{30}
$$

Deterministic classical modal analysis?

For a deterministic system, the response vector $\mathbf{u}(\omega)$ can be expressed as

$$\mathbf{u}(\omega) = \sum_{k=1}^{P} \Gamma_k(\omega)\mathbf{u}_k$$
$$\text{where} \quad \Gamma_k(\omega) = \frac{\phi_k^T \mathbf{f}}{-\omega^2 + 2\mathrm{i}\zeta_k\omega_k\omega + \omega_k^2}$$
$$\mathbf{u}_k = \phi_k \quad \text{and} \quad P \leq n \text{ (number of dominant modes)} \tag{31}$$

Can we extend this idea to stochastic systems?

Projection in the modal space

There exist a finite set of complex frequency dependent functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$ and a complete basis $\phi_k \in \mathbb{R}^n$ for $k = 1, 2, \ldots, n$ such that the solution of the discretized stochastic finite element equation (10) can be expiressed by the series

$$\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{n} \Gamma_k(\omega, \boldsymbol{\xi}(\theta)) \phi_k \tag{32}$$

Outline of the derivation: In the first step a complete basis is generated with the eigenvectors $\phi_k \in \mathbb{R}^n$ of the generalized eigenvalue problem

$$\mathbf{K}_0 \phi_k = \lambda_{0_k} \mathbf{M}_0 \phi_k; \quad k = 1, 2, \ldots n \tag{33}$$

Projection in the modal space

- We define the matrix of eigenvalues and eigenvectors

$$\boldsymbol{\lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{R}^{n\times n}; \boldsymbol{\Phi} = \left[\phi_1, \phi_2, \ldots, \phi_n\right] \in \mathbb{R}^{n\times n} \tag{34}$$

  Eigenvalues are ordered in the ascending order: $\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n}$.
- We use the orthogonality property of the modal matrix $\boldsymbol{\Phi}$ as

$$\boldsymbol{\Phi}^T \mathbf{K}_0 \boldsymbol{\Phi} = \boldsymbol{\lambda}_0, \quad \text{and} \quad \boldsymbol{\Phi}^T \mathbf{M}_0 \boldsymbol{\Phi} = \mathbf{I} \tag{35}$$

- Using these we have

$$\begin{aligned} \boldsymbol{\Phi}^T \mathbf{A}_0 \boldsymbol{\Phi} &= \boldsymbol{\Phi}^T \left([-\omega^2 + i\omega\zeta_1]\mathbf{M}_0 + [i\omega\zeta_2 + 1]\mathbf{K}_0 \right) \boldsymbol{\Phi} \\ &= \left(-\omega^2 + i\omega\zeta_1\right) \mathbf{I} + \left(i\omega\zeta_2 + 1\right) \boldsymbol{\lambda}_0 \end{aligned} \tag{36}$$

  This gives $\boldsymbol{\Phi}^T \mathbf{A}_0 \boldsymbol{\Phi} = \boldsymbol{\Lambda}_0$ and $\mathbf{A}_0 = \boldsymbol{\Phi}^{-T} \boldsymbol{\Lambda}_0 \boldsymbol{\Phi}^{-1}$, where $\boldsymbol{\Lambda}_0 = \left(-\omega^2 + i\omega\zeta_1\right) \mathbf{I} + \left(i\omega\zeta_2 + 1\right) \boldsymbol{\lambda}_0$ and $\mathbf{I}$ is the identity matrix.

Projection in the modal space

- Hence, $\mathbf{\Lambda}_0$ can also be written as

$$\mathbf{\Lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{C}^{n\times n} \tag{37}$$

where $\lambda_{0_j} = \left(-\omega^2 + i\omega\zeta_1\right) + \left(i\omega\zeta_2 + 1\right)\lambda_j$ and λ_j is as defined in Eqn. (34). We also introduce the transformations

$$\widetilde{\mathbf{A}}_i = \mathbf{\Phi}^T \mathbf{A}_i \mathbf{\Phi} \in \mathbb{C}^{n\times n}; i = 0, 1, 2, \ldots, M. \tag{38}$$

Note that $\widetilde{\mathbf{A}}_0 = \mathbf{\Lambda}_0$ is a diagonal matrix and

$$\mathbf{A}_i = \mathbf{\Phi}^{-T} \widetilde{\mathbf{A}}_i \mathbf{\Phi}^{-1} \in \mathbb{C}^{n\times n}; i = 1, 2, \ldots, M. \tag{39}$$

Projection in the modal space

Suppose the solution of Eq. (10) is given by

$$\hat{\mathbf{u}}(\omega,\theta) = \left[\mathbf{A}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i(\omega)\right]^{-1} \mathbf{f}(\omega) \tag{40}$$

Using Eqs. (34)–(39) and the mass and stiffness orthogonality of $\boldsymbol{\Phi}$ one has

$$\hat{\mathbf{u}}(\omega,\theta) = \left[\boldsymbol{\Phi}^{-T}\boldsymbol{\Lambda}_0(\omega)\boldsymbol{\Phi}^{-1} + \sum_{i=1}^{M} \xi_i(\theta)\boldsymbol{\Phi}^{-T}\widetilde{\mathbf{A}}_i(\omega)\boldsymbol{\Phi}^{-1}\right]^{-1} \mathbf{f}(\omega)$$

$$\Rightarrow \quad \hat{\mathbf{u}}(\omega,\theta) = \boldsymbol{\Phi}\underbrace{\left[\boldsymbol{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\widetilde{\mathbf{A}}_i(\omega)\right]^{-1}}_{\boldsymbol{\Psi}\,(\omega,\boldsymbol{\xi}(\theta))} \boldsymbol{\Phi}^{-T}\mathbf{f}(\omega) \tag{41}$$

$$\text{where} \qquad \boldsymbol{\xi}(\theta) = \left\{\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta)\right\}^T.$$

Projection in the modal space

Now we separate the diagonal and off-diagonal terms of the $\widetilde{\mathbf{A}}_i$ matrices as

$$\widetilde{\mathbf{A}}_i = \mathbf{\Lambda}_i + \mathbf{\Delta}_i, \quad i = 1, 2, \ldots, M \tag{42}$$

Here the diagonal matrix

$$\mathbf{\Lambda}_i = \mathrm{diag}\left[\widetilde{\mathbf{A}}\right] = \mathrm{diag}\left[\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}\right] \in \mathbb{R}^{n \times n} \tag{43}$$

and $\mathbf{\Delta}_i = \widetilde{\mathbf{A}}_i - \mathbf{\Lambda}_i$ is an off-diagonal only matrix.

$$\mathbf{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \left[\underbrace{\mathbf{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Lambda}_i(\omega)}_{\mathbf{\Lambda}(\omega,\boldsymbol{\xi}(\theta))} + \underbrace{\sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Delta}_i(\omega)}_{\mathbf{\Delta}(\omega,\boldsymbol{\xi}(\theta))}\right]^{-1} \tag{44}$$

where $\mathbf{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right) \in \mathbb{R}^{n \times n}$ is a diagonal matrix and $\mathbf{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)$ is an off-diagonal only matrix.

Projection in the modal space

We rewrite Eq. (44) as

$$\boldsymbol{\Psi}\left(\omega,\boldsymbol{\xi}(\theta)\right)=\left[\boldsymbol{\Lambda}\left(\omega,\boldsymbol{\xi}(\theta)\right)\left[\mathbf{I}_n+\boldsymbol{\Lambda}^{-1}\left(\omega,\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\omega,\boldsymbol{\xi}(\theta)\right)\right]\right]^{-1} \qquad (45)$$

The above expression can be represented using a Neumann type of matrix series as

$$\boldsymbol{\Psi}\left(\omega,\boldsymbol{\xi}(\theta)\right)=\sum_{s=0}^{\infty}(-1)^s\left[\boldsymbol{\Lambda}^{-1}\left(\omega,\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\omega,\boldsymbol{\xi}(\theta)\right)\right]^s\boldsymbol{\Lambda}^{-1}\left(\omega,\boldsymbol{\xi}(\theta)\right) \qquad (46)$$

Projection in the modal space

Taking an arbitrary r-th element of $\hat{\mathbf{u}}(\omega,\theta)$, Eq. (41) can be rearranged to have

$$\hat{u}_r(\omega,\theta) = \sum_{k=1}^{n} \Phi_{rk} \left(\sum_{j=1}^{n} \Psi_{kj}\left(\omega, \boldsymbol{\xi}(\theta)\right) \left(\phi_j^T \mathbf{f}(\omega) \right) \right) \tag{47}$$

Defining

$$\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right) = \sum_{j=1}^{n} \Psi_{kj}\left(\omega, \boldsymbol{\xi}(\theta)\right) \left(\phi_j^T \mathbf{f}(\omega) \right) \tag{48}$$

and collecting all the elements in Eq. (47) for $r = 1, 2, \ldots, n$ one has

$$\hat{\mathbf{u}}(\omega,\theta) = \sum_{k=1}^{n} \Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right) \phi_k \tag{49}$$

Spectral functions

Definition

The functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta)), k = 1, 2, \ldots n$ *are the frequency-adaptive spectral functions as they are expressed in terms of the spectral properties of the coefficient matrices at each frequency of the governing discretized equation.*

- Each of the spectral functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$ contain infinite number of terms and they are highly nonlinear functions of the random variables $\xi_i(\theta)$.
- For computational purposes, it is necessary to truncate the series after certain number of terms.
- Different order of spectral functions can be obtained by using truncation in the expression of $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$

First-order and second order spectral functions

Definition

The different order of spectral functions $\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)), k = 1, 2, \ldots, n$ are obtained by retaining as many terms in the series expansion in Eqn. (46).

Retaining one and two terms in (46) we have

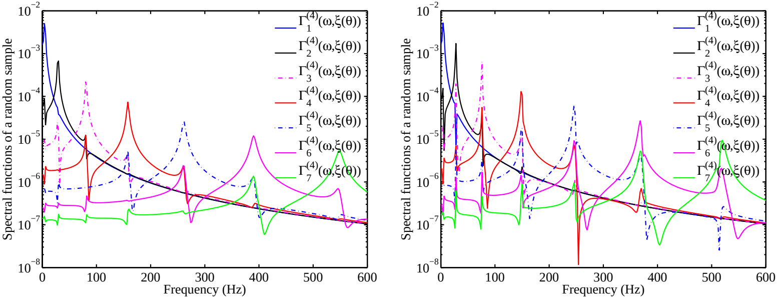
$$\boldsymbol{\Psi}^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) \tag{50}$$

$$\boldsymbol{\Psi}^{(2)}(\omega, \boldsymbol{\xi}(\theta)) = \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) - \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta))\, \boldsymbol{\Delta}(\omega, \boldsymbol{\xi}(\theta))\, \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) \tag{51}$$

which are the first and second order spectral functions respectively.

- From these we find $\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \sum_{j=1}^{n} \Psi_{kj}^{(1)}(\omega, \boldsymbol{\xi}(\theta)) \left(\phi_j^T \mathbf{f}(\omega)\right)$ are non-Gaussian random variables even if $\xi_i(\theta)$ are Gaussian random variables.

Nature of the spectral functions



(a) Spectral functions for $\sigma_a = 0.1$.

(b) Spectral functions for $\sigma_a = 0.2$.

The amplitude of first seven spectral functions of order 4 for a particular random sample under applied force. The spectral functions are obtained for two different standard deviation levels of the underlying random field: $\sigma_a = \{0.10, 0.20\}$.

Summary of the basis functions (frequency-adaptive spectral functions)

The basis functions are:

1. not polynomials in $\xi_i(\theta)$ but ratio of polynomials.
2. independent of the nature of the random variables (i.e. applicable to Gaussian, non-Gaussian or even mixed random variables).
3. not general but specific to a problem as it utilizes the eigenvalues and eigenvectors of the system matrices.
4. such that truncation error depends on the off-diagonal terms of the matrix $\mathbf{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)$.
5. showing 'peaks' when ω is near to the system natural frequencies

Next we use these frequency-adaptive spectral functions as trial functions within a Galerkin error minimization scheme.

The Galerkin approach

One can obtain constants $c_k \in \mathbb{C}$ such that the error in the following representation

$$\hat{\mathbf{u}}(\omega,\theta) = \sum_{k=1}^{n} c_k(\omega)\widehat{\Gamma}_k(\omega,\boldsymbol{\xi}(\theta))\phi_k \tag{52}$$

can be minimised in the least-square sense. It can be shown that the vector $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}^T$ satisfies the $n \times n$ complex algebraic equations $\mathbf{S}(\omega)\,\mathbf{c}(\omega) = \mathbf{b}(\omega)$ with

$$S_{jk} = \sum_{i=0}^{M} \widetilde{A}_{i_{jk}} D_{ijk}; \quad \forall\, j,k = 1,2,\ldots,n; \widetilde{A}_{i_{jk}} = \phi_j^T \mathbf{A}_i \phi_k, \tag{53}$$

$$D_{ijk} = \mathrm{E}\left[\xi_i(\theta)\widehat{\Gamma}_k(\omega,\boldsymbol{\xi}(\theta))\right], b_j = \mathrm{E}\left[\phi_j^T \mathbf{f}(\omega)\right]. \tag{54}$$

The Galerkin approach

- The error vector can be obtained as

$$\varepsilon(\omega, \theta) = \left(\sum_{i=0}^{M} \mathbf{A}_i(\omega)\xi_i(\theta) \right) \left(\sum_{k=1}^{n} c_k \widehat{\Gamma}_k(\omega, \boldsymbol{\xi}(\theta))\phi_k \right) - \mathbf{f}(\omega) \in \mathbb{C}^{N \times N} \tag{55}$$

  The solution is viewed as a projection where $\phi_k \in \mathbb{R}^n$ are the basis functions and c_k are the unknown constants to be determined. This is done for each frequency step.

- The coefficients c_k are evaluated using the Galerkin approach so that the error is made orthogonal to the basis functions, that is, mathematically

$$\varepsilon(\omega, \theta) \perp \phi_j \Rrightarrow \left\langle \phi_j, \varepsilon(\omega, \theta) \right\rangle = 0 \; \forall \, j = 1, 2, \ldots, n \tag{56}$$

The Galerkin approach

- Imposing the orthogonality condition and using the expression of the error one has

$$\mathrm{E}\left[\phi_j^T\left(\sum_{i=0}^{M}\mathbf{A}_i\xi_i(\theta)\right)\left(\sum_{k=1}^{n}c_k\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\phi_k\right)-\phi_j^T\mathbf{f}\right]=0,\forall j \qquad (57)$$

- Interchanging the $\mathrm{E}[\bullet]$ and summation operations, this can be simplified to

$$\sum_{k=1}^{n}\left(\sum_{i=0}^{M}\left(\phi_j^T\mathbf{A}_i\phi_k\right)\mathrm{E}\left[\xi_i(\theta)\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right]\right)c_k = \mathrm{E}\left[\phi_j^T\mathbf{f}\right] \qquad (58)$$

$$\text{or} \quad \sum_{k=1}^{n}\left(\sum_{i=0}^{M}\widetilde{A}_{i_{jk}}D_{ijk}\right)c_k = b_j \qquad (59)$$

Model Reduction by reduced number of basis

- Suppose the eigenvalues of $\mathbf{A}_0$ are arranged in an increasing order such that

$$\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n} \tag{60}$$

- From the expression of the spectral functions observe that the eigenvalues ($\lambda_{0_k} = \omega_{0_k}^2$) appear in the denominator:

$$\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \frac{\phi_k^T \mathbf{f}(\omega)}{\Lambda_{0_k}(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\Lambda_{i_k}(\omega)} \tag{61}$$

where $\Lambda_{0_k}(\omega) = -\omega^2 + i\omega(\zeta_1 + \zeta_2\omega_{0_k}^2) + \omega_{0_k}^2$

- The series can be truncated based on the magnitude of the eigenvalues relative to the frequency of excitation. Hence for the frequency domain analysis all the eigenvalues that cover almost twice the frequency range under consideration can be chosen.

Computational method

- The mean vector can be obtained as

$$\bar{\mathbf{u}} = \mathrm{E}\left[\hat{\mathbf{u}}(\theta)\right] = \sum_{k=1}^{p} c_k \mathrm{E}\left[\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right] \boldsymbol{\phi}_k \tag{62}$$

- The covariance of the solution vector can be expressed as

$$\boldsymbol{\Sigma}_u = \mathrm{E}\left[\left(\hat{\mathbf{u}}(\theta) - \bar{\mathbf{u}}\right)\left(\hat{\mathbf{u}}(\theta) - \bar{\mathbf{u}}\right)^T\right] = \sum_{k=1}^{p}\sum_{j=1}^{p} c_k c_j \Sigma_{\Gamma_{kj}} \boldsymbol{\phi}_k \boldsymbol{\phi}_j^T \tag{63}$$

  where the elements of the covariance matrix of the spectral functions are given by

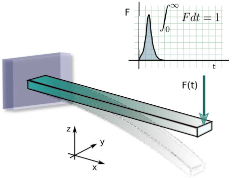
$$\Sigma_{\Gamma_{kj}} = \mathrm{E}\left[\left(\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta)) - \mathrm{E}\left[\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right]\right)\left(\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta)) - \mathrm{E}\left[\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta))\right]\right)\right] \tag{64}$$

Summary of the computational method

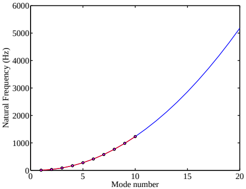
1. Solve the generalized eigenvalue problem associated with the mean mass and stiffness matrices to generate the orthonormal basis vectors: $\mathbf{K}_0\mathbf{\Phi} = \mathbf{M}_0\mathbf{\Phi}\boldsymbol{\lambda}_0$
2. Select a number of samples, say $N_{\rm samp}$. Generate the samples of basic random variables $\xi_i(\theta), i = 1, 2, \ldots, M$.
3. Calculate the spectral basis functions (for example, first-order): $\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right) = \frac{\phi_k^T \mathbf{f}(\omega)}{\Lambda_{0_k}(\omega) + \sum_{i=1}^M \xi_i(\theta)\Lambda_{i_k}(\omega)}$, for $k = 1, \cdots p$, $p < n$
4. Obtain the coefficient vector: $\mathbf{c}(\omega) = \mathbf{S}^{-1}(\omega)\mathbf{b}(\omega) \in \mathbb{R}^n$, where $\mathbf{b}(\omega) = \widetilde{\mathbf{f}(\omega)} \odot \overline{\mathbf{\Gamma}(\omega)}$, $\mathbf{S}(\omega) = \mathbf{\Lambda}_0(\omega) \odot \mathbf{D}_0(\omega) + \sum_{i=1}^M \widetilde{\mathbf{A}}_i(\omega) \odot \mathbf{D}_i(\omega)$ and $\mathbf{D}_i(\omega) = \mathrm{E}\left[\mathbf{\Gamma}(\omega,\theta)\xi_i(\theta)\mathbf{\Gamma}^T(\omega,\theta)\right], \forall\, i = 0, 1, 2, \ldots, M$
5. Obtain the samples of the response from the spectral series: $\hat{\mathbf{u}}(\omega,\theta) = \sum_{k=1}^p c_k(\omega)\Gamma_k(\boldsymbol{\xi}(\omega,\theta))\phi_k$

The Euler-Bernoulli beam example

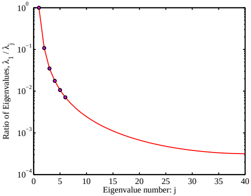
- An Euler-Bernoulli cantilever beam with stochastic bending modulus for a specified value of the correlation length and for different degrees of variability of the random field.



(c) Euler-Bernoulli beam



(d) Natural frequency distribution.



(e) Eigenvalue ratio of KL decomposition

- Length : 1.0 m, Cross-section : 39×5.93 mm^2, Young's Modulus: 2×10^{11} Pa.
- Load: Unit impulse at $t = 0$ on the free end of the beam.

Problem details

- The bending modulus of the cantilever beam is taken to be a homogeneous stationary Gaussian random field of the form

$$EI(x, \theta) = EI_0(1 + a(x, \theta)) \tag{65}$$

  where x is the coordinate along the length of the beam, EI_0 is the estimate of the mean bending modulus, $a(x, \theta)$ is a zero mean stationary random field.
- The covariance kernel associated with this random field is

$$C_a(x_1, x_2) = \sigma_a^2 e^{-(|x_1 - x_2|)/\mu_a} \tag{66}$$

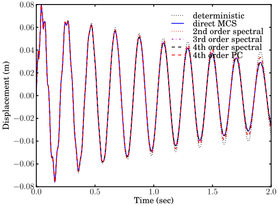
  where μ_a is the correlation length and σ_a is the standard deviation.
- A correlation length of $\mu_a = L/5$ is considered in the present numerical study.

Problem details

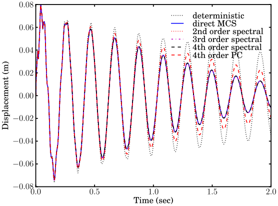
The random field is assumed to be Gaussian. The results are compared with the polynomial chaos expansion.

- The number of degrees of freedom of the system is n = 200.
- The K.L. expansion is truncated at a finite number of terms such that 90% variability is retained.
- direct MCS have been performed with 10,000 random samples and for three different values of standard deviation of the random field, $\sigma_a = 0.05, 0.1, 0.2$.
- Constant modal damping is taken with 1% damping factor for all modes.
- Time domain response of the free end of the beam is sought under the action of a unit impulse at $t = 0$
- Upto 4^{th} order spectral functions have been considered in the present problem. Comparison have been made with 4^{th} order Polynomial chaos results.

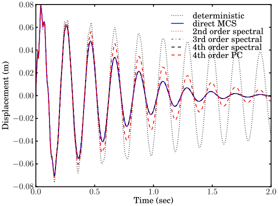
Mean of the response



(f) Mean, $\sigma_a = 0.05$.



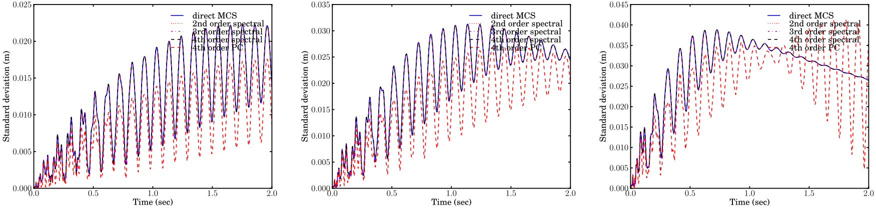
(g) Mean, $\sigma_a = 0.1$.



(h) Mean, $\sigma_a = 0.2$.

- Time domain response of the deflection of the tip of the cantilever for three values of standard deviation σ_a of the underlying random field.
- Spectral functions approach approximates the solution accurately.
- For long time-integration, the discrepancy of the 4^{th} order PC results increases.

Standard deviation of the response



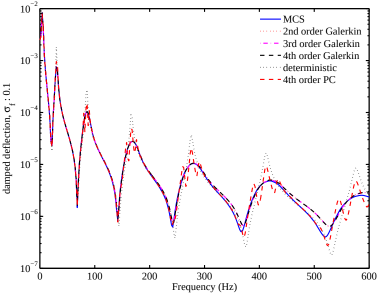
(i) Standard deviation of deflection, $\sigma_a = 0.05$.

(j) Standard deviation of deflection, $\sigma_a = 0.1$.

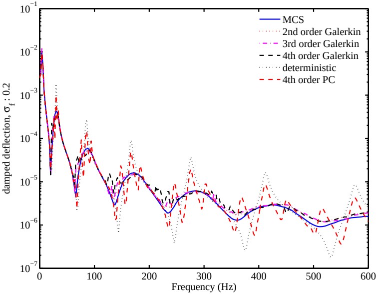
(k) Standard deviation of deflection, $\sigma_a = 0.2$.

- The standard deviation of the tip deflection of the beam.
- Since the standard deviation comprises of higher order products of the Hermite polynomials associated with the PC expansion, the higher order moments are less accurately replicated and tend to deviate more significantly.

Frequency domain response: mean



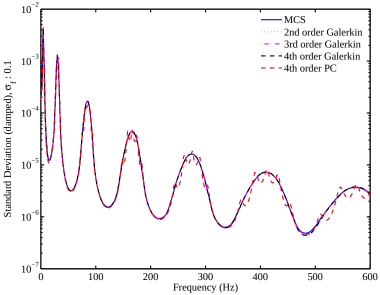
(l) Beam deflection for $\sigma_a = 0.1$.



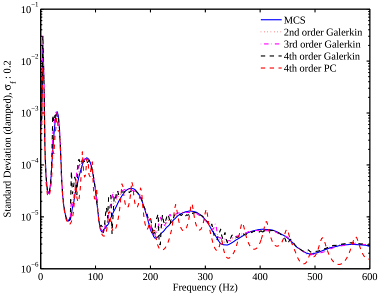
(m) Beam deflection for $\sigma_a = 0.2$.

The frequency domain response of the deflection of the tip of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with 10,000 sample MCS and for $\sigma_a = \{0.10, 0.20\}$.

Frequency domain response: standard deviation



(n) Standard deviation of the response for $\sigma_a = 0.1$.



(o) Standard deviation of the response for $\sigma_a = 0.2$.

The standard deviation of the tip deflection of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$.

Experimental investigations

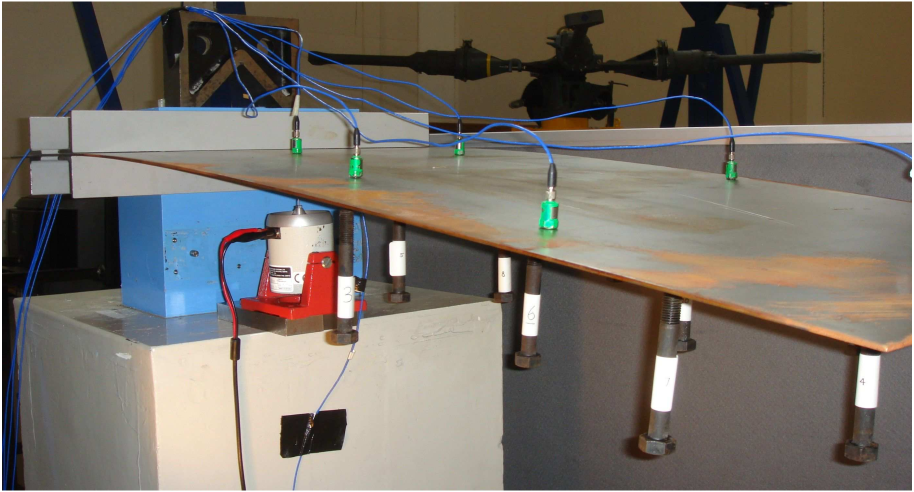
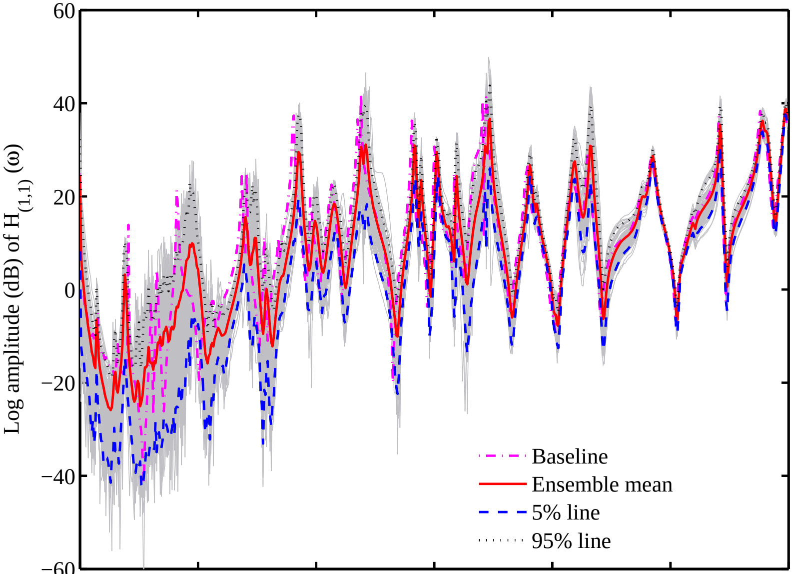


Figure : A cantilever plate with randomly attached oscillators - Probabilistic

Measured frequency response function



Conclusions

- The mean response of a damped stochastic system is more damped than the underlying baseline system
- For small damping, $\xi_e \approx \frac{3^{1/4}\sqrt{\epsilon}}{\sqrt{\pi}}\sqrt{\xi}$
- Random modal analysis may not be practical or physically intuitive for stochastic multiple degrees of freedom systems
- Conventional response surface based methods fails to capture the physics of damped dynamic systems
- Proposed spectral function approach uses the undamped modal basis and can capture the statistical trend of the dynamic response of stochastic damped MDOF systems

Conclusions

- The solution is projected into the modal basis and the associated stochastic coefficient functions are obtained at each frequency step (or time step).
- The coefficient functions, called as the spectral functions, are expressed in terms of the spectral properties (natural frequencies and mode shapes) of the system matrices.
- The proposed method takes advantage of the fact that for a given maximum frequency only a small number of modes are necessary to represent the dynamic response. This modal reduction leads to a significantly smaller basis.

Assimilation with experimental measurements

In the frequency domain, the response can be simplified as

$$\mathbf{u}(\omega,\theta) \approx \sum_{k=1}^{n_r} \frac{\phi_k^T \mathbf{f}(\omega)}{-\omega^2 + 2i\omega\zeta_k\omega_{0_k} + \omega_{0_k}^2 + \sum_{i=1}^{M} \xi_i(\theta)\Lambda_{i_k}(\omega)}\phi_k$$

Some parts can be obtained from experiments while other parts can come from stochastic modelling.