Uncertainty propagation is structural dynamics: Theory and Applications

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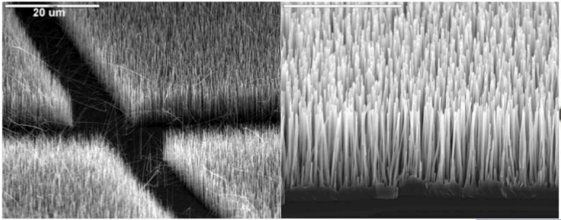
Stochastic dynamic systems

Stochastic dynamical systems across the length-scale



Outline of the talk

Objectives

- How does system stochasticity impact the dynamic response? Does it matter?
- What is the underlying physics?
- How can we efficiently quantify uncertainty in the dynamic response for large dynamic systems?
- What about using ‘black box’ type response surface methods?
- Can we use modal analysis for stochastic systems?

Stochastic SDOF systems

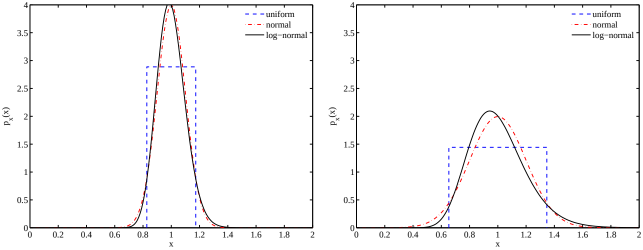
Consider a normalised single degrees of freedom system (SDOF):

$$\ddot{u}(t) + 2\zeta\omega_n \dot{u}(t) + \omega_n^2 u(t) = f(t)/m \tag{1}$$

Here $\omega_n = \sqrt{k/m}$ is the natural frequency and $\zeta = c/2\sqrt{km}$ is the damping ratio.

- We are interested in understanding the motion when the natural frequency of the system is perturbed in a stochastic manner.
- Stochastic perturbation can represent statistical scatter of measured values or a lack of knowledge regarding the natural frequency.

Frequency variability

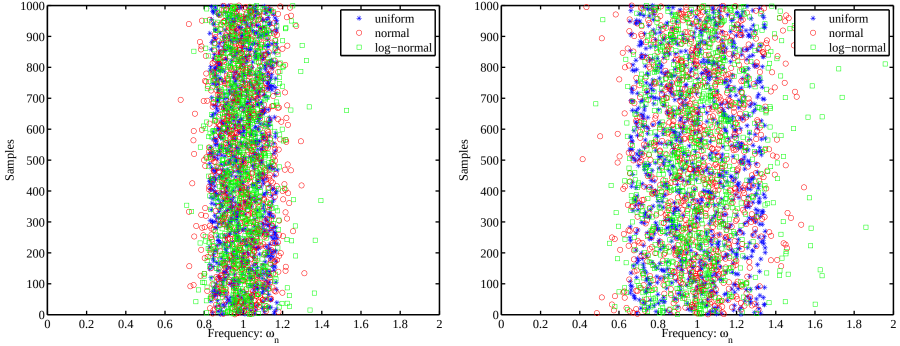


(a) Pdf: $\sigma_a = 0.1$ (b) Pdf: $\sigma_a = 0.2$

Figure : We assume that the mean of r is 1 and the standard deviation is σ_a.

- Suppose the natural frequency is expressed as $\omega_n^2 = \omega_{n_0}^2 r$, where ω_{n_0} is deterministic frequency and r is a random variable with a given probability distribution function.

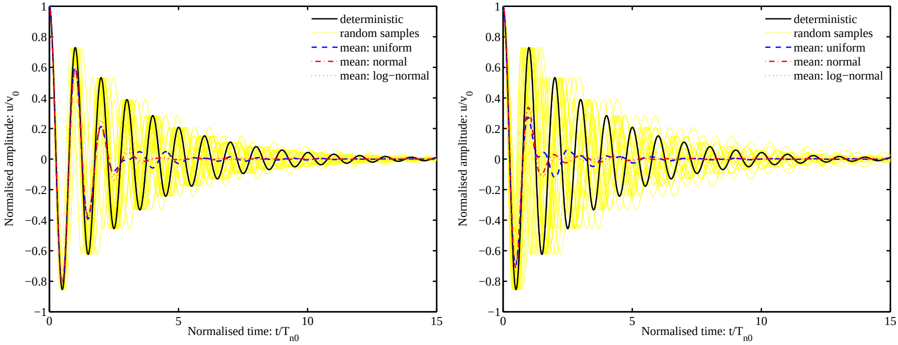
Frequency samples



(a) Frequencies: $\sigma_a = 0.1$ (b) Frequencies: $\sigma_a = 0.2$

Figure : 1000 sample realisations of the frequencies for the three distributions

Response in the time domain

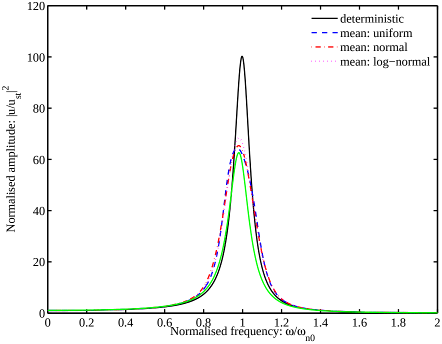


(a) Response: $\sigma_a = 0.1$

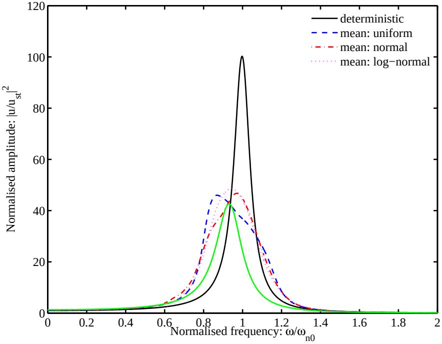
(b) Response: $\sigma_a = 0.2$

Figure : Response due to initial velocity v_0 with 5% damping

Frequency response function



(a) Response: $\sigma_a = 0.1$



(b) Response: $\sigma_a = 0.2$

Figure : Normalised frequency response function $|u/u_{st}|^2$, where $u_{st} = f/k$

Key observations

- The mean response response is more damped compared to deterministic response.
- The higher the randomness, the higher the ”effective damping’.
- The qualitative features are almost independent of the distribution random frequencies.
- We often use averaging to obtain more reliable experimental results - is it always true?

Equivalent damping

- The mean response is more damped compared to deterministic response
- The higher the randomness, the higher the ‘effective damping’
- The qualitative features are almost independent of the distribution random frequencies

Assuming uniform random variable, we aim to explain some of these observations.

Equivalent damping

- Assume that the random natural frequencies are $\omega_n^2 = \omega_{n_0}^2(1+\epsilon x)$, where x has zero mean and unit standard deviation.
- The amplitude of the normalised dynamic response at $\omega = \omega_{n_0}$ in the frequency domain can be obtained as

$$\hat{U} = \left(\frac{|u|}{f/k}\right)^2 = \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} \tag{2}$$

- Since x is zero mean unit standard deviation uniform random variable, its pdf is given by $p_x(x) = 1/2\sqrt{3}, -\sqrt{3} \leq x \leq \sqrt{3}$
- The mean is therefore

$$\mathrm{E}\left[\hat{U}\right] = \int \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} p_x(x)\mathrm{d}x = \frac{\tan^{-1}(\sqrt{3}\epsilon/2\xi)}{2\sqrt{3}\epsilon\xi} \tag{3}$$

Equivalent damping

- For small damping, the maximum amplitude at $\omega = \omega_{n_0}$ is $1/4\xi_e^2$ where ξ_e is the equivalent damping for the mean response
- Therefore, the equivalent damping for the mean response is given by

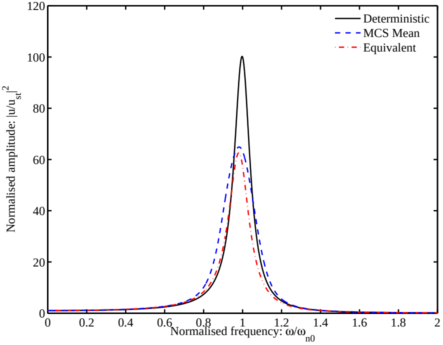
$$(2\xi_e)^2 = \frac{2\sqrt{3}\epsilon\xi}{\tan^{-1}(\sqrt{3}\epsilon/2\xi)} \tag{4}$$

- For small damping, taking the limit we can obtain

$$\xi_e \approx \frac{3^{1/4}\sqrt{\epsilon}}{\sqrt{\pi}}\sqrt{\xi} \tag{5}$$

- *The equivalent damping factor of the mean system is proportional to the square root of the damping factor of the underlying baseline system*

Equivalent frequency response function



(a) Response: $\sigma_a = 0.1$

(b) Response: $\sigma_a = 0.2$

Figure : Normalised frequency response function with equivalent damping

Equation for motion

- The equation for motion for stochastic linear MDOF dynamic systems:

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}(\theta,t) + \mathbf{C}(\theta)\dot{\mathbf{u}}(\theta,t) + \mathbf{K}(\theta)\mathbf{u}(\theta,t) = \mathbf{f}(t) \quad (6)$$

- $\mathbf{M}(\theta) = \mathbf{M}_0 + \sum_{i=1}^{p} \mu_i(\theta_i)\mathbf{M}_i \in \mathbb{R}^{n\times n}$ is the random mass matrix, $\mathbf{K}(\theta) = \mathbf{K}_0 + \sum_{i=1}^{p} \nu_i(\theta_i)\mathbf{K}_i \in \mathbb{R}^{n\times n}$ is the random stiffness matrix, $\mathbf{C}(\theta) \in \mathbb{R}^{n\times n}$ as the random damping matrix and $\mathbf{f}(t)$ is the forcing vector
- The mass and stiffness matrices have been expressed in terms of their deterministic components ($\mathbf{M}_0$ and $\mathbf{K}_0$) and the corresponding random contributions ($\mathbf{M}_i$ and $\mathbf{K}_i$). These can be obtained from discretising stochastic fields with a finite number of random variables ($\mu_i(\theta_i)$ and $\nu_i(\theta_i)$) and their corresponding spatial basis functions.
- Proportional damping model is considered for which $\mathbf{C}(\theta) = \zeta_1\mathbf{M}(\theta) + \zeta_2\mathbf{K}(\theta)$, where ζ_1 and ζ_2 are scalars.

Stochastic modal analysis

Idea: *Extend conventional modal analysis to diagonalise the system and use the SDOF results presented earlier*
Difficulty: Need to solve a random eigenvalue problem

$$\mathbf{K}(\theta)\phi_j(\theta) = \omega_j^2(\theta)\mathbf{M}(\theta)\phi_j(\theta), \quad j = 1, 2, \cdots \tag{7}$$

- Computationally very challenging, probably more challenging than the solution problem itself! A research field in its own right.
- Normalising the modes is an open problem
- We also have the conceptual problem of 'statistical overlap' of the modes.

Stochastic modal analysis

- Stochastic modal analysis to obtain the dynamic response in general is not a good idea
- Consider the following 3DOF example:

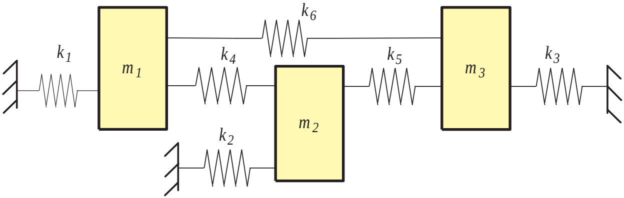
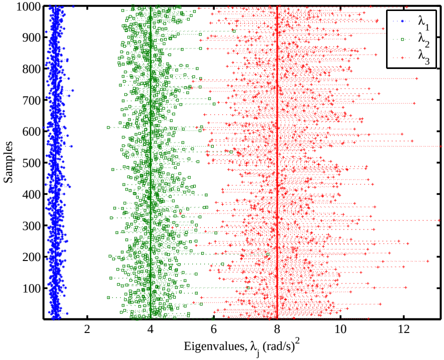
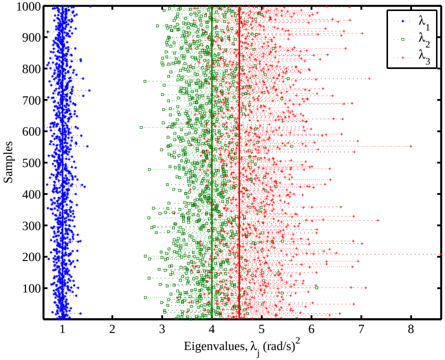


Figure : A 3DOF system with parametric uncertainty in m_i and k_i

Statistical overlap



(a) Eigenvalues are seperated



(b) Someeigenvalues are close

Figure : Scatter of the eigenvalues due to parametric uncertainties

Time domain representation

If the time steps are fixed to Δt, then the equation of motion can be written as

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{C}(\theta)\dot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{K}(\theta)\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t}. \tag{8}$$

Following the Newmark method based on constant average acceleration scheme, the above equations can be represented as

$$[a_0\mathbf{M}(\theta) + a_1\mathbf{C}(\theta) + \mathbf{K}(\theta)]\,\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta) \tag{9}$$

$$\text{and,}\quad \mathbf{p}^{eqv}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t} + f(\mathbf{u}_t(\theta), \dot{\mathbf{u}}_t(\theta), \ddot{\mathbf{u}}_t(\theta), \mathbf{M}(\theta), \mathbf{C}(\theta)) \tag{10}$$

where $\mathbf{p}^{eqv}_{t+\Delta t}(\theta)$ is the equivalent force at time $t+\Delta t$ which consists of contributions of the system response at the previous time step.

Newmark's method

The expressions for the velocities $\dot{\mathbf{u}}_{t+\Delta t}(\theta)$ and accelerations $\ddot{\mathbf{u}}_{t+\Delta t}(\theta)$ at each time step is a linear combination of the values of the system response at previous time steps (Newmark method) as

$$\ddot{\mathbf{u}}_{t+\Delta t}(\theta) = a_0 \left[\mathbf{u}_{t+\Delta t}(\theta) - \mathbf{u}_t(\theta)\right] - a_2 \dot{\mathbf{u}}_t(\theta) - a_3 \ddot{\mathbf{u}}_t(\theta) \tag{11}$$

$$\text{and,} \quad \dot{\mathbf{u}}_{t+\Delta t}(\theta) = \dot{\mathbf{u}}_t(\theta) + a_6 \ddot{\mathbf{u}}_t(\theta) + a_7 \ddot{\mathbf{u}}_{t+\Delta t}(\theta) \tag{12}$$

where the integration constants $a_i,\ i = 1, 2, \ldots, 7$ are independent of system properties and depends only on the chosen time step and some constants:

$$a_0 = \frac{1}{\alpha \Delta t^2}; \qquad a_1 = \frac{\delta}{\alpha \Delta t}; \qquad a_2 = \frac{1}{\alpha \Delta t}; \qquad a_3 = \frac{1}{2\alpha} - 1; \tag{13}$$

$$a_4 = \frac{\delta}{\alpha} - 1; \qquad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right); \qquad a_6 = \Delta t(1 - \delta); \qquad a_7 = \delta \Delta t \tag{14}$$

Newmark's method

Following this development, the linear structural system in (9) can be expressed as

$$\underbrace{\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right]}_{\mathbf{A}(\theta)} \mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta). \tag{15}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. For the case of proportional damping, the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written similar to the case of frequency domain as

$$\mathbf{A}_0 = [a_0 + a_1\zeta_1]\,\mathbf{M}_0 + [a_1\zeta_2 + 1]\,\mathbf{K}_0 \tag{16}$$

$$\text{and,} \quad \mathbf{A}_i = [a_0 + a_1\zeta_1]\,\mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p_1 \tag{17}$$

$$= [a_1\zeta_2 + 1]\,\mathbf{K}_i \quad \text{for} \quad i = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\,.$$

General mathematical representation

- Whether time-domain or frequency domain methods were used, in general the main equation which need to be solved can be expressed as

$$\left(\mathbf{A}_0 + \sum_{i=1}^{M} \Gamma_i(\boldsymbol{\xi}(\theta))\mathbf{A}_i\right) \mathbf{u}(\theta) = \mathbf{f}(\theta) \qquad (18)$$

  where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. These can be real or complex matrices.

- The functions $\Gamma_i(\boldsymbol{\xi}(\theta))$ can be used to introduce non-Gaussian random variables. In the special case $\Gamma_i(\boldsymbol{\xi}(\theta)) = \xi_i(\theta)$
- Generic response surface based methods have been used in literature

Polynomial Chaos expansion

After the finite truncation, the polynomial chaos expansion can be written as

$$\hat{\mathbf{u}}(t,\theta) = \sum_{k=1}^{P} H_k(\xi(\theta))\mathbf{u}_k(t) \tag{19}$$

where $H_k(\xi(\theta))$ are the polynomial chaoses. We need to solve a $nP \times nP$ linear equation to obtain all $\mathbf{u}_k \in \mathbb{R}^n$.

$$\begin{bmatrix} \mathbf{A}_{0,0} & \cdots & \mathbf{A}_{0,P-1} \\ \mathbf{A}_{1,0} & \cdots & \mathbf{A}_{1,P-1} \\ \vdots & \vdots & \vdots \\ \mathbf{A}_{P-1,0} & \cdots & \mathbf{A}_{P-1,P-1} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{P-1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{P-1} \end{Bmatrix} \tag{20}$$

The number of terms P increases exponentially with M:

M	2	3	5	10	20	50	100
2nd order PC	5	9	20	65	230	1325	5150
3rd order PC	9	19	55	285	1770	23425	176850

Some Observations

- The basis is a function of the pdf of the random variables only. For example, Hermite polynomials for Gaussian pdf, Legender's polynomials for uniform pdf.
- The physics of the underlying problem (static, dynamic, heat conduction, transients....) cannot be incorporated in the basis.
- The physical interpretation of the coefficient vectors $\mathbf{u}_k$ is not immediately obvious.
- The functional form of the response is a pure polynomial in random variables.

Projection in the modal space

Suppose the solution of the general equation (18) is given by

$$\hat{\mathbf{u}}_{t+\Delta t}(\theta) = \left[\mathbf{A}_0 + \sum_{i=1}^{M} \Gamma_i(\boldsymbol{\xi}(\theta))\mathbf{A}_i\right]^{-1} \mathbf{f}^{eqv}_{t+\Delta t}(\theta) \tag{21}$$

Orthogonal decompostion of the deterministic system yields

$$\boldsymbol{\lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{R}^{n\times n}; \boldsymbol{\Phi} = [\phi_1, \phi_2, \ldots, \phi_n] \in \mathbb{R}^{n\times n} \tag{22}$$

where the eigenpairs are ordered in the ascending order: $\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n}$. Introducing the transformations $\widetilde{\mathbf{A}}_i = \boldsymbol{\Phi}^T \mathbf{A}_i \boldsymbol{\Phi}; i = 0, 1, 2, \ldots, M$ and with the orthonormality of $\boldsymbol{\Phi}$

$$\hat{\mathbf{u}}_{t+\Delta t}(\theta) = \left[\boldsymbol{\Phi}^{-T}\boldsymbol{\Lambda}_0\boldsymbol{\Phi}^{-1} + \sum_{i=1}^{M} \Gamma_i(\boldsymbol{\xi}(\theta))\boldsymbol{\Phi}^{-T}\widetilde{\mathbf{A}}_i\boldsymbol{\Phi}^{-1}\right]^{-1} \mathbf{f}^{eqv}_{t+\Delta t}(\theta) \tag{23}$$

$$\text{where} \qquad \boldsymbol{\xi}(\theta) = \left[\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta)\right]^T.$$

Projection in the modal space

Now we separate the diagonal and off-diagonal terms of the $\widetilde{\mathbf{A}}_i$ matrices as

$$\widetilde{\mathbf{A}}_i = \boldsymbol{\Lambda}_i + \boldsymbol{\Delta}_i, \quad i = 1, 2, \ldots, M \tag{24}$$

Here the diagonal matrix

$$\boldsymbol{\Lambda}_i = \mathrm{diag}\left[\widetilde{\mathbf{A}}\right] = \mathrm{diag}\left[\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}\right] \in \mathbb{R}^{n\times n} \tag{25}$$

and $\boldsymbol{\Delta}_i = \widetilde{\mathbf{A}}_i - \boldsymbol{\Lambda}_i$ is an off-diagonal only matrix. We can write :

$$\boldsymbol{\Psi}\left(\boldsymbol{\xi}(\theta)\right) = \left[\underbrace{\boldsymbol{\Lambda}_0 + \sum_{i=1}^{M} \Gamma_i(\boldsymbol{\xi}(\theta))\boldsymbol{\Lambda}_i}_{\boldsymbol{\Lambda}(\Gamma_i(\boldsymbol{\xi}(\theta)))} + \underbrace{\sum_{i=1}^{M} \Gamma_i(\boldsymbol{\xi}(\theta))\boldsymbol{\Delta}_i}_{\boldsymbol{\Delta}(\Gamma_i(\boldsymbol{\xi}(\theta)))}\right]^{-1}. \tag{26}$$

Projection in the modal space

The diagonal matrix $\mathbf{\Lambda}\,(\xi(\theta))$ is treated as the preconditioner in the stochastic Krylov space, such that the solution can be projected onto a very few basis functions.
Hence the left preconditioned stochastic Krylov space becomes

$$\mathcal{K}_m(\mathbf{\Lambda}^{-1}\mathbf{\Psi}, \mathbf{\Lambda}^{-1}\mathbf{f}^{eqv}_{t+\Delta t}) = \text{span}\{\mathbf{\Phi}^T\mathbf{\Lambda}^{-1}\mathbf{\Phi}\mathbf{f}^{eqv}_{t+\Delta t}, \mathbf{\Phi}^T(\mathbf{\Lambda}^{-1}\mathbf{\Delta})\mathbf{\Lambda}^{-1}\mathbf{\Phi}\mathbf{f}^{eqv}_{t+\Delta t}, \mathbf{\Phi}^T(\mathbf{\Lambda}^{-1}\mathbf{\Delta})^2\mathbf{\Lambda}^{-1}\mathbf{\Phi}\mathbf{f}^{eqv}_{t+\Delta t}, \ldots, \mathbf{\Phi}^T(\mathbf{\Lambda}^{-1}\mathbf{\Delta})^{m-1}\mathbf{\Lambda}^{-1}\mathbf{\Phi}\mathbf{f}^{eqv}_{t+\Delta t}\} \tag{27}$$

The equivalent infinite Neumann matrix series representation of the above equation is

$$\mathbf{\Psi}\,(\xi(\theta)) = \sum_{s=0}^{\infty}(-1)^s\left[\mathbf{\Lambda}^{-1}\,(\xi(\theta))\,\mathbf{\Delta}\,(\xi(\theta))\right]^s\mathbf{\Lambda}^{-1}\,(\xi(\theta)) \tag{28}$$

Projection in the modal space

Taking an arbitrary r-th element of $\mathbf{u}(t,\theta)$, Eqn. (23) can be rearranged to have

$$u^r_{t+\Delta t}(\theta) = \sum_{k=1}^{n} \Phi_{rk} \left(\sum_{j=1}^{n} \Psi_{kj}\left(\boldsymbol{\xi}(\theta)\right) \left(\phi_j^T \mathbf{f}^{eqv}_{t+\Delta t}\right) \right) \tag{29}$$

Defining the spectral functions

$$\mathcal{L}_k\left(t, \boldsymbol{\xi}(\theta)\right) = \sum_{j=1}^{n} \Psi_{kj}\left(\boldsymbol{\xi}(\theta)\right) \left(\phi_j^T \mathbf{f}^{eqv}_{t+\Delta t}\right) \tag{30}$$

and collecting all the elements in Eqn. (29) for $r = 1, 2, \ldots, n$ one has

$$\mathbf{u}_{t+\Delta t}(\theta) = \sum_{k=1}^{n} \mathcal{L}_k\left(t, \boldsymbol{\xi}(\theta)\right) \phi_k \tag{31}$$

Projection in the modal space

A few observations:

- The matrix power series is different from the classical Neumann series in that the elements of the former are not simple polynomials in $\xi_i(\theta)$ but are in terms of the ratio of polynomials.
- The convergence of the series depends on the spectral radius of

$$\mathbf{R}\left(\boldsymbol{\xi}(\theta)\right) = \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right) \tag{32}$$

- A generic term of the matrix **R** is

$$R_{rs} = \frac{\Delta_{rs}}{\Lambda_{rr}} = \frac{\sum_{i=1}^{M}\Gamma_i(\xi)\Delta_{i_{rs}}}{\Lambda_{0_r} + \sum_{i=1}^{M}\Gamma_i(\xi)\Lambda_{i_r}} = \frac{\sum_{i=1}^{M}\Gamma_i(\xi)\widetilde{A}_{i_{rs}}}{\Lambda_{0_r} + \sum_{i=1}^{M}\Gamma_i(\xi)\widetilde{A}_{i_{rr}}}; r \neq s \tag{33}$$

  which shows that the spectral radius of **R** is controlled by the diagonal dominance of the $\widetilde{\mathbf{A}}_i$ matrices.

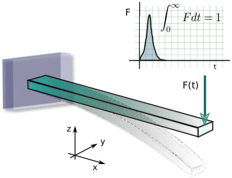
Frequency domain representation

Using a similar approach, in the frequency domain, the response can be simplified as

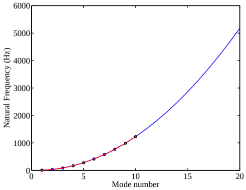
$$\mathbf{u}(\omega,\theta) = \sum_{k=1}^{n_r} \frac{\phi_k^T \mathbf{f}(\omega)}{-\omega^2 + 2i\omega\zeta_k\omega_0^2 + \omega_{0_k}^2 + \sum_{i=1}^{M} \xi_i(\theta)\Lambda_{i_k}(\omega)} \phi_k$$

The Euler-Bernoulli beam example

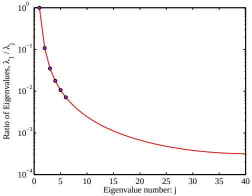
- An Euler-Bernoulli cantilever beam with stochastic bending modulus for a specified value of the correlation length and for different degrees of variability of the random field.



(a) Euler-Bernoulli beam



(b) Natural frequency distribution.



(c) Eigenvalue ratio of KL decomposition

- Length : 1.0 m, Cross-section : 39×5.93 mm^2, Young's Modulus: 2×10^{11} Pa.
- Load: Unit impulse at $t = 0$ on the free end of the beam.

Problem details

- The bending modulus of the cantilever beam is taken to be a homogeneous stationary Gaussian random field of the form

$$EI(x,\theta) = EI_0(1 + a(x,\theta)) \tag{34}$$

where x is the coordinate along the length of the beam, EI_0 is the estimate of the mean bending modulus, $a(x,\theta)$ is a zero mean stationary random field.

- The covariance kernel associated with this random field is

$$C_a(x_1, x_2) = \sigma_a^2 e^{-(|x_1 - x_2|)/\mu_a} \tag{35}$$

where μ_a is the correlation length and σ_a is the standard deviation.

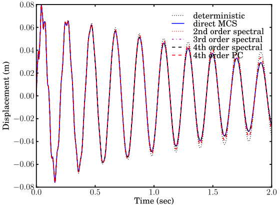
- A correlation length of $\mu_a = L/5$ is considered in the present numerical study.

Problem details

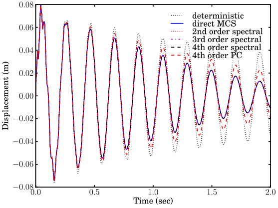
The random field is assumed to be Gaussian. The results are compared with the polynomial chaos expansion.

- The number of degrees of freedom of the system is n = 200.
- The K.L. expansion is truncated at a finite number of terms such that 90% variability is retained.
- direct MCS have been performed with 10,000 random samples and for three different values of standard deviation of the random field, $\sigma_a = 0.05, 0.1, 0.2$.
- Constant modal damping is taken with 1% damping factor for all modes.
- Time domain response of the free end of the beam is sought under the action of a unit impulse at $t = 0$
- Upto 4^{th} order spectral functions have been considered in the present problem. Comparison have been made with 4^{th} order Polynomial chaos results.

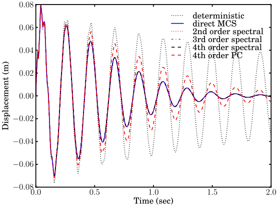
Mean of the response



(d) Mean, $\sigma_a = 0.05$.



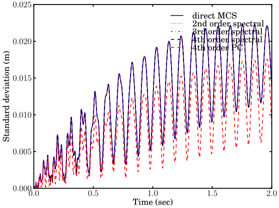
(e) Mean, $\sigma_a = 0.1$.



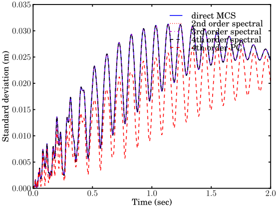
(f) Mean, $\sigma_a = 0.2$.

- Time domain response of the deflection of the tip of the cantilever for three values of standard deviation σ_a of the underlying random field.
- Spectral functions approach approximates the solution accurately.
- For long time-integration, the discrepancy of the 4^{th} order PC results increases.

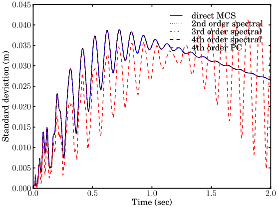
Standard deviation of the response



(g) Standard deviation of deflection, $\sigma_a = 0.05$.



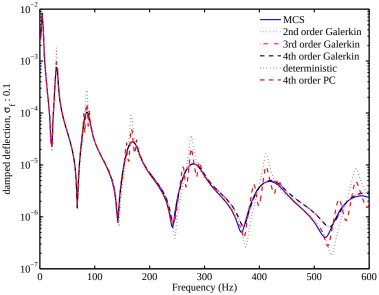
(h) Standard deviation of deflection, $\sigma_a = 0.1$.



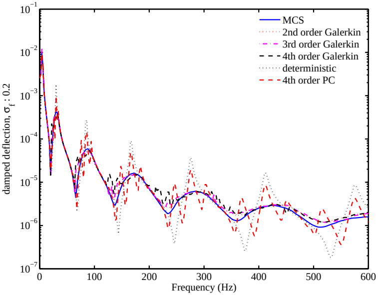
(i) Standard deviation of deflection, $\sigma_a = 0.2$.

- The standard deviation of the tip deflection of the beam.
- Since the standard deviation comprises of higher order products of the Hermite polynomials associated with the PC expansion, the higher order moments are less accurately replicated and tend to deviate more significantly.

Frequency domain response: mean



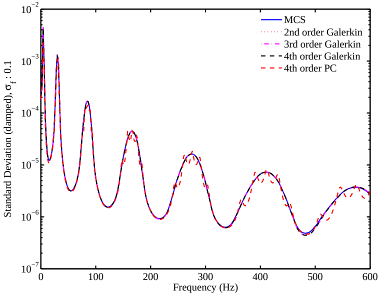
(j) Beam deflection for $\sigma_a = 0.1$.



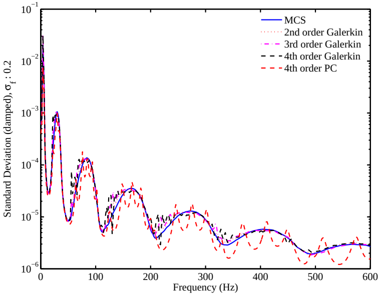
(k) Beam deflection for $\sigma_a = 0.2$.

The frequency domain response of the deflection of the tip of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$.

Frequency domain response: standard deviation



(l) Standard deviation of the response for $\sigma_a = 0.1$.



(m) Standard deviation of the response for $\sigma_a = 0.2$.

The standard deviation of the tip deflection of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.10, 0.20\}$.

Experimental investigations

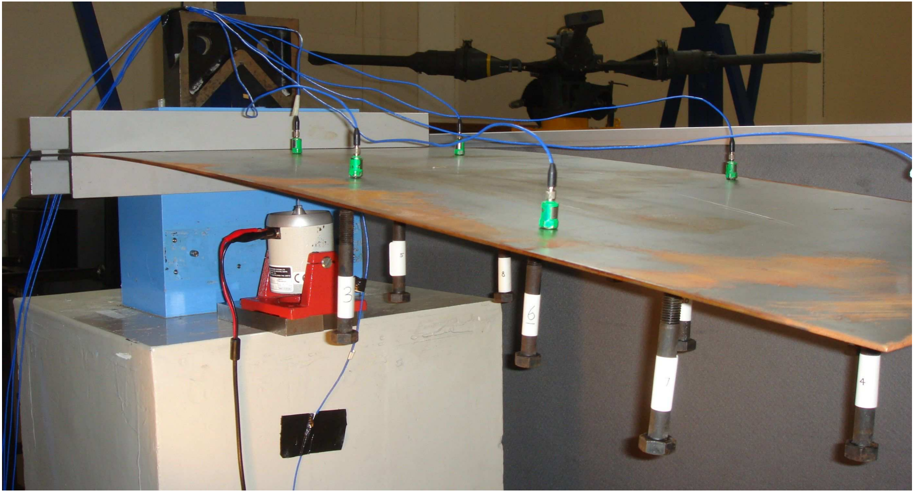
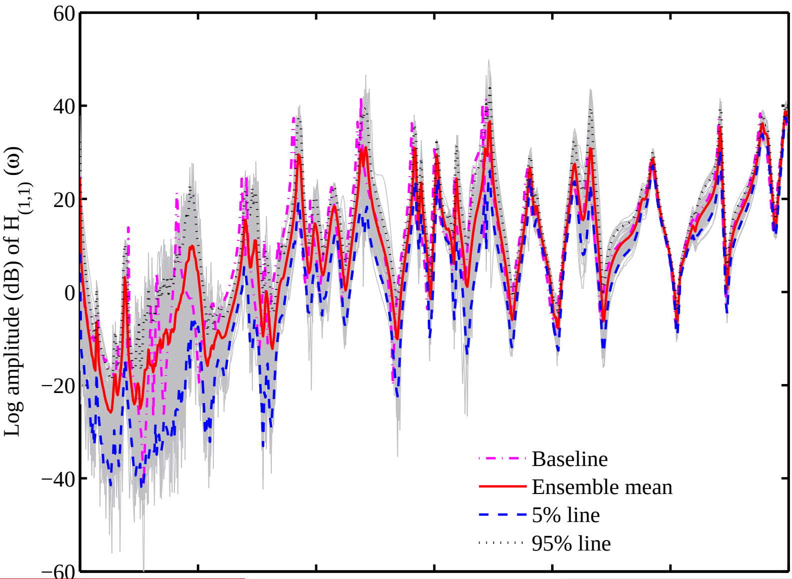


Figure : A cantilever plate with randomly attached oscillators - Probabilistic

Measured frequency response function



Conclusions

- The mean response of a damped stochastic system is more damped than the underlying baseline system
- For small damping, $\xi_e \approx \frac{3^{1/4}\sqrt{\epsilon}}{\sqrt{\pi}}\sqrt{\xi}$
- Random modal analysis may not be practical or physically intuitive for stochastic multiple degrees of freedom systems
- Conventional response surface based methods fails to capture the physics of damped dynamic systems
- Proposed spectral function approach uses the undamped modal basis and can capture the statistical trend of the dynamic response of stochastic damped MDOF systems