Uncertainty propagation in structural dynamics

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Uncertainty Quantification and Management in Aircraft Design

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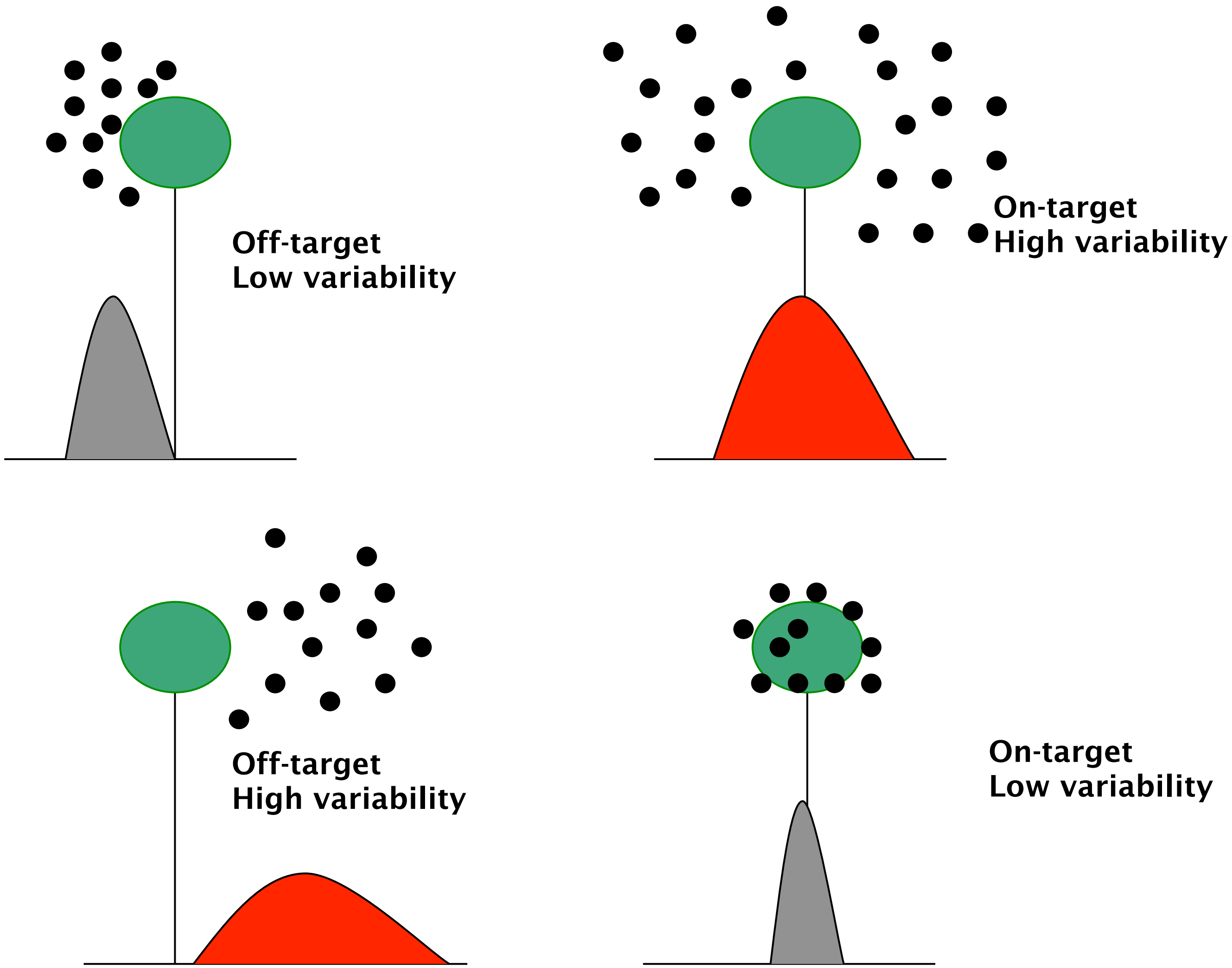
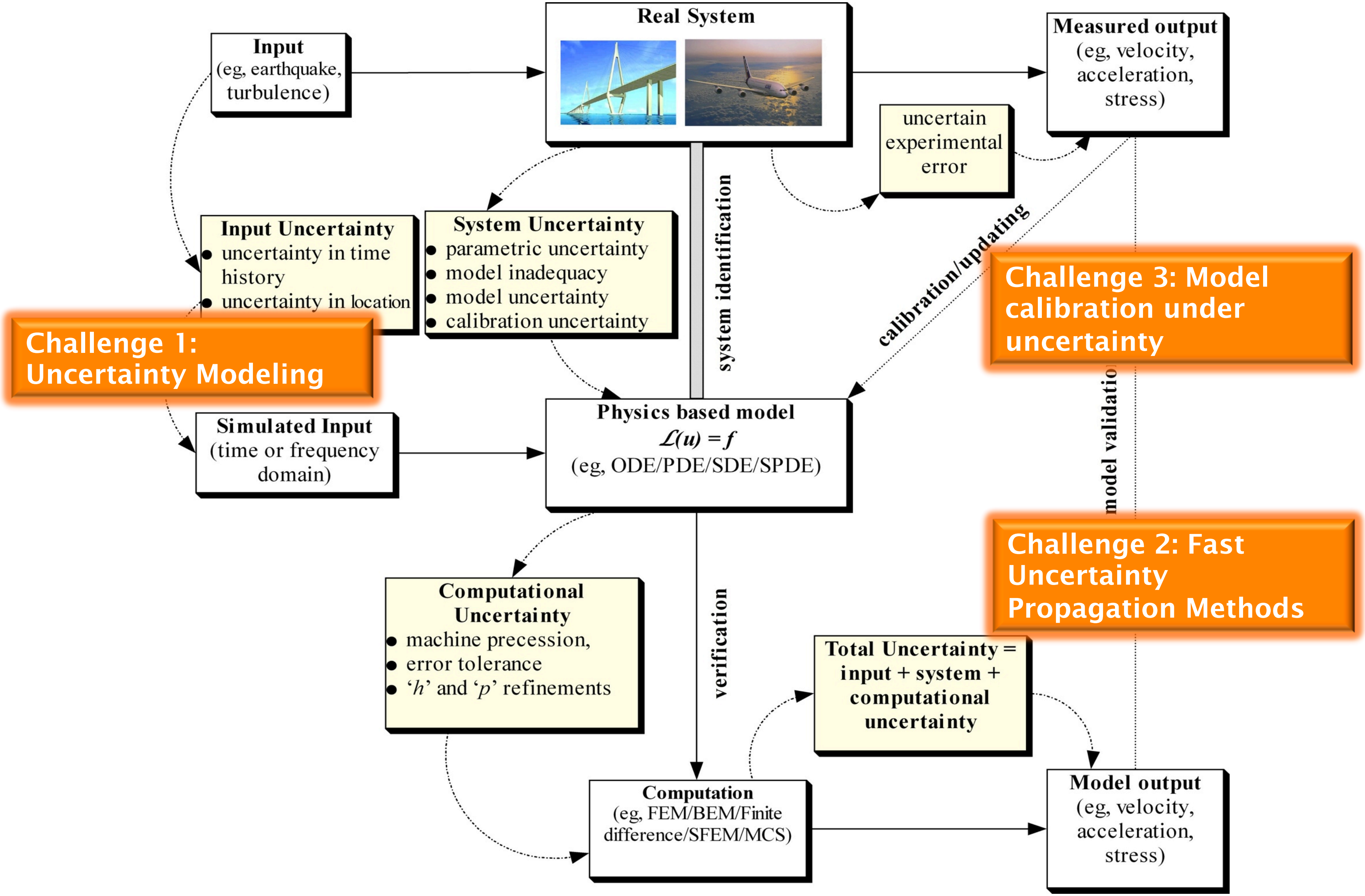
Outline of the Talk

- Background & Motivation
- Uncertainty Quantification
- Uncertainty propagation in complex dynamical systems
  - Parametric uncertainty propagation
  - Nonparametric uncertainty propagation
  - Unified representation
- Computational method and validation
  - Representative experimental results
  - Software integration
- Conclusions

Actual Performance of Engineering Designs



Overview of Computational Modeling



Why Uncertainty: The Sources

Experimental error

uncertain and unknown error percolate into the model when they are calibrated against experimental results

Parametric Uncertainty

uncertainty in the geometric parameters, boundary conditions, forces, strength of the materials involved

Computational uncertainty

machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis

Model Uncertainty

arising from the lack of scientific knowledge about the model which is a-priori unknown (damping, nonlinearity, joints)

A low-fidelity answer with known uncertainty bounds is more valuable than a high-fidelity answer with unknown uncertainty bounds (NASA White Paper, 2002).

Uncertainty Modeling

Parametric Uncertainty →

- Random variables
- Random fields

Non-parametric Uncertainty →

- Probabilistic Approach
  - Random matrix theory
- Possibilistic Approaches
  - Fuzzy variable
  - Interval algebra
  - Convex modeling

Equation of Motion of Dynamical Systems

- The Equation of motion of all these systems (and many other) about an equilibrium point can be expressed by:

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}(\theta,t)+\mathbf{C}(\theta)\dot{\mathbf{u}}(\theta,t)+\mathbf{K}(\theta)\mathbf{u}(\theta,t)=\mathbf{f}(t)$$

- $\mathbf{M}(\theta)\in\mathbb{R}^{n\times n}$ is the random mass matrix, $\mathbf{K}(\theta)\in\mathbb{R}^{n\times n}$ is the random stiffness matrix, $\mathbf{C}(\theta)\in\mathbb{R}^{n\times n}$ is the random damping matrix and $\mathbf{f}(t)$ is the forcing vector. We use (θ) to denote that the quantity is random.

The uncertainty propagation problem:
Given the stochastic description of the three systems matrices and the input forcing function, obtain the stochastic description of the response

Uncertainty modeling in structural dynamics

- Uncertainty modeling
  - Parametric uncertainty: mean matrices + random field/variable information
    - Random variables

      $$\mathbf{A} = \left(\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta) \mathbf{A}_i \right)$$
  - Nonparametric uncertainty: mean matrices + a single dispersion parameter for each matrices
    - Random matrix model

      $$\mathbf{A} \sim W_n(\mathbf{A}_0, \delta_A^2)$$

Parametric uncertainty propagation

Frequency domain analysis

- Taking the Fourier transform of the equation of motion

$$\left[-\omega^2\mathbf{M}(\theta)+i\omega\mathbf{C}(\theta)+\mathbf{K}(\theta)\right]\widetilde{\mathbf{u}}(\omega,\theta)=\widetilde{\mathbf{f}}(\omega)$$

  where $\widetilde{\mathbf{u}}(\omega,\theta)$ is the complex frequency domain system response amplitude, $\widetilde{\mathbf{f}}(\omega)$ is the amplitude of the harmonic force.
- $\mathbf{M}(\theta)=\mathbf{M}_0+\sum_{i=1}^{p_1}\mu_i(\theta_i)\mathbf{M}_i\in\mathbb{R}^{n\times n}$ is the random mass matrix, $\mathbf{K}(\theta)=\mathbf{K}_0+\sum_{i=1}^{p_2}\nu_i(\theta_i)\mathbf{K}_i\in\mathbb{R}^{n\times n}$ is the random stiffness matrix, $\mathbf{C}(\theta)\in\mathbb{R}^{n\times n}$ as the random damping matrix
- Proportional damping model is considered: $\mathbf{C}(\theta)=\zeta_1\mathbf{M}(\theta)+\zeta_2\mathbf{K}(\theta)$, where ζ_1 and ζ_2 are scalars.
- For convenience we group the random variables associated with the mass and stiffness matrices as

$$\xi_i(\theta)=\mu_i(\theta)\quad\text{and}\quad\xi_{j+p_1}(\theta)=\nu_j(\theta)\quad\text{for}\quad i=1,2,\ldots,p_1$$
$$\text{and}\quad j=1,2,\ldots,p_2$$

Frequency domain analysis

- Using $M = p_1 + p_2$ which we have

$$\left(\mathbf{A}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i(\omega)\right)\widetilde{\mathbf{u}}(\omega,\theta) = \widetilde{\mathbf{f}}(\omega)$$

  where $\mathbf{A}_0$ and $\mathbf{A}_i \in \mathbb{C}^{n\times n}$ represent the complex deterministic and stochastic parts respectively of the mass, the stiffness and the damping matrices ensemble.

- For the case of proportional damping the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written as

$$\mathbf{A}_0(\omega) = \left[-\omega^2 + i\omega\zeta_1\right]\mathbf{M}_0 + \left[i\omega\zeta_2 + 1\right]\mathbf{K}_0,$$

$$\mathbf{A}_i(\omega) = \left[-\omega^2 + i\omega\zeta_1\right]\mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p_1$$

$$\text{and} \quad \mathbf{A}_{j+p_1}(\omega) = \left[i\omega\zeta_2 + 1\right]\mathbf{K}_j \quad \text{for} \quad j = 1, 2, \ldots, p_2\,.$$

General mathematical representation

- In general the main equation which need to be solved for parametric uncertainty propagation, can be expressed as

$$\left(\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta_i)\mathbf{A}_i\right)\mathbf{u}(\theta) = \mathbf{f}(\theta)$$

- Here $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. These are symmetric matrices and can be real or complex.
- The mathematical form of this equation is valid for static or dynamic problems, and also for time-domain or frequency domain representation.

What should be the form of the response?

- The frequency domain equation of the stochastic system $\left[-\omega^2\mathbf{M}(\xi(\theta)) + i\omega\mathbf{C}(\xi(\theta)) + \mathbf{K}(\xi(\theta))\right]\mathbf{u}(\omega,\theta) = \mathbf{f}(\omega)$.
- Some possibilities for the expression $\mathbf{u}(\omega,\theta)$ of are

$$
\begin{aligned}
\mathbf{u}(\omega,\theta) &= \sum_{k=1}^{P_1} H_k(\xi(\theta))\mathbf{u}_k(\omega) \\
\text{or} \quad &= \sum_{k=1}^{P_2} \Gamma_k(\omega,\xi(\theta))\phi_k \\
\text{or} \quad &= \sum_{k=1}^{P_3} a_k(\omega)H_k(\xi(\theta))\phi_k \\
\text{or} \quad &= \sum_{k=1}^{P_4} a_k(\omega)H_k(\xi(\theta))\mathbf{U}_k(\xi(\theta)) \quad \ldots \text{ etc.}
\end{aligned}
$$

Classical Modal Analysis?

For a deterministic system, the response vector $\mathbf{u}(\omega)$ can be expressed as

$$\mathbf{u}(\omega) = \sum_{k=1}^{P} \Gamma_k(\omega)\mathbf{u}_k$$

$$\text{where} \quad \Gamma_k(\omega) = \frac{\phi_k^T \mathbf{f}}{-\omega^2 + 2\mathrm{i}\zeta_k\omega_k\omega + \omega_k^2}$$

$$\mathbf{u}_k = \phi_k \quad \text{and} \quad P \leq n \text{ (number of dominant modes)}$$

where ω_k: natural frequencies, ϕ_k: mode shapes.

Can we extend this idea to stochastic systems?

Projection in the Modal Basis

There exist a finite set of complex frequency dependent functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$ and a complete basis $\phi_k \in \mathbb{R}^n$ for $k = 1, 2, \ldots, n$ such that the solution of the discretized stochastic finite element equation can be expressed by the series

$$\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{n} \Gamma_k(\omega, \boldsymbol{\xi}(\theta))\phi_k$$

Here $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$ are the spectral functions and $\phi_k \in \mathbb{R}^n$ are the eigenvectors arising from the generalized eigenvalue problem

$$\mathbf{K}_0\phi_k = \lambda_{0_k}\mathbf{M}_0\phi_k; \quad k = 1, 2, \ldots n$$

Adhikari, S., "A reduced spectral function approach for the stochastic finite element analysis", Computer Methods in Applied Mechanics and Engineering, 200[21-22] (2011), pp. 1804-1821.

Outline of the derivation

- Transform the equation of motion into the modal domain by using the matrix of the eigenvectors $\mathbf{\Phi}$.
- Separate the diagonal and off-diagonal terms of the resulting matrix.
- Expand the inverse of the matrix in terms of the inverse of the diagonal term in a Neumann-like series for a given frequency value.

Projection in the Modal Basis

The solution of the frequency-domain equation is given by

$$\hat{\mathbf{u}}(\omega,\theta) = \left[\mathbf{A}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i(\omega)\right]^{-1} \mathbf{f}(\omega)$$

Using the mass and stiffness orthogonality property of the modal matrix $\mathbf{\Phi}$ one has

$$\hat{\mathbf{u}}(\omega,\theta) = \left[\mathbf{\Phi}^{-T}\mathbf{\Lambda}_0(\omega)\mathbf{\Phi}^{-1} + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Phi}^{-T}\widetilde{\mathbf{A}}_i(\omega)\mathbf{\Phi}^{-1}\right]^{-1} \mathbf{f}(\omega)$$

$$\Rightarrow \quad \hat{\mathbf{u}}(\omega,\theta) = \mathbf{\Phi} \underbrace{\left[\mathbf{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\widetilde{\mathbf{A}}_i(\omega)\right]^{-1}}_{\mathbf{\Psi}(\omega,\boldsymbol{\xi}(\theta))} \mathbf{\Phi}^{-T}\mathbf{f}(\omega)$$

where $\quad \boldsymbol{\xi}(\theta) = \{\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta)\}^T.$

Projection in the Modal Basis

Now we separate the diagonal and off-diagonal terms of the $\widetilde{\mathbf{A}}_i$ matrices as

$$\widetilde{\mathbf{A}}_i = \mathbf{\Lambda}_i + \mathbf{\Delta}_i, \quad i = 1, 2, \ldots, M$$

Here the diagonal matrix

$$\mathbf{\Lambda}_i = \mathrm{diag}\left[\widetilde{\mathbf{A}}\right] = \mathrm{diag}\left[\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}\right] \in \mathbb{R}^{n \times n}$$

and $\mathbf{\Delta}_i = \widetilde{\mathbf{A}}_i - \mathbf{\Lambda}_i$ is an off-diagonal only matrix.

$$\mathbf{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \left[\underbrace{\mathbf{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Lambda}_i(\omega)}_{\mathbf{\Lambda}(\omega, \boldsymbol{\xi}(\theta))} + \underbrace{\sum_{i=1}^{M} \xi_i(\theta)\mathbf{\Delta}_i(\omega)}_{\mathbf{\Delta}(\omega, \boldsymbol{\xi}(\theta))}\right]^{-1}$$

where $\mathbf{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right) \in \mathbb{R}^{n \times n}$ is a diagonal matrix and $\mathbf{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)$ is an off-diagonal only matrix.

Projection in the Modal Basis

We rewrite this equation as

$$\boldsymbol{\Psi}\left(\omega,\boldsymbol{\xi}(\theta)\right)=\left[\boldsymbol{\Lambda}\left(\omega,\boldsymbol{\xi}(\theta)\right)\left[\mathbf{I}_n+\boldsymbol{\Lambda}^{-1}\left(\omega,\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\omega,\boldsymbol{\xi}(\theta)\right)\right]\right]^{-1}$$

The above expression can be represented using a Neumann type of matrix series as

$$\boldsymbol{\Psi}\left(\omega,\boldsymbol{\xi}(\theta)\right)=\sum_{s=0}^{\infty}(-1)^s\left[\boldsymbol{\Lambda}^{-1}\left(\omega,\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\omega,\boldsymbol{\xi}(\theta)\right)\right]^s\boldsymbol{\Lambda}^{-1}\left(\omega,\boldsymbol{\xi}(\theta)\right)$$

Projection in the Modal Basis

Taking an arbitrary r-th element, the expression of $\hat{\mathbf{u}}(\omega,\theta)$ can be rearranged to have

$$\hat{u}_r(\omega,\theta) = \sum_{k=1}^{n} \Phi_{rk} \left(\sum_{j=1}^{n} \Psi_{kj}\left(\omega, \boldsymbol{\xi}(\theta)\right) \left(\phi_j^T \mathbf{f}(\omega)\right) \right)$$

Defining

$$\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right) = \sum_{j=1}^{n} \Psi_{kj}\left(\omega, \boldsymbol{\xi}(\theta)\right) \left(\phi_j^T \mathbf{f}(\omega)\right)$$

and collecting all the elements for $r = 1, 2, \ldots, n$ we have the complete solution

$$\hat{\mathbf{u}}(\omega,\theta) = \sum_{k=1}^{n} \Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right) \phi_k$$

Spectral functions

Definition

The functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$, $k = 1, 2, \ldots n$ are the frequency-adaptive spectral functions as they are expressed in terms of the spectral properties of the coefficient matrices at each frequency of the governing discretized equation.

- Each of the spectral functions $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$ contain infinite number of terms and they are highly nonlinear functions of the random variables $\xi_i(\theta)$.
- For computational purposes, it is necessary to truncate the series after certain number of terms.
- Different order of spectral functions can be obtained by using truncation in the expression of $\Gamma_k(\omega, \boldsymbol{\xi}(\theta))$

Spectral functions

Definition

The different order of spectral functions $\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)), k = 1, 2, \ldots, n$ are obtained by retaining different order of terms in the series expansion.

Retaining one and two terms we have

$$\boldsymbol{\Psi}^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta))$$
$$\boldsymbol{\Psi}^{(2)}(\omega, \boldsymbol{\xi}(\theta)) = \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta)) - \boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta))\,\boldsymbol{\Delta}(\omega, \boldsymbol{\xi}(\theta))\,\boldsymbol{\Lambda}^{-1}(\omega, \boldsymbol{\xi}(\theta))$$

which are the first and second order spectral functions respectively.

- From these we find $\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \sum_{j=1}^{n} \Psi_{kj}^{(1)}(\omega, \boldsymbol{\xi}(\theta)) \left(\phi_j^T \mathbf{f}(\omega)\right)$ are non-Gaussian random variables even if $\xi_i(\theta)$ are Gaussian random variables.

Model reduction by a reduced basis

- The eigenvalues are arranged in an increasing order such that

$$\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n}$$

- From the expression of the spectral functions observe that the eigenvalues ($\lambda_{0_k} = \omega_{0_k}^2$) appear in the denominator:

$$\Gamma_k^{(1)}(\omega, \boldsymbol{\xi}(\theta)) = \frac{\phi_k^T \mathbf{f}(\omega)}{\Lambda_{0_k}(\omega) + \sum_{i=1}^M \xi_i(\theta)\Lambda_{i_k}(\omega)}$$

where $\Lambda_{0_k}(\omega) = -\omega^2 + i\omega(\zeta_1 + \zeta_2\omega_{0_k}^2) + \omega_{0_k}^2$

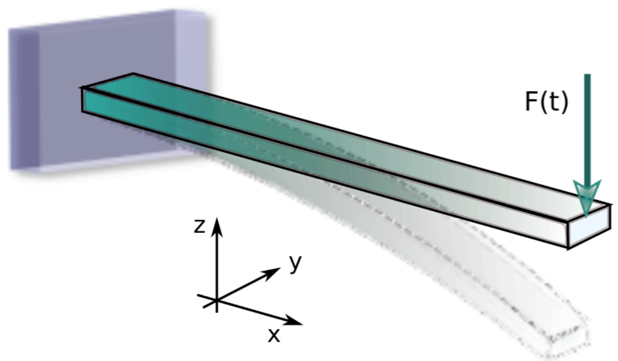
- The series can be truncated based on the magnitude of the eigenvalues relative to the frequency of excitation. The approximate solution can be represented with a reduced number (n_r) of modal basis as

$$\widetilde{\mathbf{u}}(\omega, \theta) \approx \sum_{k=1}^{n_r} \widehat{\Gamma}_k^{(m)}(\omega, \boldsymbol{\xi}(\theta))\phi_k$$

Summary of the spectral functions

- Not polynomials in random variables, but ratio of polynomials
- Independent of the nature of the random variables (i.e. applicable to Gaussian, non-Gaussian or even mixed random variables
- Not general, but specific to a problem as it utilizes the eigenvalues and eigenvectors of the system matrices.
- The truncation error depends on the off-diagonal terms of the random part of the modal system matrix
- Show 'peaks' when the frequency is close to the system natural frequencies

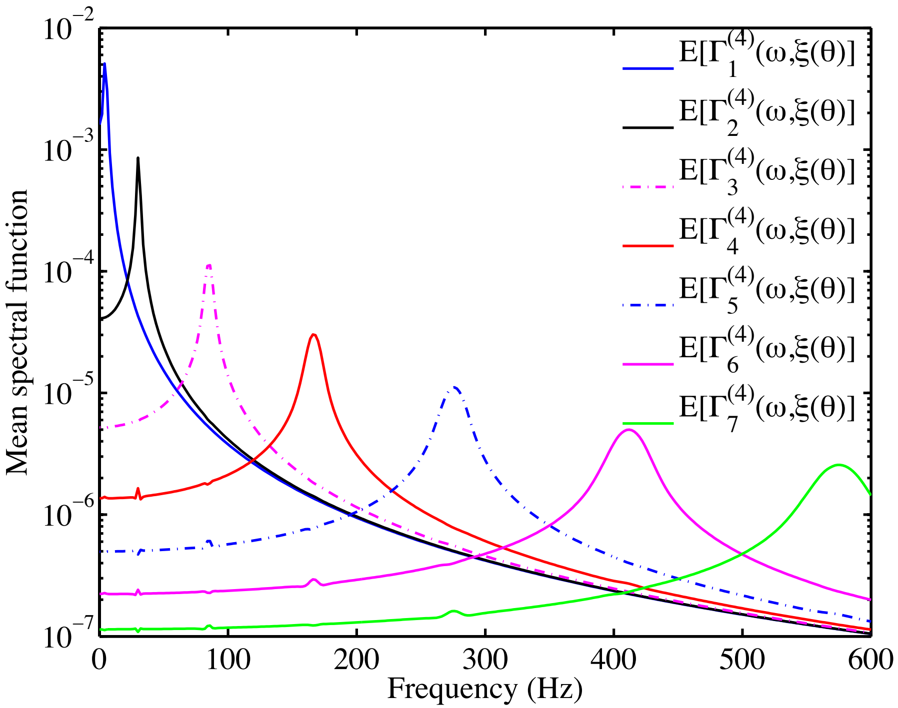
Numerical illustration



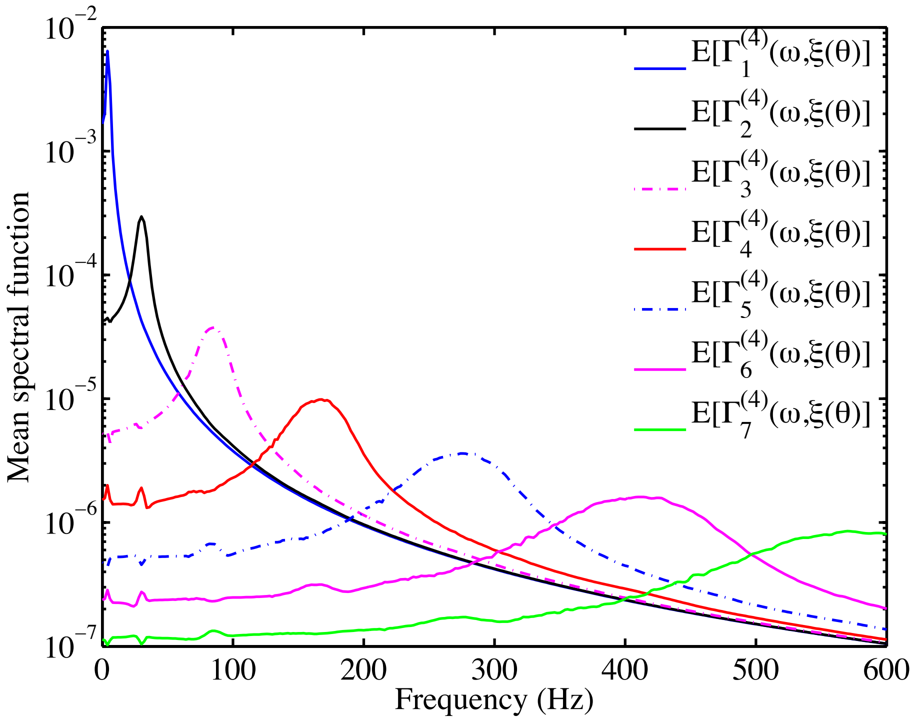
- An Euler-Bernoulli cantilever beam with stochastic bending modulus (nominal properties L=1m, A=39 x 5.93mm^{2}, E=2 x 10^{11} Pa)
- We use n=200, M=2

- We study the deflection of the beam under the action of a point load on the free end.
- The bending modulus of the cantilever beam is taken to be a homogeneous stationary Gaussian random field with exponential autocorrelation function (correlation length L/2)
- Constant modal damping is taken with 1% damping factor for all modes.
- The standard deviation of the random field σ_a is varied up to 0.2 times the mean.

Spectral functions



$\sigma_a = 0.05$



$\sigma_a = 0.2$

Mean of the spectral functions (4th order)

Galerkin Approach

One can obtain constants $c_k \in \mathbb{C}$ such that the error in the following representation

$$\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{n_r} c_k(\omega) \widehat{\Gamma}_k(\omega, \boldsymbol{\xi}(\theta)) \boldsymbol{\phi}_k$$

can be minimised in the least-square sense. It can be shown that the vector $\mathbf{c} = \{c_1, c_2, \ldots, c_{n_r}\}^T$ satisfies the $n_r \times n_r$ complex algebraic equations $\mathbf{S}(\omega)\ \mathbf{c}(\omega) = \mathbf{b}(\omega)$ with

$$S_{jk} = \sum_{i=0}^{M} \widetilde{A}_{i_{jk}} D_{ik}; \quad \forall\ j, k = 1, 2, \ldots, n_r; \widetilde{A}_{i_{jk}} = \boldsymbol{\phi}_j^T \mathbf{A}_i \boldsymbol{\phi}_k,$$

$$D_{ik} = \mathrm{E}\left[\xi_i(\theta) \widehat{\Gamma}_k(\omega, \boldsymbol{\xi}(\theta))\right], b_j = \mathrm{E}\left[\boldsymbol{\phi}_j^T \mathbf{f}(\omega)\right].$$

Galerkin approach

- The error vector can be obtained as

$$\varepsilon(\omega,\theta) = \left(\sum_{i=0}^{M} \mathbf{A}_i(\omega)\xi_i(\theta)\right)\left(\sum_{k=1}^{n_r} c_k \widehat{\Gamma}_k(\omega,\boldsymbol{\xi}(\theta))\phi_k\right) - \mathbf{f}(\omega) \in \mathbb{C}^{N\times N}$$

  The solution is viewed as a projection where $\phi_k \in \mathbb{R}^n$ are the basis functions and c_k are the unknown constants to be determined. This is done for each frequency step.

- The coefficients c_k are evaluated using the Galerkin approach so that the error is made orthogonal to the basis functions, that is, mathematically

$$\varepsilon(\omega,\theta) \perp \phi_j \Rrightarrow \left\langle \phi_j, \varepsilon(\omega,\theta)\right\rangle = 0 \,\forall\, j = 1,2,\ldots,n_r$$

Galerkin approach

- Imposing the orthogonality condition and using the expression of the error one has

$$\mathrm{E}\left[\phi_j^T\left(\sum_{i=0}^{M}\mathbf{A}_i\xi_i(\theta)\right)\left(\sum_{k=1}^{n_r}c_k\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\phi_k\right)-\phi_j^T\mathbf{f}\right]=0,\forall j$$

- Interchanging the $\mathrm{E}\left[\bullet\right]$ and summation operations, this can be simplified to

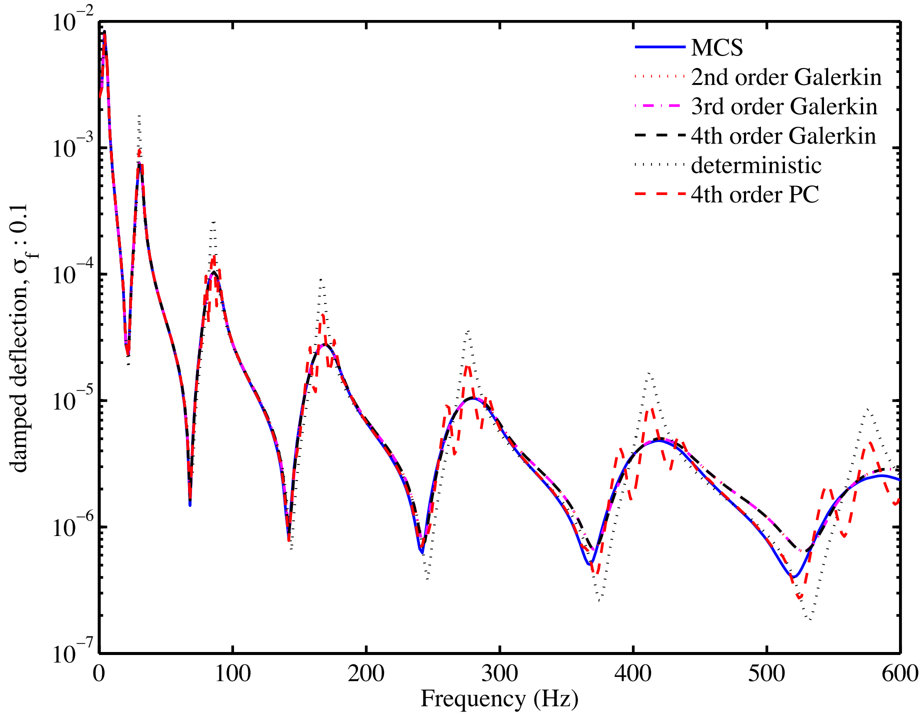
$$\sum_{k=1}^{n_r}\left(\sum_{i=0}^{M}\left(\phi_j^T\mathbf{A}_i\phi_k\right)\mathrm{E}\left[\xi_i(\theta)\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right]\right)c_k=\mathrm{E}\left[\phi_j^T\mathbf{f}\right]$$

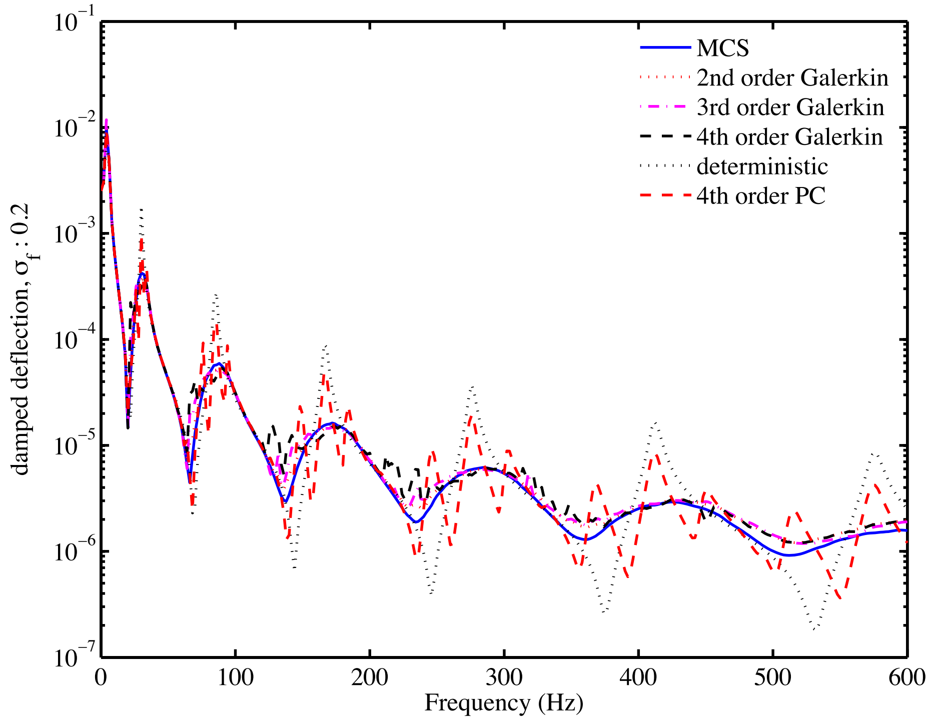
$$\text{or}\quad \sum_{k=1}^{n_r}\left(\sum_{i=0}^{M}\widetilde{A}_{i_{jk}}D_{ik}\right)c_k=b_j$$

Summary of the Proposed Method

1. Solve the generalised eigenvalue problem associated with the mean mass and stiffness matrices to generate the orthonormal basis vectors: $\mathbf{K}_0\mathbf{\Phi} = \mathbf{M}_0\mathbf{\Phi}\boldsymbol{\lambda}_0$
2. Select a number of samples, say N_{samp}. Generate the samples of basic random variables $\xi_i(\theta), i = 1, 2, \ldots, M$.
3. Calculate the spectral basis functions (for example, first-order): $\Gamma_k\left(\omega, \boldsymbol{\xi}(\theta)\right) = \frac{\boldsymbol{\phi}_k^T \mathbf{f}(\omega)}{\Lambda_{0_k}(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\Lambda_{i_k}(\omega)}$, for $k = 1, \cdots n_r$, $n_r < n$
4. Obtain the coefficient vector: $\mathbf{c}(\omega) = \mathbf{S}^{-1}(\omega)\mathbf{b}(\omega)$, where $\mathbf{b}(\omega) = \widetilde{\mathbf{f}(\omega)} \odot \overline{\mathbf{\Gamma}(\omega)}$, $\mathbf{S}(\omega) = \mathbf{\Lambda}_0(\omega) \odot \mathbf{D}_0(\omega) + \sum_{i=1}^{M} \widetilde{\mathbf{A}}_i(\omega) \odot \mathbf{D}_i(\omega)$ and $\mathbf{D}_i(\omega) = \mathrm{E}\left[\mathbf{\Gamma}(\omega, \theta)\xi_i(\theta)\mathbf{\Gamma}^T(\omega, \theta)\right], \forall\, i = 0, 1, 2, \ldots, M$
5. Obtain the samples of the response from the spectral series: $\hat{\mathbf{u}}(\omega, \theta) = \sum_{k=1}^{n_r} c_k(\omega)\Gamma_k(\boldsymbol{\xi}(\omega, \theta))\boldsymbol{\phi}_k$

Frequency domain response of the beam



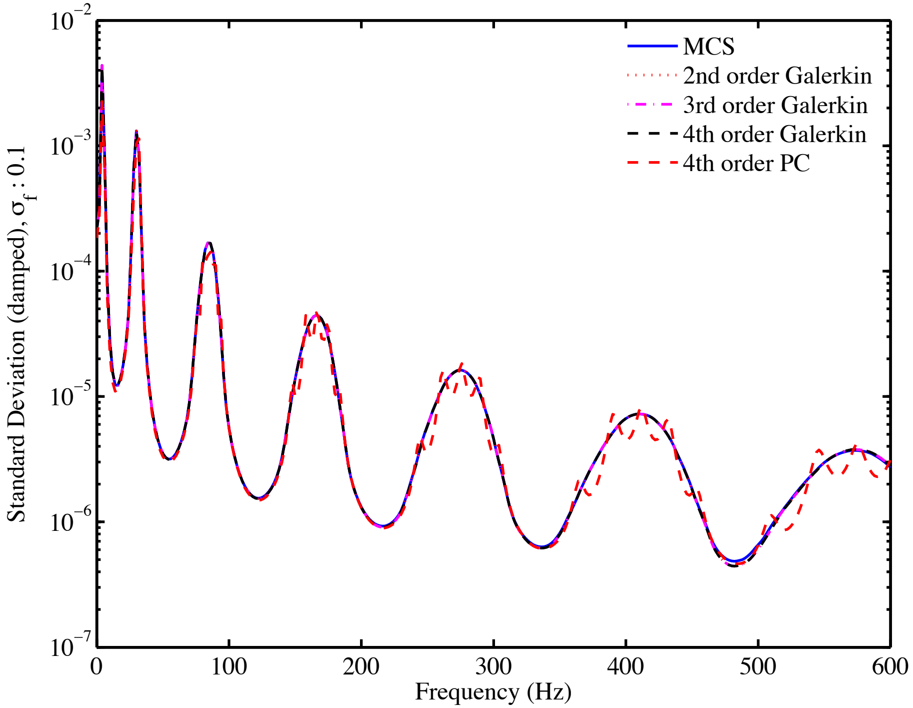


$\sigma_a = 0.1$

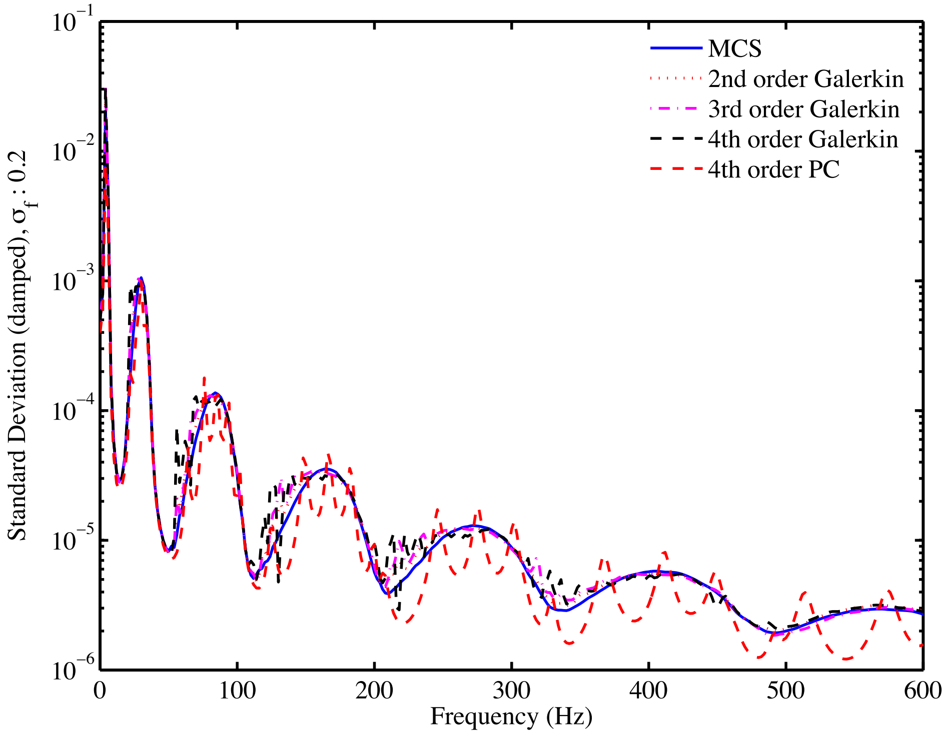
$\sigma_a = 0.2$

Mean of the dynamic response (m)

Frequency domain response of the beam



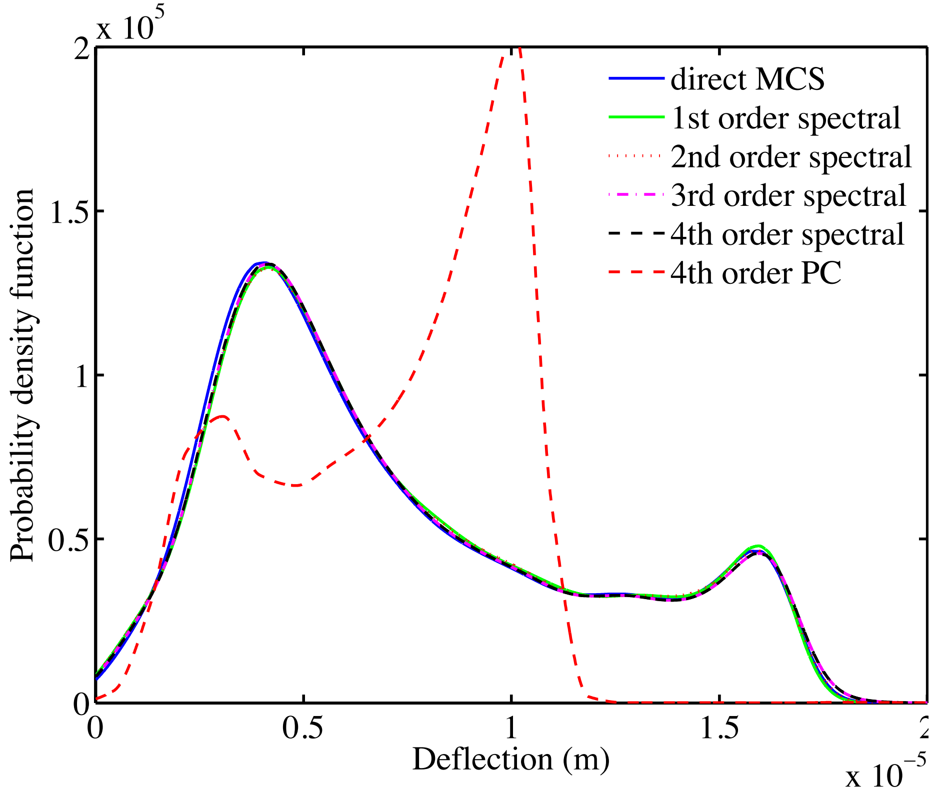
$\sigma_a = 0.1$



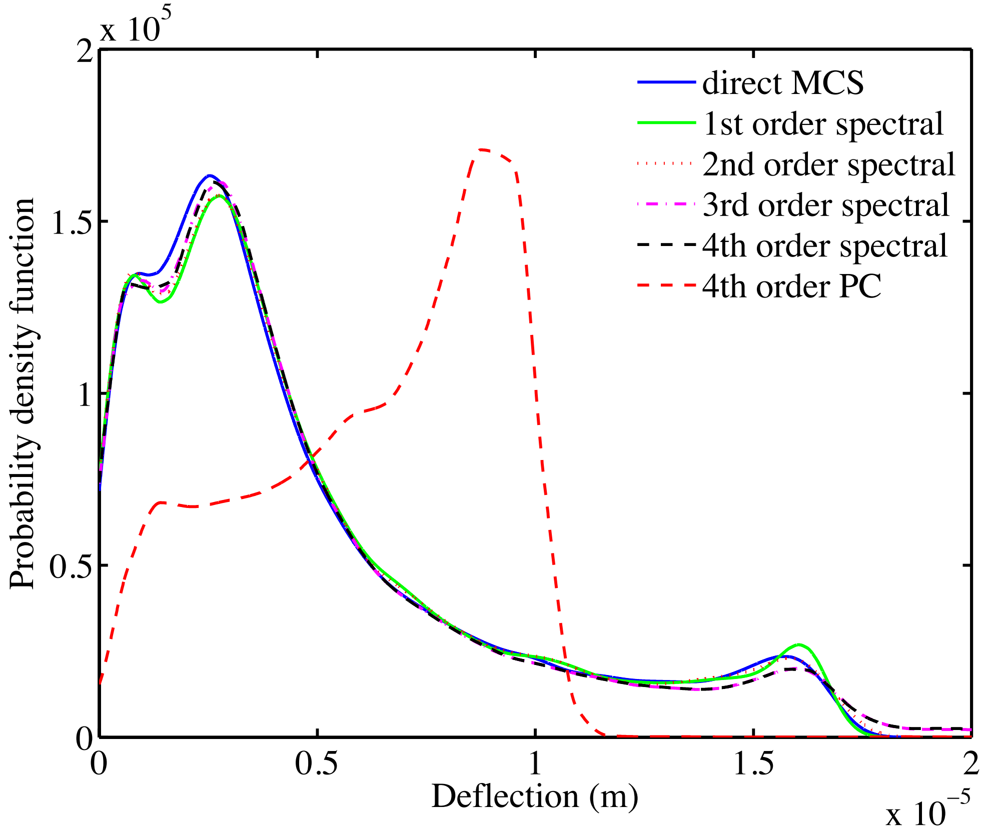
$\sigma_a = 0.2$

Standard deviation of the dynamic response (m)

PDF of the Response Amplitude



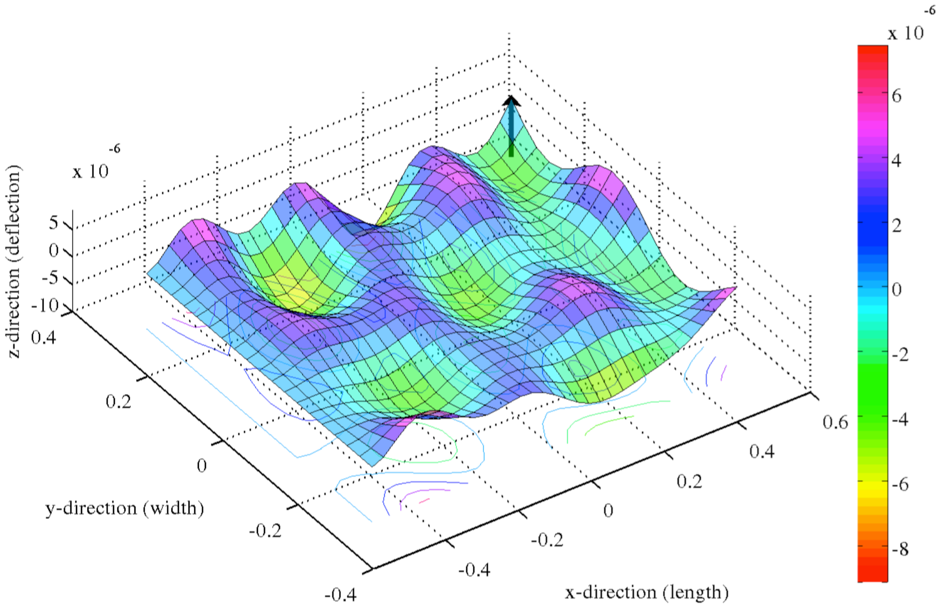
$\sigma_a = 0.1$



$\sigma_a = 0.2$

Standard deviation of the dynamic response (m)

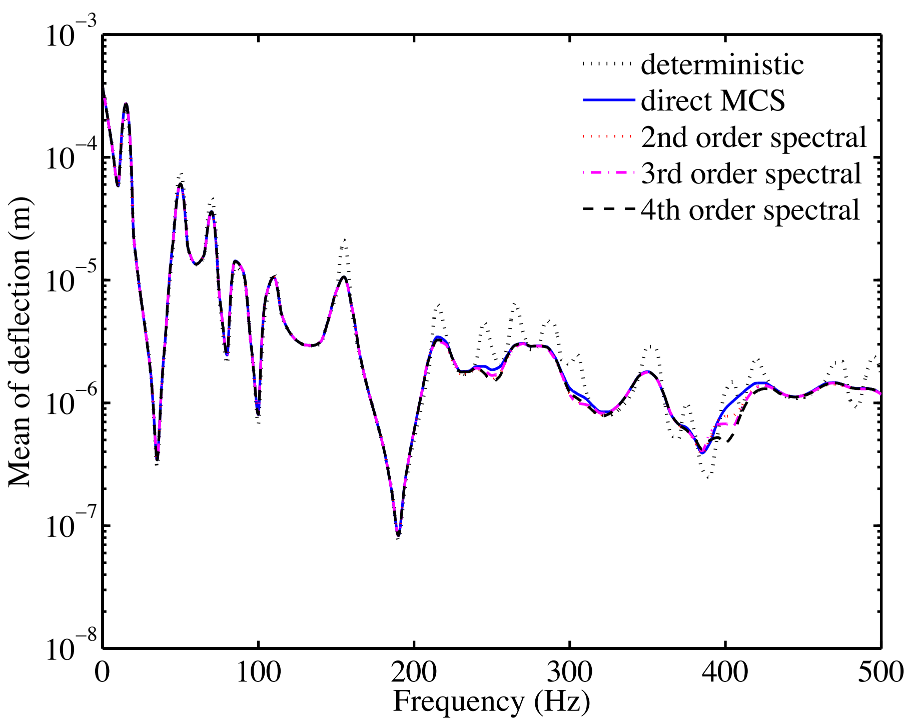
Plate with Stochastic Properties



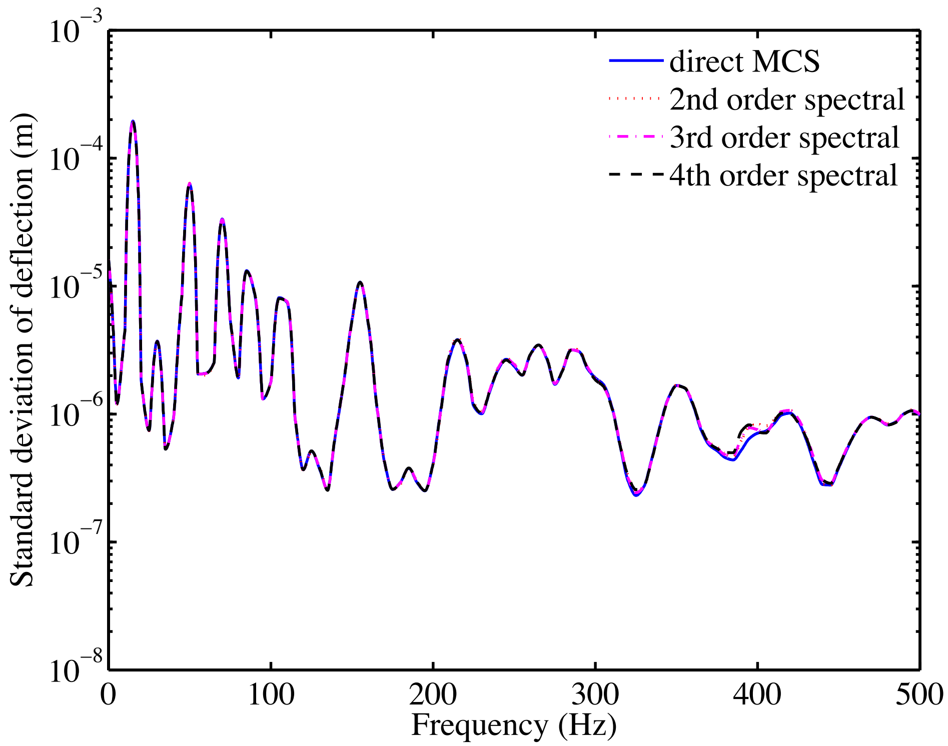
- An Euler-Bernoulli cantilever beam with stochastic bending modulus (nominal properties 1m x 0.6m, t=03mm, E=2 x 10^{11} Pa)
- We use n=1881, M=16

- We study the deflection of the beam under the action of a point load on the free end.
- The bending modulus is taken to be a homogeneous stationary Gaussian random field with exponential autocorrelation function (correlation lengths L/5)
- Constant modal damping is taken with 1% damping factor for all modes.

Response Statistics



Mean with σ_a = 0.1



Standard deviation with σ_a = 0.1

Proposed approach: 150 x 150 equations
4^{th} order Polynomial Chaos: 9113445 x 9113445 equations

Non-parametric uncertainty propagation

Wishart random matrix model

Distribution of the systems matrices should be such that they are

- Symmetric, and
- Positive definite

Using these as constraints, it can be shown that the mass, stiffness and damping matrices can be represented by Wishart random matrices such that

$$\mathbf{A} \sim W_n(\mathbf{A}_0, \delta_A^2), \quad \delta_A^2 = \frac{\mathrm{E}\left[\|\mathbf{A} - \mathrm{E}[\mathbf{A}]\|_\mathrm{F}^2\right]}{\|\mathrm{E}[\mathbf{A}]\|_\mathrm{F}^2}$$

[1] Adhikari, S., Pastur, L., Lytova, A. and Du Bois, J. L., "Eigenvalue-density of linear stochastic dynamical systems: A random matrix approach", *Journal of Sound and Vibration*, 331[5] (2012), pp. 1042-1058.
[2] Adhikari, S. and Chowdhury, R., "A reduced-order random matrix approach for stochastic structural dynamics", *Computers and Structures*, 88[21-22] (2010), pp. 1230-1238.
[3] Adhikari, S., "Generalized Wishart distribution for probabilistic structural dynamics", Computational Mechanics, 45[5] (2010), pp. 495-511.
[4 Adhikari, S., and Sarkar, A., "Uncertainty in structural dynamics: experimental validation of a Wishart random matrix model", *Journal of Sound and Vibration*, 323[3-5] (2009), pp. 802-825.
[5] Adhikari, S., "Matrix variate distributions for probabilistic structural mechanics", *AIAA Journal*, 45[7] (2007), pp. 1748-1762.
[6] Adhikari, S., "Wishart random matrices in probabilistic structural mechanics", ASCE Journal of Engineering Mechanics, 134[12] (2008), pp. 1029-1044.

How to obtain the dispersion parameters?

Suppose a random system matrix is expressed as

$$\mathbf{A} = \mathbf{A}_0 + \sum_{j=1}^{M} \epsilon \xi_j(\theta) \mathbf{A}_j$$

It can be shown that the dispersion parameter is given by

$$\delta_A^2 = \frac{\epsilon_A^2 \mathrm{Trace}\left(\left(\sum_{j=1}^{M} \sum_{k=1}^{M} \mathrm{E}\left[\xi_j(\theta) \xi_k(\theta) \right] \mathbf{A}_j \mathbf{A}_k \right) \right)}{\| \mathbf{A}_0 \|_{\mathrm{F}}^2}$$

$$= \frac{\epsilon_A^2 \mathrm{Trace}\left(\left(\sum_{j=1}^{M} \mathbf{A}_j^2 \right) \right)}{\| \mathbf{A}_0 \|_{\mathrm{F}}^2} = \epsilon_A^2 \frac{\sum_j^M \| (\mathbf{A}_j) \|_{\mathrm{F}}^2}{\| \mathbf{A}_0 \|_{\mathrm{F}}^2}$$

Therefore, it can be calculated using sensitivity matrices within a finite element formulation

Dynamic Response

- Taking the Fourier transform of the equation of motion

$$\left[-\omega^2\mathbf{M}(\theta) + \mathrm{i}\omega\mathbf{C}(\theta) + \mathbf{K}(\theta)\right]\bar{\mathbf{u}}(\mathrm{i}\omega) = \bar{\mathbf{f}}(\mathrm{i}\omega)$$

- Transforming into a reduced modal coordinate we have

$$\left[-\omega^2\mathbf{I}_m + \mathrm{i}\omega\mathbf{C}' + \mathbf{\Omega}^2\right]\bar{\mathbf{u}}' = \bar{\mathbf{f}}'$$

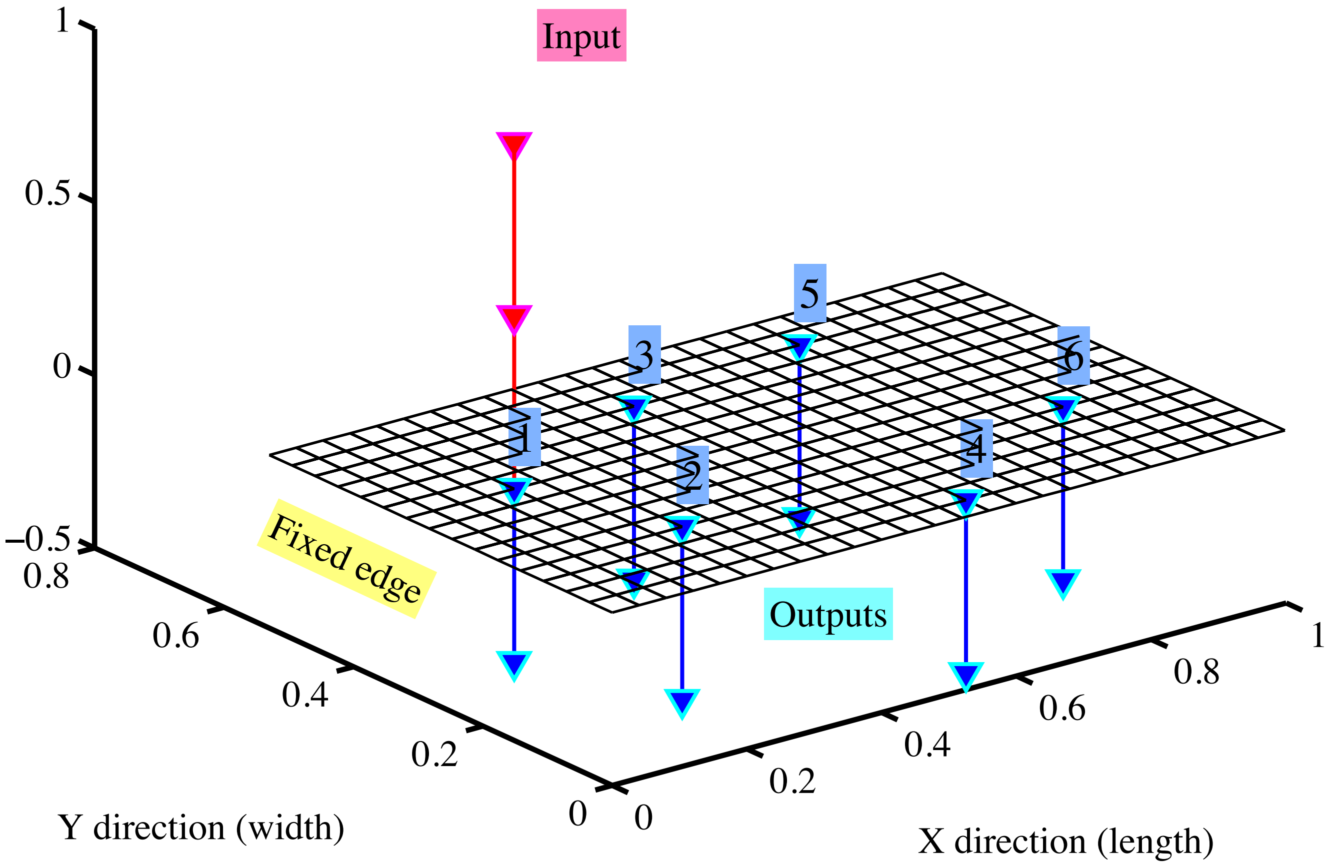
- Solving a random eigenvalue problem for the random matrix Ω^2, the uncertainty propagation can be expressed

$$\mathbf{u}(\omega,\theta) = \sum_{k=1}^{n_r} \frac{\mathbf{x}_{r_k}(\theta)^T \bar{\mathbf{f}}(\omega)}{-\omega^2 + 2i\omega\zeta_k\omega_{r_k}(\theta) + \omega_{r_k}^2(\theta)} \mathbf{x}_{r_k}(\theta)$$

$$\mathbf{X}_r(\theta) = \mathbf{\Phi}\mathbf{\Psi}_r(\theta), \quad \mathbf{\Psi}_r^T\mathbf{W}\mathbf{\Psi}_r = \mathbf{\Omega}_r^2$$

- The matrix Ω^2 is a Wishart matrix (called as a reduced diagonal Wishart matrix) who's parameters can be obtained explicitly from the dispersions parameters of the mass and stiffness matrices.

An example: A vibrating plate



A thin cantilever plate with random properties and 0.7% fixed modal damping.

Physical properties

Plate Properties	Numerical values
Length (L_x)	998 mm
Width (L_y)	530 mm
Thickness (t_h)	3.0 mm
Mass density (ρ)	7860 kg/m^3
Young's modulus (E)	2.0×10^5 MPa
Poisson's ratio (μ)	0.3
Total weight	12.47 kg

The data presented here are available from:

http://engweb.swan.ac.uk/~adhikaris/uq

Uncertainty type 1

The Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\left(1 + \epsilon_E f_1(\mathbf{x})\right)$$

$$\mu(\mathbf{x}) = \bar{\mu}\left(1 + \epsilon_\mu f_2(\mathbf{x})\right)$$

$$\rho(\mathbf{x}) = \bar{\rho}\left(1 + \epsilon_\rho f_3(\mathbf{x})\right)$$

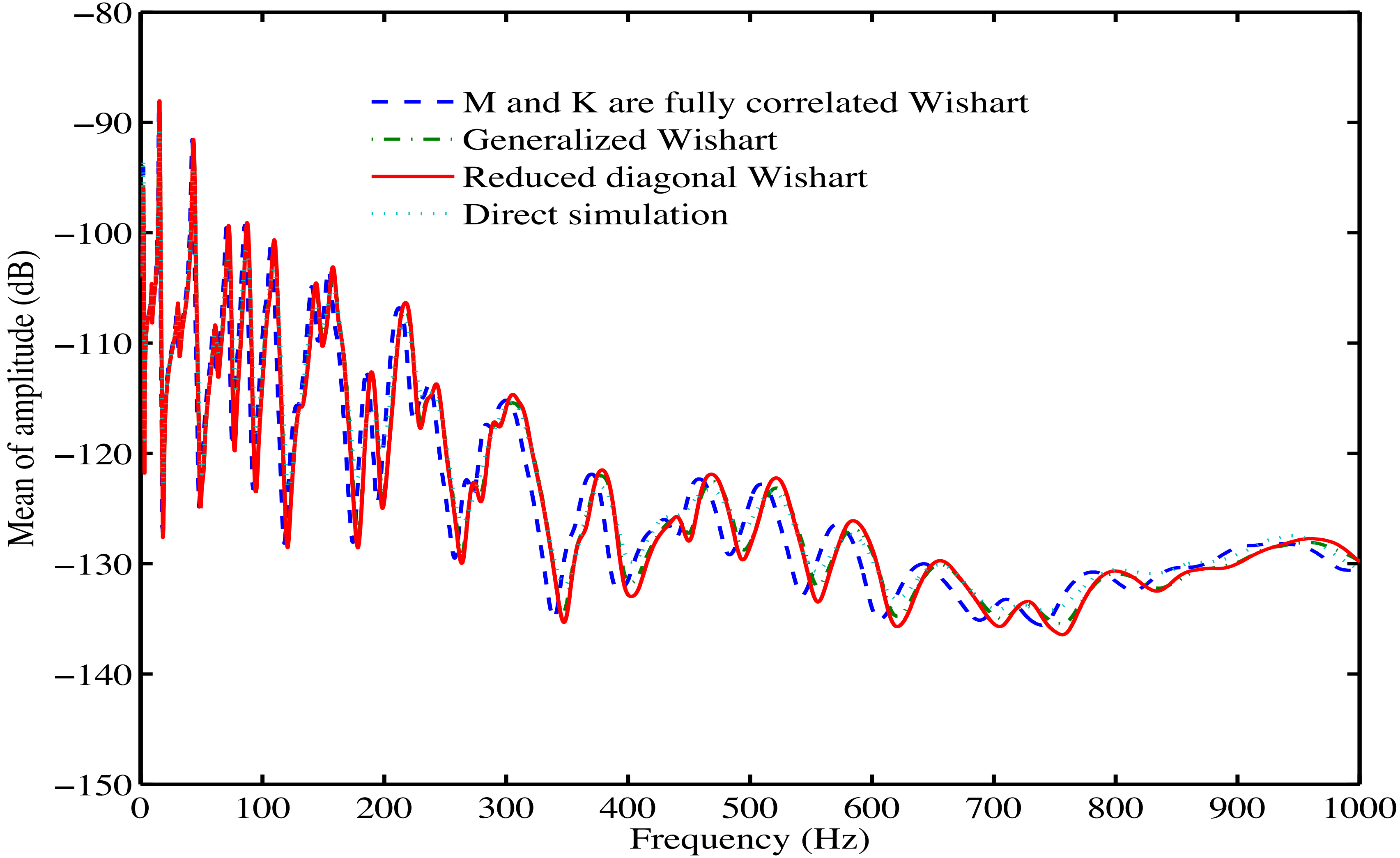
and $$t(\mathbf{x}) = \bar{t}\left(1 + \epsilon_t f_4(\mathbf{x})\right)$$

- The strength parameters: $\epsilon_E = 0.15$, $\epsilon_\mu = 0.15$, $\epsilon_\rho = 0.10$ and $\epsilon_t = 0.15$.
- The random fields $f_i(\mathbf{x}), i = 1, \cdots, 4$ are delta-correlated homogenous Gaussian random fields.

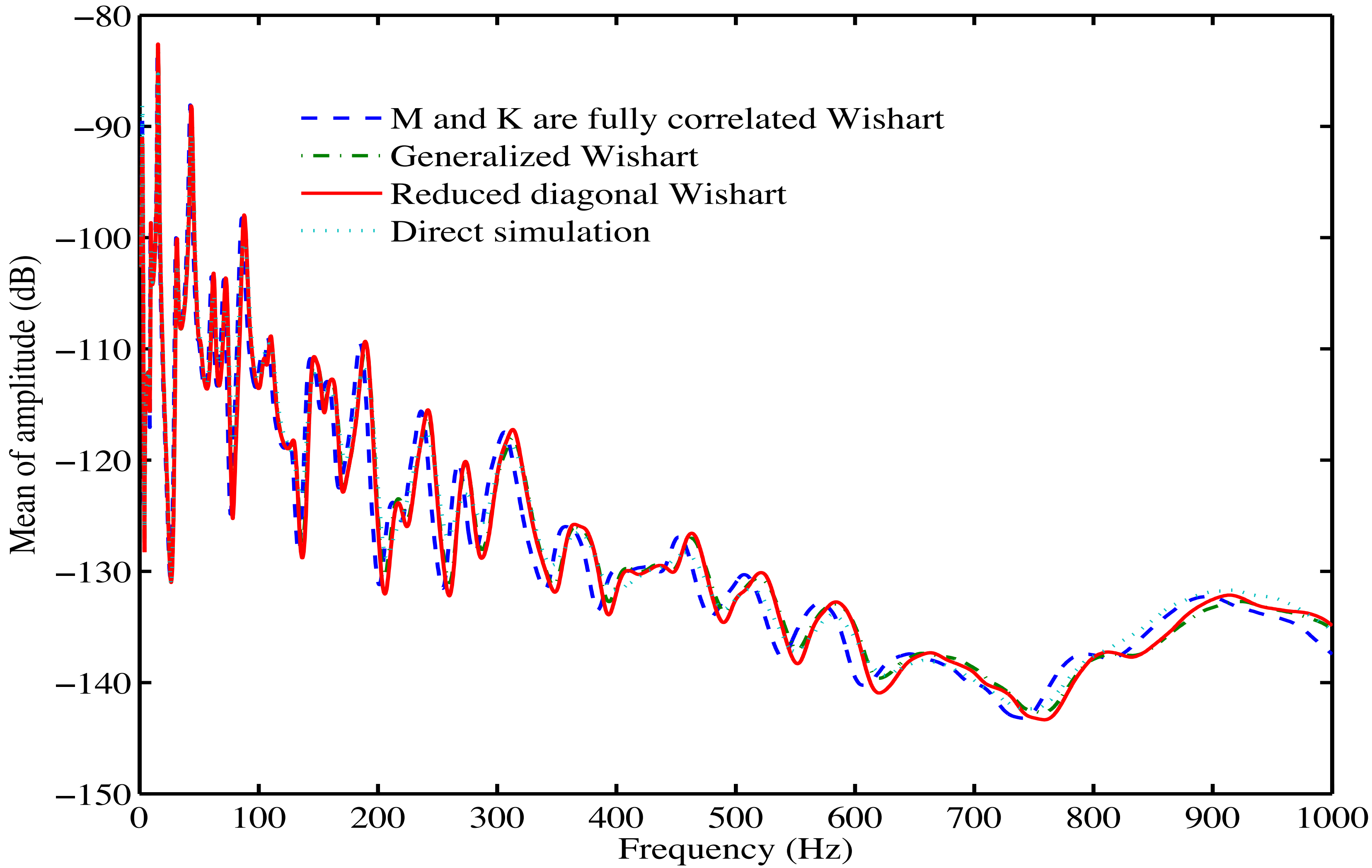
Uncertainty type 2

- Here we consider that the baseline plate is `perturbed' by attaching 10 oscillators with random spring stiffnesses at random locations
- This is aimed at modeling non-parametric uncertainty only.
- This case will be investigated experimentally also.

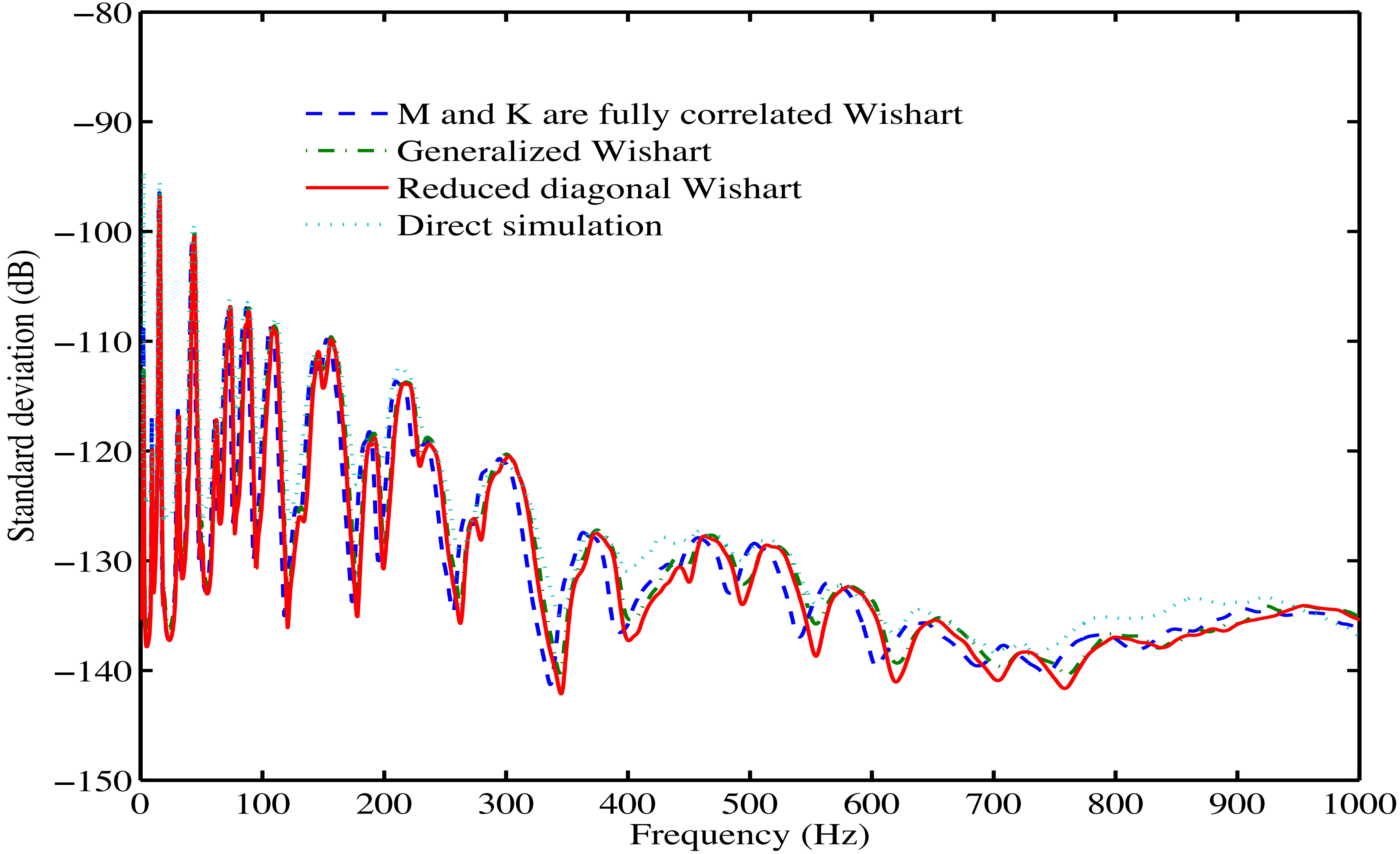
Mean of a cross-FRF: Utype 1



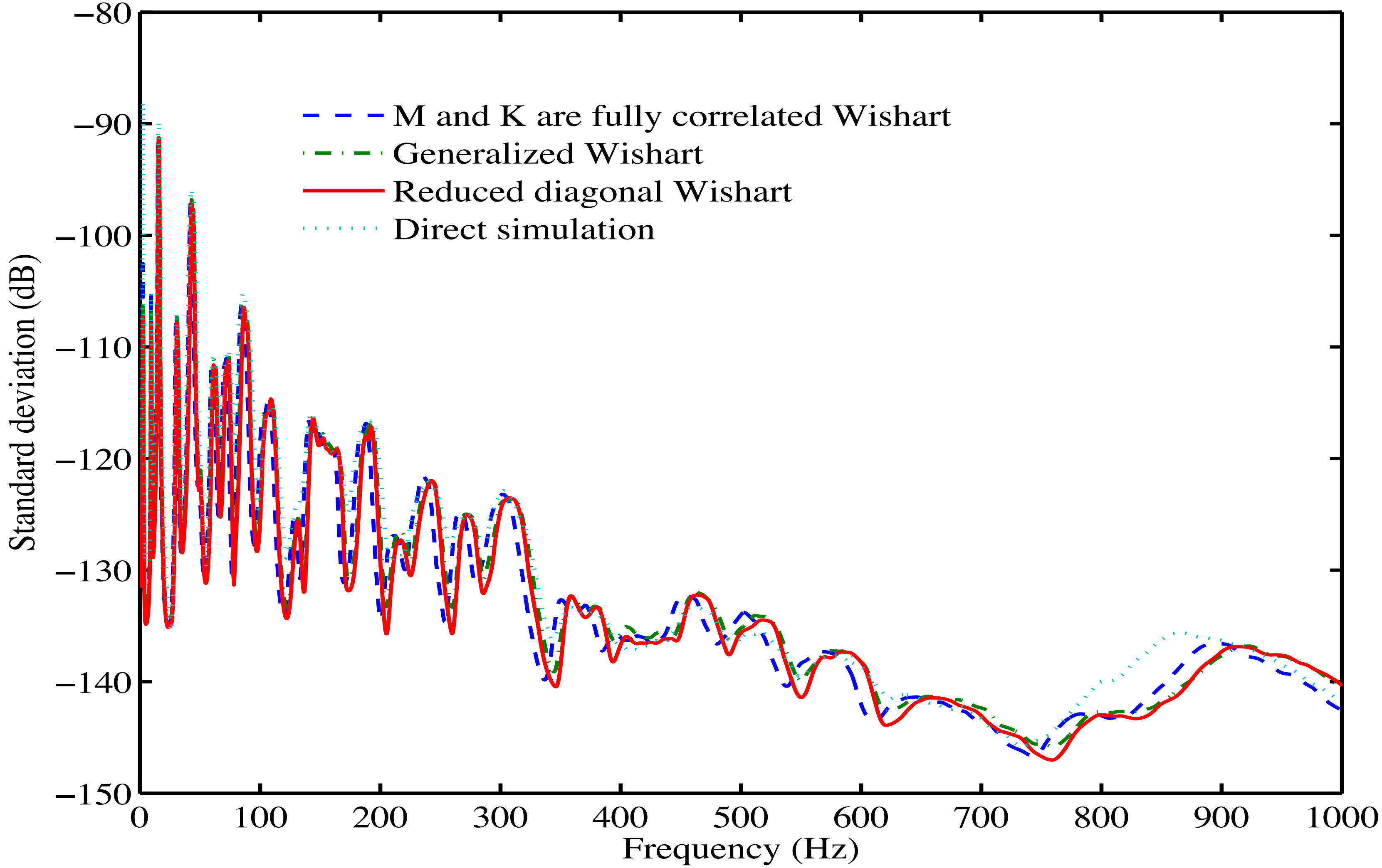
Mean of the driving-point-FRF: Utype 1



Standard deviation of a cross-FRF: Utype 1



Standard deviation of the driving-point-FRF: Utype 1



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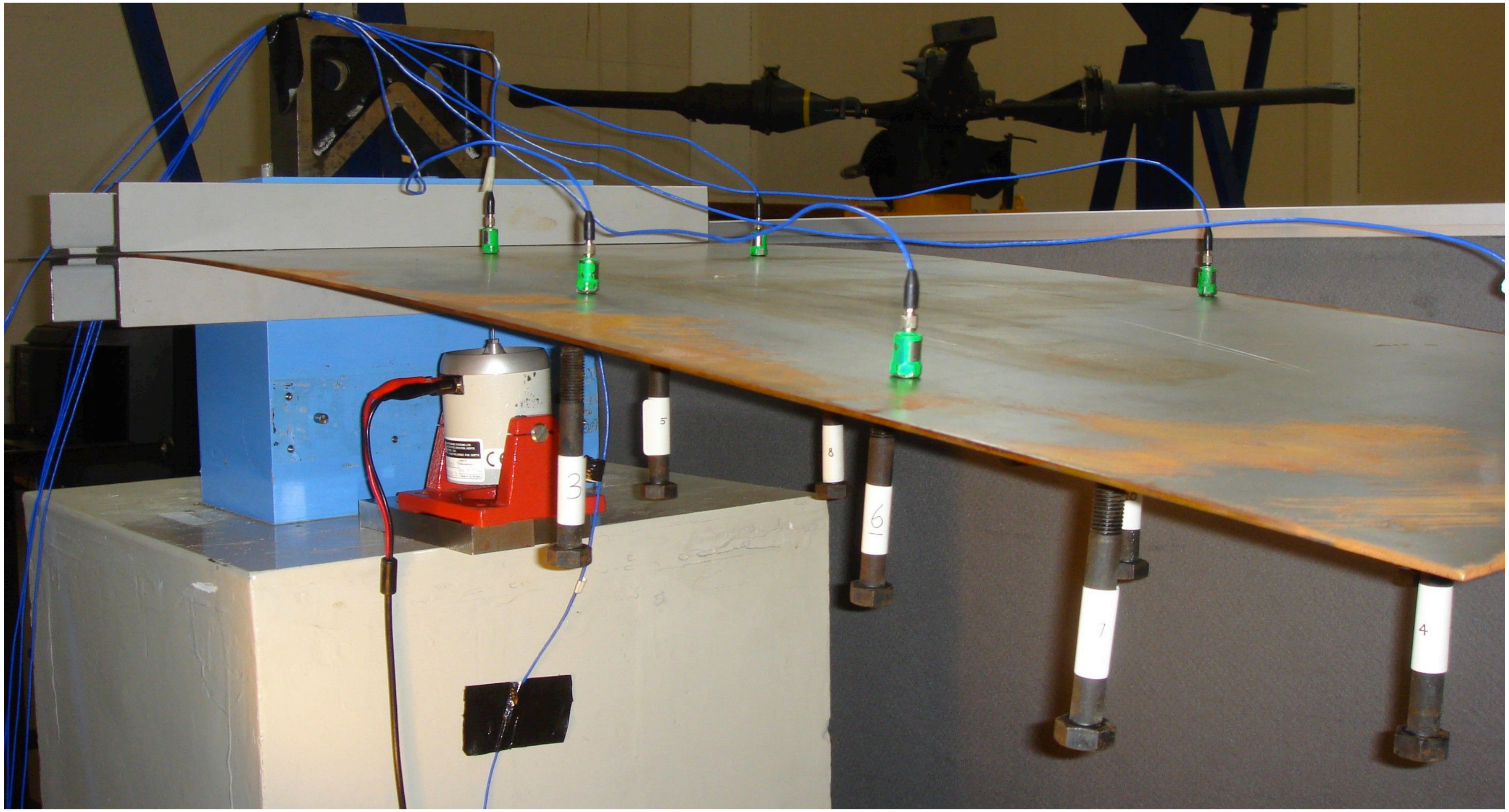
Computational method and validation

- Representative experimental results
  - Plate with randomly placed oscillator

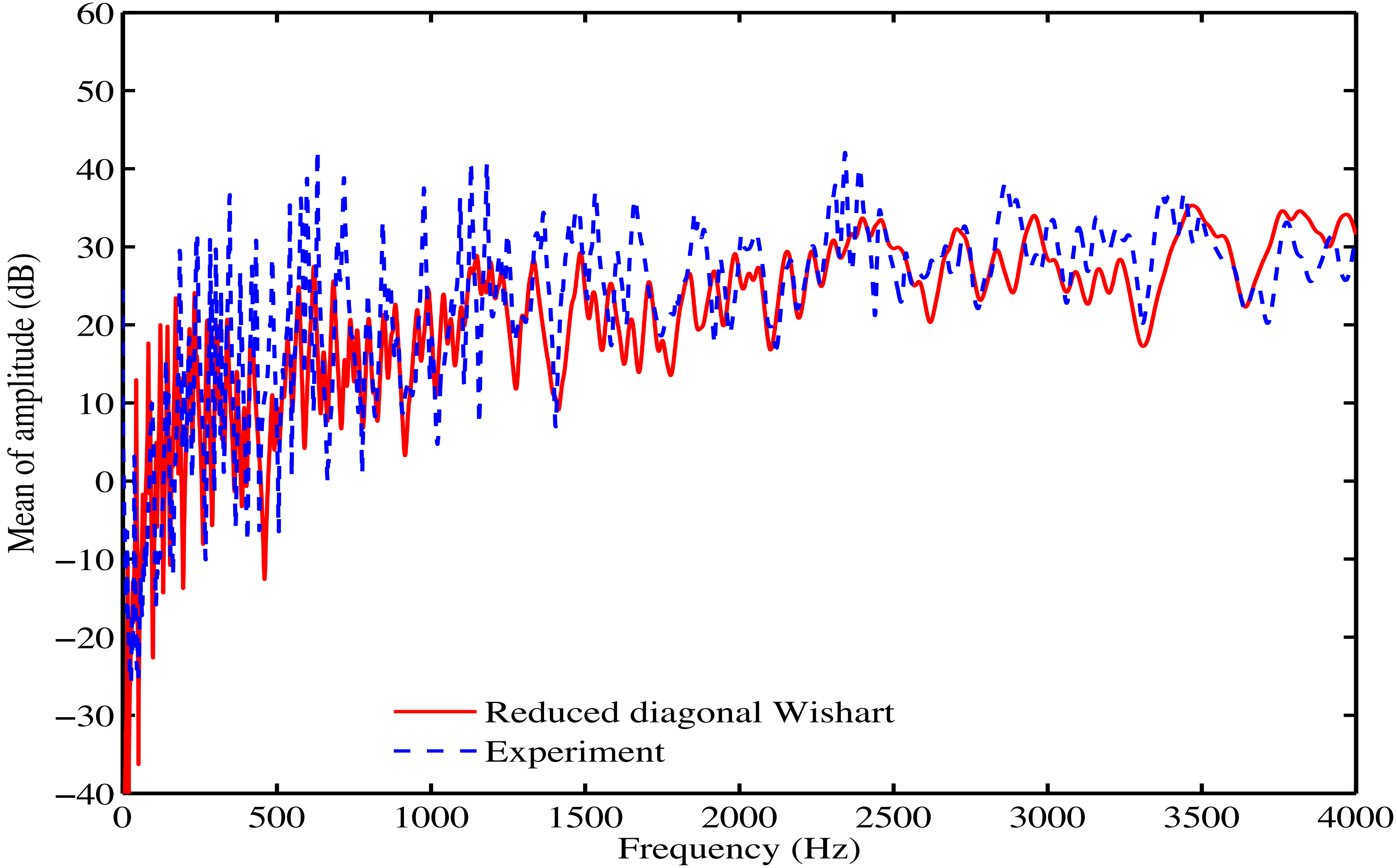
- Software integration
  - Integration with ANSYS

Plate with randomly placed oscillators

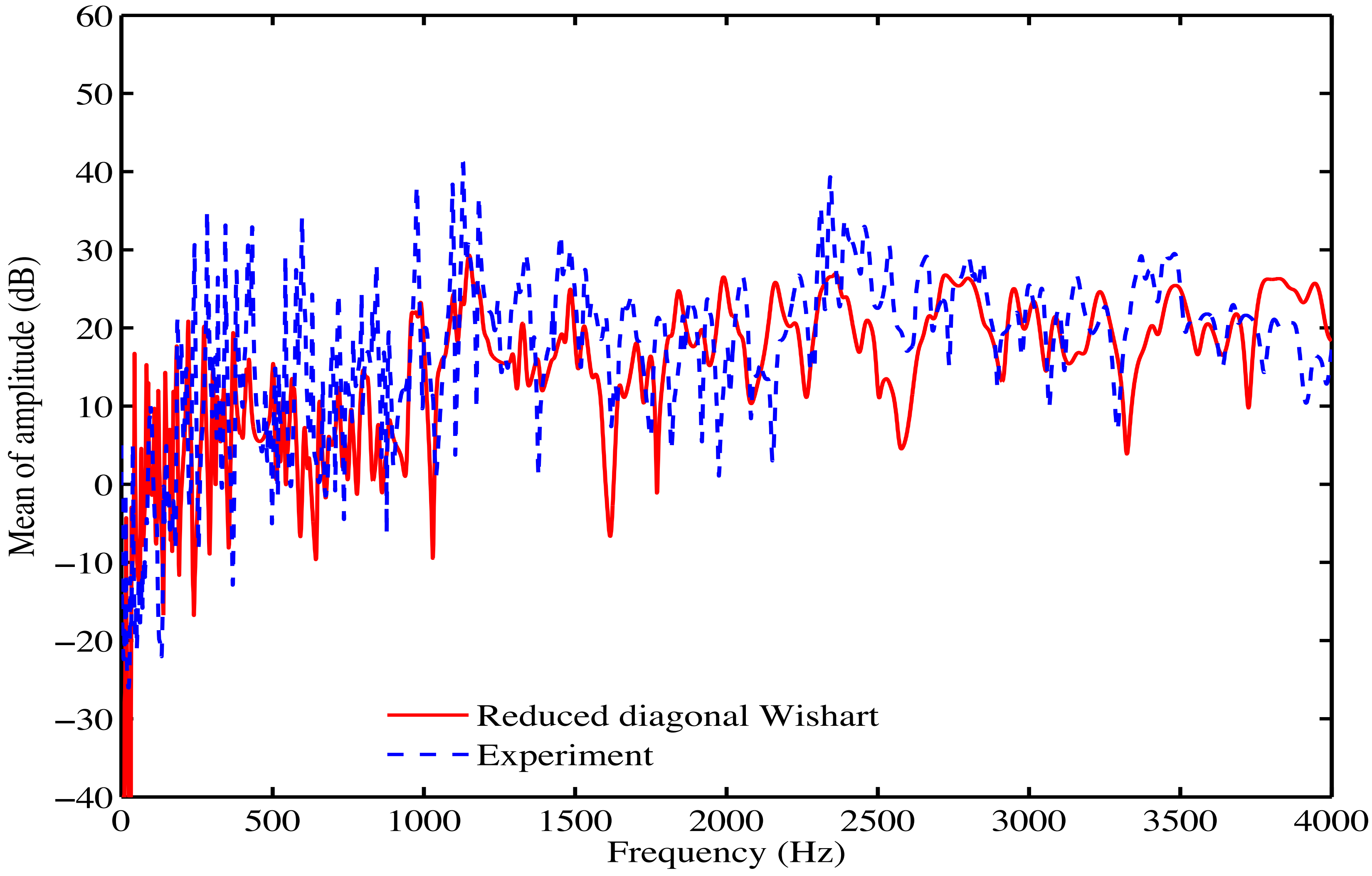


10 oscillators with random stiffness values are attached at random locations in the plate by magnet

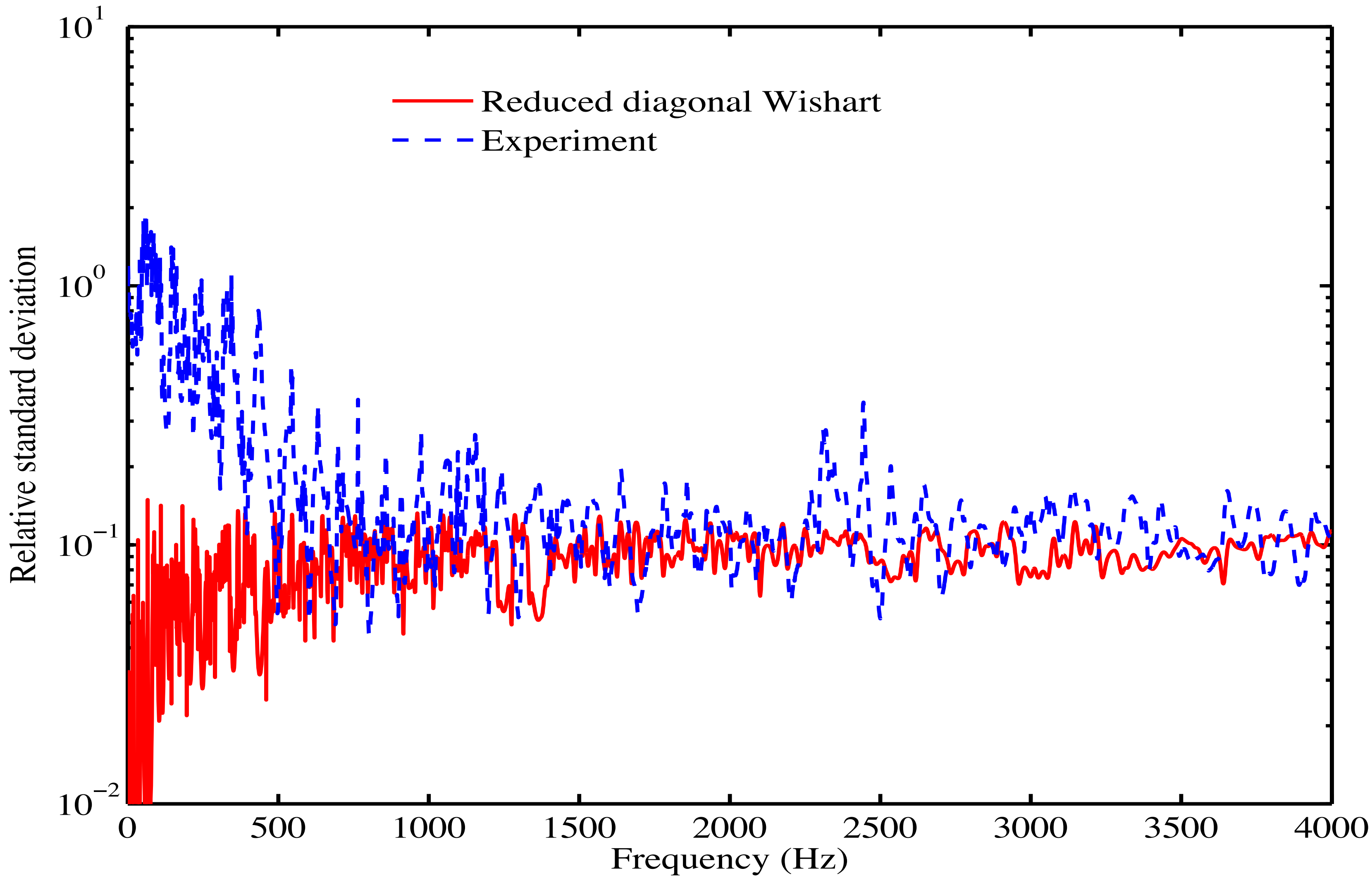
Mean of a cross-FRF



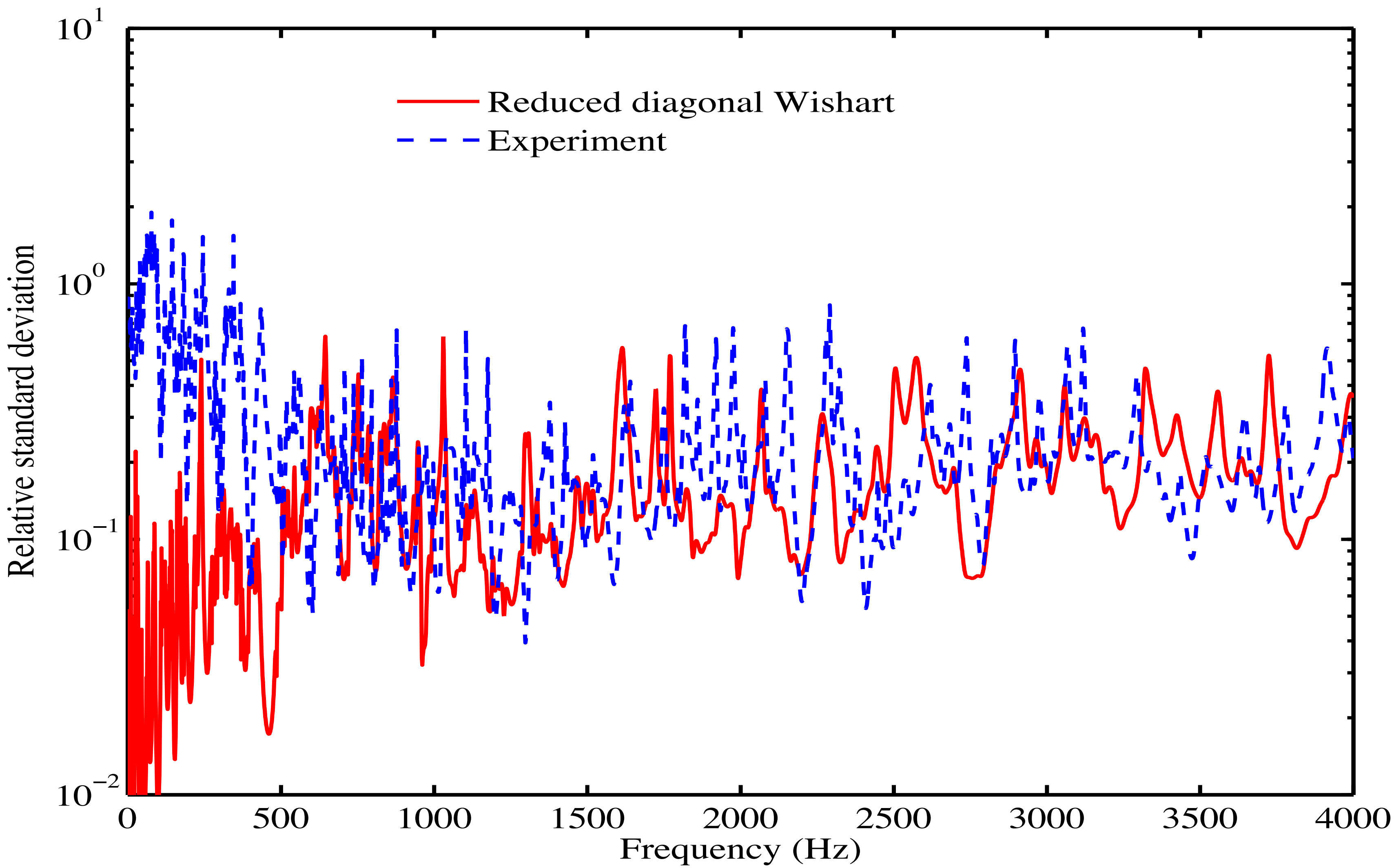
Mean of the driving-point-FRF



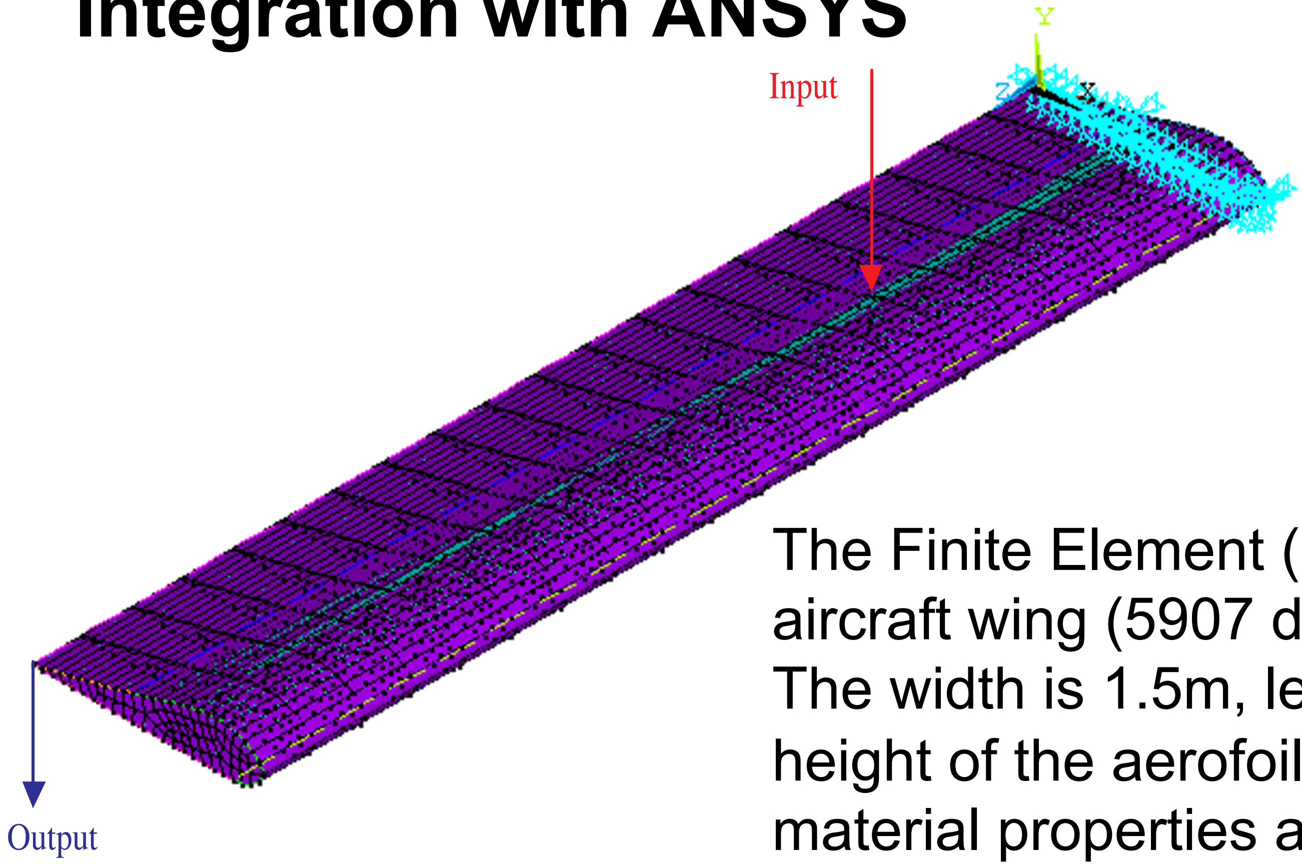
Standard deviation of a cross-FRF



Standard deviation of the driving-point-FRF

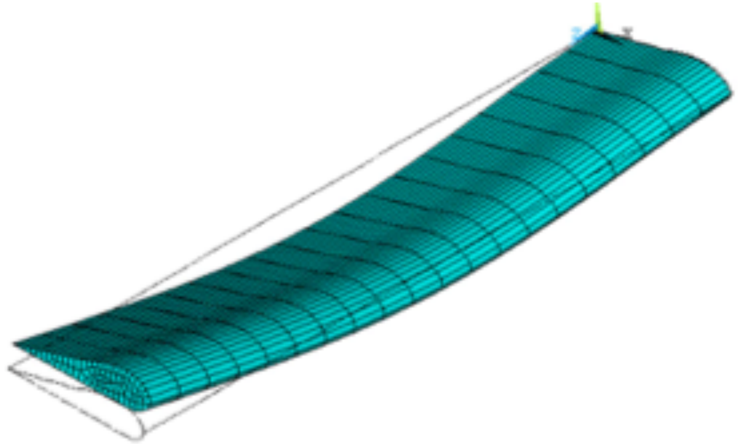


Integration with ANSYS

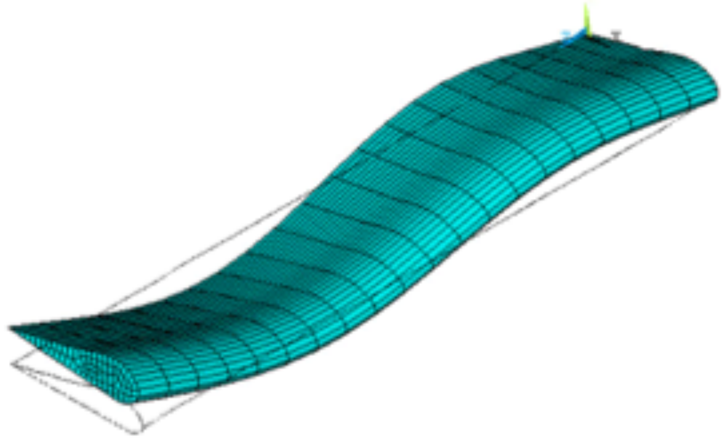


The Finite Element (FE) model of an aircraft wing (5907 degrees-of-freedom). The width is 1.5m, length is 20.0m and the height of the aerofoil section is 0.3m. The material properties are: Young's modulus 262Mpa, Poisson's ratio 0.3 and mass density 888.10kg/m^3. Input node number: 407 and the output node number 96. A 2% modal damping factor is assumed for all modes.

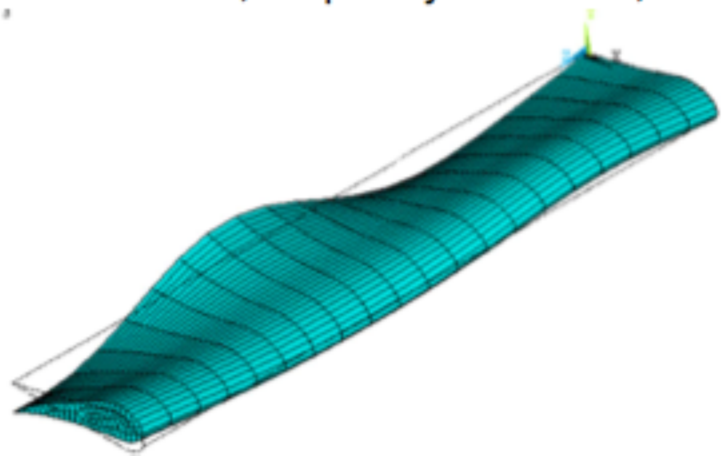
Vibration modes



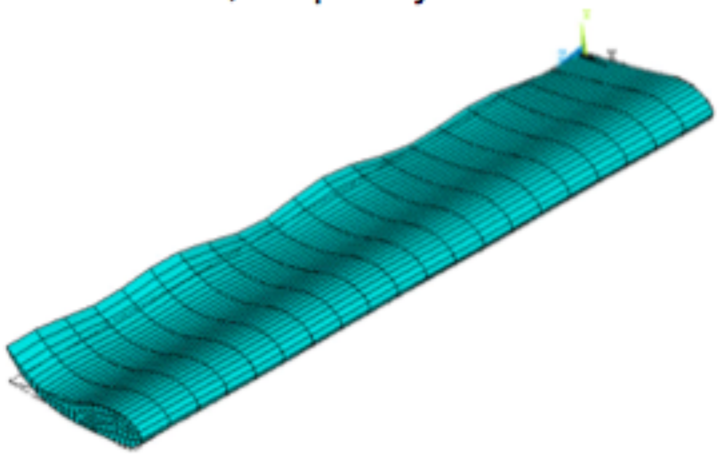
Mode 3, frequency 19.047Hz,



Mode 5, frequency 53.628Hz

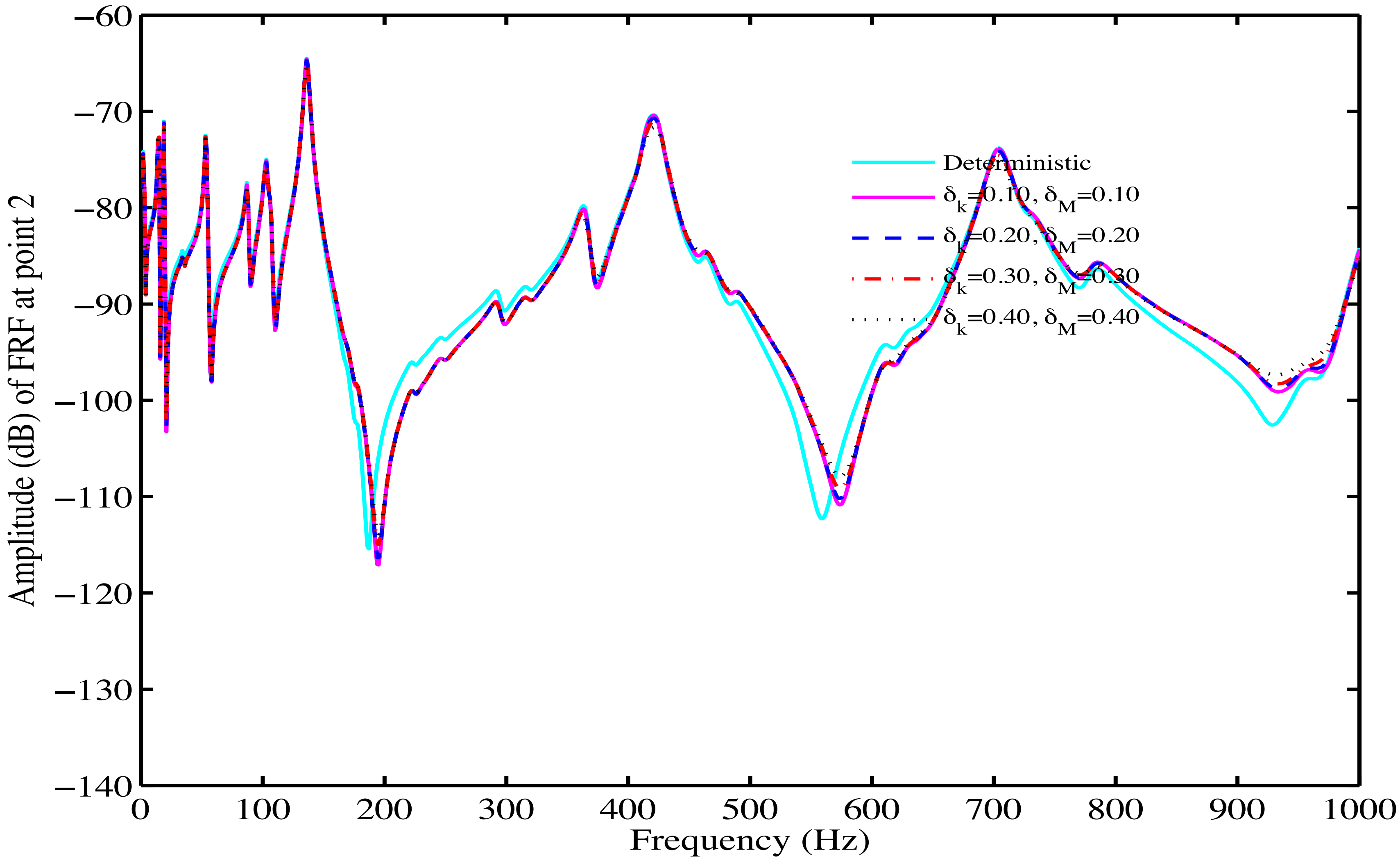


Mode 10, frequency 168.249Hz,

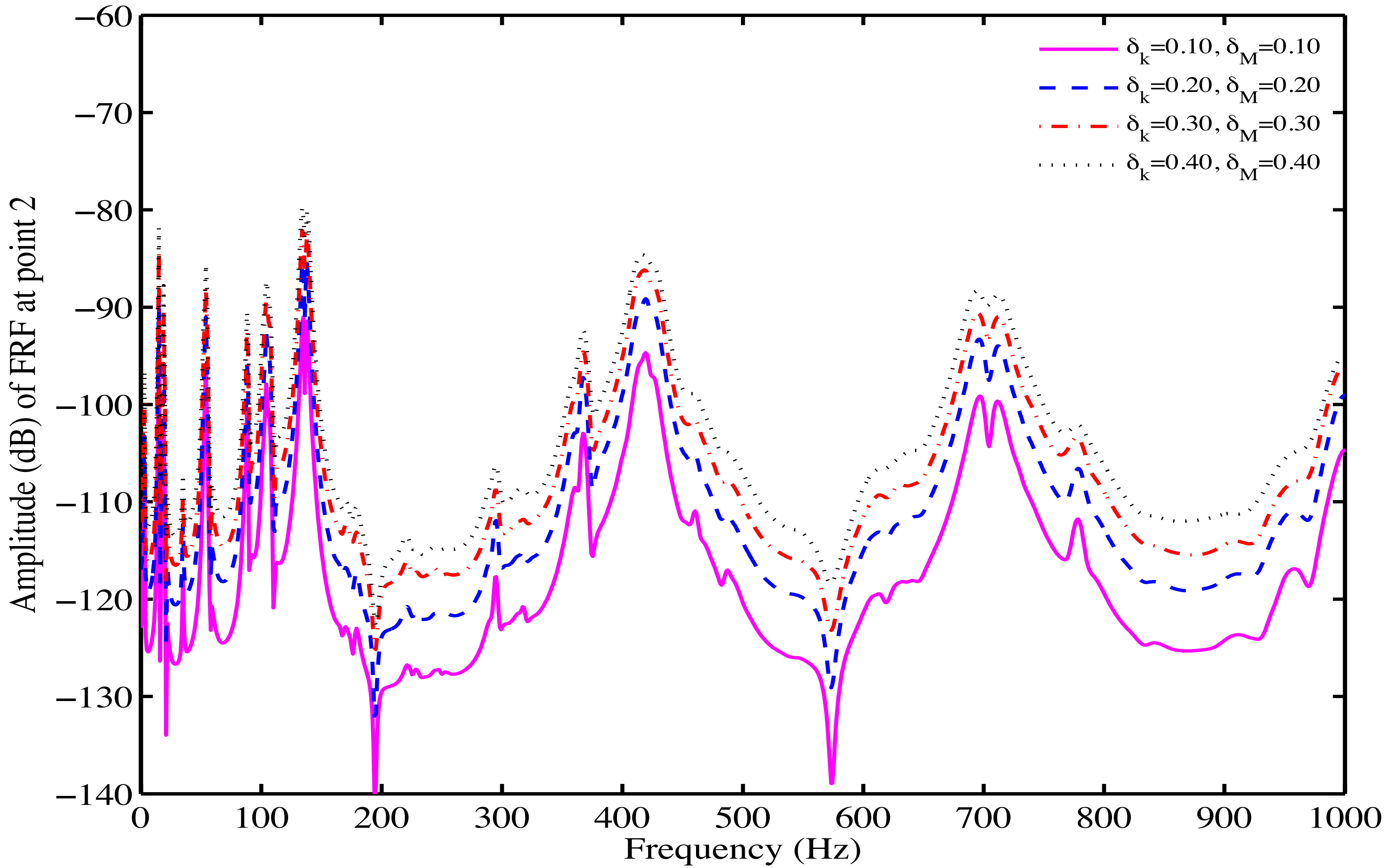


Mode 20, frequency 403.711Hz

Mean of a cross-FRF



Standard deviation of a cross-FRF



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Summary and Conclusions

Dynamic Response

- For parametric uncertainty propagation:

$$\mathbf{u}(\omega,\theta) = \sum_{k=1}^{n_r} \frac{\phi_k^T \mathbf{f}(\omega)}{-\omega^2 + 2i\omega\zeta_k\omega_0^2 + \omega_{0_k}^2 + \sum_{i=1}^{M} \xi_i(\theta)\Lambda_{i_k}(\omega)} \phi_k$$

- For nonparametric uncertainty propagation

$$\mathbf{u}(\omega,\theta) = \sum_{k=1}^{n_r} \frac{\mathbf{x}_{r_k}(\theta)^T \mathbf{f}(s)}{-\omega^2 + 2i\omega\zeta_k\omega_{r_k}(\theta) + \omega_{r_k}^2(\theta)} \mathbf{x}_{r_k}(\theta)$$

$$\mathbf{X}_r(\theta) = \mathbf{\Phi}\mathbf{\Psi}_r, \quad \mathbf{\Psi}_r^T \mathbf{W} \mathbf{\Psi}_r = \mathbf{\Omega}_r^2$$

- Unified mathematical representation
- Can be useful for hybrid experimental-simulation approach for uncertainty quantification

Summary

- Response of stochastic dynamical systems is projected in to the basis of undamped modes
- The coefficient functions, called as the spectral functions, are expressed in terms of the spectral properties of the system matrices in the frequency domain.
- The proposed method takes advantage of the fact that for a given maximum frequency only a small number of modes are necessary to represent the dynamic response. This modal reduction leads to a significantly smaller basis.
- Wishart random matrix model can used to represent non-parametric uncertainty directly at the system matrix level.
- Reduced computational approach can be implemented within the conventional finite element environment

Summary

- Dispersion parameters necessary for the Wishart model can be obtained, for example, using sensitivity matrices
- Both parametric and nonparametric uncertainty can be propagated via an unified mathematical framework.
- Future work will exploit this novel representation for model validation and updating in conjunction with measured data.